GENERALIZATIONS OF CLINE’S FORMULA FOR THREE GENERALIZED INVERSES

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Abstract. It is shown that an element $a$ in a ring is Drazin invertible if and only if so is $a^n$; the Drazin inverse of $a$ is given by that of $a^n$, and vice versa. Using this result, we prove that, in the presence of $aba = aca$, for any natural numbers $n$ and $m$, $(ac)^n$ is Drazin invertible in a ring if and only if so is $(ba)^m$; the Drazin inverse of $(ac)^n$ is expressed by that of $(ba)^m$, and vice versa. Also, analogous results for the pseudo Drazin inverse and the generalized Drazin inverse are established on Banach algebras.

1. Introduction

Throughout this paper, $A$ will denote a complex Banach algebra with identity $1$ and $R$ will denote an associative ring with identity $1$. $J(R)$ (resp. $J(A)$) denotes the Jacobson radical of $R$ (resp. $A$). The commutant and double commutant of an element $a \in R$ are defined as usual by

$$\text{comm}(a) = \{x \in R, ax = xa\}$$

and

$$\text{comm}^2(a) = \{x \in R, xy = yx \text{ for all } y \in \text{comm}(a)\},$$

respectively. We say that $b \in R$ is an outer generalized inverse of an element $a \in R$ provided that $bab = b$. The element $b \in R$ is not unique in general. In order to force its uniqueness, further conditions have to be imposed.

In 1958, Drazin [4] introduced the following generalized inverse. An element $a \in R$ is called Drazin invertible provided that there is a common solution to the equations

$$b \in \text{comm}(a), \quad bab = b \quad \text{and} \quad a^k ba = a^k \text{ for some } k \geq 0. \quad (1.1)$$

If such a solution exists, then it is unique and is called a Drazin inverse of $a$, denoted as usual by $b = a^D$. The minimal $k$ for which (1.1) holds is called the Drazin index $i(a)$ of $a$. Moreover, $a^D \in \text{comm}^2(a)$ (see [4, Theorem 1]). If $i(a) \leq 1$, then $a$

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is group invertible and $b$ is called a group inverse of $a$. Some applications of the group and Drazin inverses can be found in [5, 7, 8, 9, 10, 13, 16].

In 2002, Koliha and Patr´ıcio [12] introduced the notion of generalized Drazin inverse. The generalized Drazin inverse is the unique common solution to the equations

$$ b \in \text{comm}^2(a), \ bab = b \quad \text{and} \quad aba - a \text{ is quasinilpotent.} \quad (1.2) $$

Here an element $s \in R$ is said to be quasinilpotent [6] if $1 + st$ is invertible for all $t \in \text{comm}(s)$. If such a solution to (1.2) exists, then it is denoted as usual by $b = a^{gD}$ and we shall call the element $a \in R$ generalized Drazin invertible. According to [11, Theorem 4.4], the condition $b \in \text{comm}^2(a)$ in (1.2) can be weakened as $b \in \text{comm}(a)$ in the Banach algebra case.

An intermediate between the Drazin inverse and the generalized Drazin inverse is the pseudo Drazin inverse, which was introduced by Wang and Chen [17] in 2012. The pseudo Drazin inverse is the unique common solution to the equations

$$ b \in \text{comm}^2(a), \ bab = b \quad \text{and} \quad a^k ba - a^k \in J(R) \quad (1.3) $$

for some $k \geq 0$. The minimal such $k$ is called the pseudo Drazin index $i(a)$ of $a$. If such a solution to (1.3) exists, then it is denoted as usual by $b = a^{pD}$ and we shall call the element $a \in R$ pseudo Drazin invertible. Also, in a Banach algebra, the condition $b \in \text{comm}^2(a)$ in (1.3) can be weakened to $b \in \text{comm}(a)$ (see [17, Remark 5.1]).

In 1965, Cline [2] showed that if $ab$ is Drazin invertible then so is $ba$ and $(ba)^D = b((ab)^D)^2 a$. This equation is the so-called Cline’s formula. Cline’s formulas for the generalized Drazin inverse and the pseudo Drazin inverse were recently proved in [15] and [17], respectively. As extensions of Jacobson’s Lemma, in 2013 Corach, Duggal and Harte [3] firstly investigated common properties of $ac - 1$ and $ba - 1$ under the assumption

$$ aba = aca, $$

where $a, b, c \in R$. Recently, we extended Cline’s formula for the Drazin inverse, the pseudo Drazin inverse and the generalized Drazin inverse to the case when $aba = aca$ (see [14, 18]).

In this paper, we show that the Drazin invertibility of an element $a \in R$ is equivalent to that of $a^n$; the Drazin inverse of $a$ is given by that of $a^n$, and vice versa. Using this result, we prove that, in the presence of $aba = aca$, for any natural numbers $n$ and $m$, the Drazin invertibility of $(ac)^n$ in a ring is equivalent to that of $(ba)^m$; the Drazin inverse of $(ac)^n$ is expressed by that of $(ba)^m$, and vice versa. Also, we establish analogous results for the pseudo Drazin inverse and the generalized Drazin inverse on Banach algebras.

2. Main results

In [11 Theorem 2.3], Berkani and Sarih showed that an element $a$ in an algebra with unit is Drazin invertible if and only if $a^n$ is Drazin invertible. In the following, we give a different proof which holds in the frame of rings.

Theorem 2.1. Let $a \in R$ and $n \in \mathbb{N}$. Then $a$ is Drazin invertible if and only if $a^n$ is Drazin invertible. In this case, we have

$$ (a^n)^D = (a^D)^n, \quad (2.1) $$

and

$$ a^D = (a^n)^D a^{n-1} \quad (2.2) $$

and

$$ \frac{i(a)}{n} \leq i(a^n) < \frac{i(a)}{n} + 1. \quad (2.3) $$

Proof. Since (2.1) and (2.3) have been proved by Drazin (see [4, Theorem 2]), it suffices to show (2.2). Suppose that $a^n$ is Drazin invertible and let $b = (a^n)^D$. Next, we show that $a^D = ba^{n-1}$.

(i) Since $a^n a = aa^n$, $a(ba^{n-1}) = baa^{n-1} = (ba^{n-1})a$.

(ii) We have $(ba^{n-1})a = (ba^n)b = ba^{n-1}$.

(iii) Let $i(a^n) = k$ and $l = nk$. Since $a^{n-1}a^n = a^n a^{n-1}$, we have

$$ a^{l+1}(ba^{n-1}) = a^{nk+1}(ba^{n-1}) $$

$$ = a^{nk+1}a^{n-1}b $$

$$ = (a^n)^{k+1}b $$

$$ = (a^n)^k = a^l. $$

This completes the proof. \qed

Let us remark that the Drazin index of $a^n$ is uniquely determined by that of $a$, but not vice versa.

Lemma 2.2. ([18, Theorem 2.7]) Suppose that $a, b, c \in R$ satisfy $aba = ac$. Then $ac$ is Drazin invertible if and only if $ba$ is Drazin invertible. In this case, we have

1. $|i(ac) - i(ba)| \leq 1$;
2. $(ba)^D = b((ac)^D)^2a$ and $(ac)^D = a((ba)^D)^2c$.

In the following theorem, in the presence of $aba = ac$, we give explicit expressions for the Drazin inverses of $(ac)^n$ and $(ba)^m$, both in terms of each other.

Theorem 2.3. Suppose that $a, b, c \in R$ satisfy $aba = ac$ and let $n, m \in \mathbb{N}$.

1. If $(ac)^n$ is Drazin invertible, then $(ba)^m$ is Drazin invertible. In this case, we have

$$ ((ba)^m)^D = b((ac)^n)^D a(ba)^{n-m-1} \quad \text{if } n \geq m + 1, $$

$$ ((ba)^m)^D = b[(ac)^n]^m+2-n a(ba)^{(n-1)(m+1-n)} \quad \text{if } n < m + 1. $$

2. If $(ba)^n$ is Drazin invertible, then $(ac)^m$ is Drazin invertible. In this case, we have

$$ ((ac)^m)^D = a((ba)^n)^D c(ac)^{n-m-1} \quad \text{if } n \geq m + 1, $$

$$ ((ac)^m)^D = a[(ba)^n]^m+2-n c(ac)^{(n-1)(m+1-n)} \quad \text{if } n < m + 1. $$
Proof. (1) If \( n \geq m + 1 \), by Theorem 2.1 and Lemma 2.2 we have

\[
((ba)^m)^D = ((ba)^D)^m
= [b((ac)^D)^2 a]^{m}
= [b((ac)^D)^2 a][b((ac)^D)^2 a] \cdots [b((ac)^D)^2 a]
= [b((ac)^D)^2 a][bac((ac)^D)^3 a][b((ac)^D)^2 a] \cdots [b((ac)^D)^2 a]
= [b((ac)^D)^2 a][cac((ac)^D)^3 a][b((ac)^D)^2 a] \cdots [b((ac)^D)^2 a]
= [b((ac)^D)^2][((ac)^D a)[b((ac)^D)^2 a] \cdots [b((ac)^D)^2 a]
= \cdots
= b((ac)^D)^{m+1} a
= b((ac)^D)^n (ac)^{n-m-1} a
= b((ac)^n)^D a (ba)^{n-m-1}.
\]

If \( n < m + 1 \), by Theorem 2.1 and Lemma 2.2 we have

\[
((ba)^m)^D = ((ba)^D)^m
= [b((ac)^D)^2 a]^{m}
= [b((ac)^D)^2 a][b((ac)^D)^2 a] \cdots [b((ac)^D)^2 a]
= b((ac)^D)^{m+1} a
= b((ac)^D)^n ((ac)^D)^{m+1-n} a
= b((ac)^n)^D ((ac)^n-1)^{m+1-n} a
= b((ac)^n)^D^{m+2-n} (ac)^{(n-1)(m+1-n)} a
= b((ac)^n)^D^{m+2-n} a (ba)^{(n-1)(m+1-n)}.
\]

(2) The proof is similar to that of (1). \( \square \)

The following result concerns explicit expressions for the pseudo Drazin inverses of \( a \) and \( a^n \), both in terms of each other.

Theorem 2.4.

(1) Let \( a \in R \) and \( n \in \mathbb{N} \). If \( a^n \) is pseudo Drazin invertible, then \( a \) is pseudo Drazin invertible and \( a^p^D = (a^n)^{p^D} a^{n-1} \).

(2) Let \( a \in A \) and \( n \in \mathbb{N} \). If \( a \) is pseudo Drazin invertible, then \( a^n \) is pseudo Drazin invertible and \( (a^n)^{p^D} = (a^D)^n \).
Moreover,\[
\frac{i(a)}{n} \leq i(a^n) < \frac{i(a)}{n} + 1.
\]

**Proof.** (1) Suppose that \(a^n\) is pseudo Drazin invertible and let \(b = (a^n)^{pD}\). Next, we show that \(a^{pD} = ba^{n-1}\).

(i) We have \((ba^{n-1})a(ba^{n-1}) = (ba^n)b) = ba^{n-1}\).

(ii) Let \(c \in \text{comm}(a)\). Then \(ca^n = a^n c\), and since \(b \in \text{comm}^2(a^n)\), \(bc = cb\). Therefore \(c(ba^{n-1}) = b(ca^{n-1}) = (ba^{n-1})c\). Consequently, \(ba^{n-1} \in \text{comm}^2(a)\).

(iii) Let \(i(a^n) = k\) and \(l = nk\). Since \(a^{n-1}a^n = a^n a^{n-1}\), we have
\[
\begin{align*}
(a^{n-1}b^{n-1}) - a^l &= (a^{nk+1}b^{n-1}) - a^k \\
&= a^{nk+1}a^{n-1}b - a^k \\
&= (a^n)^{k+1}b - (a^n)^{k} \in J(R).
\end{align*}
\]

From the above argument, one can also infer that \(\frac{i(a^n)}{n} \leq \frac{i(a)}{n} + 1\), we also let \(i(a^n) = k\). Then we need to show that \(i(a) > nk - n\). Otherwise, \(i(a) \leq nk - n\) would mean that
\[
\begin{align*}
(a^n)^k(a^n)^{pD} - (a^n)^{k-1} &= a^{nk-n+1}a^{n-1}(a^n)^{pD} - a^{nk-n} \\
&= a^{nk-n+1}(a^n)^{pD}a^{n-1} - a^{nk-n} \\
&= a^{nk-n+1}a^{pD} - a^{nk-n} \in J(R),
\end{align*}
\]
which contradicts the fact that \(i(a^n) = k\).

(2) Suppose that \(a\) is pseudo Drazin invertible and let \(b = a^{pD}\). Next, we show that \((a^n)^{pD} = b^n\). Evidently, we have (i) \(b^n a^n b^n = b^n\), and (ii) \(b^n a^n = a^n b^n\). For (iii), let \(i(a) = m\) and \(q\) be an integer satisfying \(q \geq \frac{m}{n}\). Since \(ab = ba\) and \(bab = b\), \(a^{m+n}b^n = a^m b^n = a^{m+n}a^m = a^m \in J(A)\). Then
\[
(a^n)^{q+1}(b^n) - (a^n)^q = a^{ mq-n}a^{m+n}b^n - a^{mq-m}a^m \\
= a^{mq-m}(a^{m+n}b^n - a^m) \in J(A).
\]

This completes the proof. \(\square\)

Similarly, the pseudo Drazin index of \(a^n\) is uniquely determined by that of \(a\), but not vice versa.

**Lemma 2.5.** ([14 Theorem 2.4]) Suppose that \(a, b, c \in R\) satisfy \(aba = aca\). Then \(ac\) is pseudo Drazin invertible if and only if \(ba\) is pseudo Drazin invertible. In this case, we have

1. \(|i(ac) - i(ba)| \leq 1\);
2. \((ba)^{pD} = b((ac)^{pD})^2a\) and \((ac)^{pD} = a((ba)^{pD})^2c\).

In the Banach algebra case and with the presence of \(aba = aca\), we give in the following theorem explicit expressions for the pseudo Drazin inverses of \((ac)^n\) and \((ba)^n\), both in terms of each other.

**Theorem 2.6.** Suppose that \(a, b, c \in A\) satisfy \(aba = aca\) and let \(n, m \in \mathbb{N}\).
(1) If \((ac)^n\) is pseudo Drazin invertible, then \((ba)^m\) is pseudo Drazin invertible. In this case,
\[
((ba)^m)^{pD} = b((ac)^n)^{pD}a(ba)^{n-m-1} \quad \text{if } n \geq m + 1,
\]
\[
((ba)^m)^{pD} = b[((ac)^n)^{pD}]^{m+2-n}a(ba)^{(n-1)(m+1-n)} \quad \text{if } n < m + 1.
\]
(2) If \((ba)^n\) is pseudo Drazin invertible, then \((ac)^m\) is pseudo Drazin invertible. In this case,
\[
((ac)^m)^{pD} = a((ba)^n)^{pD}c(ac)^{n-m-1} \quad \text{if } n \geq m + 1,
\]
\[
((ac)^m)^{pD} = a[((ba)^n)^{pD}]^{m+2-n}c(ac)^{(n-1)(m+1-n)} \quad \text{if } n < m + 1.
\]

Proof. Apply the proof of Theorem 2.3 to the pseudo Drazin inverse, using Theorem 2.4 and Lemma 2.5. □

In the following theorem we give explicit expression for the generalized Drazin inverse of \(a\) in terms of \(a^n\).

Theorem 2.7.

(1) Let \(a \in R\) and \(n \in \mathbb{N}\). If \(a^n\) is generalized Drazin invertible, then \(a\) is generalized Drazin invertible and \(a^{gD} = (a^n)^{gD}a^{n-1}\).

(2) Let \(a \in A\) and \(n \in \mathbb{N}\). If \(a\) is generalized Drazin invertible, then \(a^n\) is generalized Drazin invertible and \((a^n)^{gD} = (a^{gD})^n\).

Proof. For part (2), see [11 Theorem 5.4(i)]. (1) Suppose that \(a^n\) is generalized Drazin invertible and let \(b = (a^n)^{gD}\). Next, we show that \(a^{gD} = ba^{n-1}\). As in the proof of Theorem 2.4, we get (i) \((ba^{n-1})a(ba^{n-1}) = ba^{n-1}\), and (ii) \(ba^{n-1} \in \text{comm}^2(a)\). (iii) Since \(b\) is a generalized Drazin inverse of \(a^n\), \(p = 1 - a^n b\) is an idempotent and commutes with \(a\), and hence \((pa)^n = pa^n\) is quasinilpotent. Let \(c \in \text{comm}(pa)\). Then \(c^n \in \text{comm}((pa)^n)\) and \(1 - (pa)^n c^n = (1 - pac)(1 + pac + (pac)^2 + \cdots + (pac)^{n-1})\) is invertible, and hence \(1 - pac\) is invertible. Therefore, \(a - a(ba^{n-1})a = (1 - a^n b)a = pa\) is quasinilpotent. □

Lemma 2.8. ([14 Theorem 2.3]) Suppose that \(a, b, c \in R\) satisfy \(aba = aca\). Then \(ac\) is generalized Drazin invertible if and only if \(ba\) is generalized Drazin invertible. In this case, we have \((ba)^{gD} = b((ac)^{gD})^2 a\) and \((ac)^{gD} = a((ba)^{gD})^2 c\).

In the following theorem, in the Banach algebra case and under the hypothesis \(aba = aca\), we give explicit expressions for the generalized Drazin inverses of \((ac)^n\) and \((ba)^m\), both in terms of each other.

Theorem 2.9. Suppose that \(a, b, c \in A\) satisfy \(aba = aca\) and let \(n, m \in \mathbb{N}\).

(1) If \((ac)^n\) is generalized Drazin invertible, then \((ba)^m\) is generalized Drazin invertible. In this case,
\[
((ba)^m)^{gD} = b((ac)^n)^{gD}a(ba)^{n-m-1} \quad \text{if } n \geq m + 1,
\]
\[
((ba)^m)^{gD} = b[((ac)^n)^{gD}]^{m+2-n}a(ba)^{(n-1)(m+1-n)} \quad \text{if } n < m + 1.
\]
(2) If \((ba)^n\) is generalized Drazin invertible, then \((ac)^m\) is generalized Drazin invertible. In this case,
\[
((ac)^m)^{gD} = a((ba)^n)^{gD} c(ac)^{n-m-1} \quad \text{if } n \geq m + 1, \\
((ac)^n)^{gD} = a[((ba)^n)^{gD}]^{m+2-n} c(ac)^{(n-1)(m+1-n)} \quad \text{if } n < m + 1.
\]

Proof. Apply the proof of Theorem 2.3 to the generalized Drazin inverse, using Theorem 2.7 and Lemma 2.8.

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