ELEMENTARY PROOF OF THE CONTINUITY OF THE TOPOLOGICAL ENTROPY AT $\theta = 1001$ IN THE MILNOR–THURSTON WORLD

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Abstract. In 1965, Adler, Konheim and McAndrew introduced the topological entropy of a given dynamical system, which consists of a real number that explains part of the complexity of the dynamics of the system. In this context, a good question could be if the topological entropy $H_{\text{top}}(f)$ changes continuously with $f$. For continuous maps this problem was studied by Misiurewicz, Slenk and Urbanski. Recently, and related with the lexicographic and the Milnor–Thurston worlds, this problem was studied by Labarca and others. In this paper we will prove, by elementary methods, the continuity of the topological entropy in a maximal periodic orbit ($\theta = 1001$) in the Milnor–Thurston world. Moreover, by using dynamical methods, we obtain interesting relations and results concerning the largest eigenvalue of a sequence of square matrices whose lengths grow up to infinity.

Introduction

It is well know that one of the purposes of the topological theory of dynamical systems is to find universal models describing the topological dynamics of a large class of systems [16]. One of these models is the shift on $n$-symbols $(\Sigma_n, \tau, \sigma)$, where $\Sigma_n$ is the set of sequences $\{\theta : \mathbb{N}_0 \to \{0, 1, 2, \ldots, n - 1\}\}$ endowed with a certain topology, $\tau$, and $\sigma : \Sigma_n \to \Sigma_n$ is the shift map defined by $(\sigma(\theta))(i) = \theta(i + 1)$ (see [2, 13, 7, 12]). These models have been extensively used to obtain a great amount of information about maps defined in an interval, vector fields on three dimensional manifolds, and other kinds of dynamical systems (see, for instance, [3, 6, 9, 10]).

By using a result in [8] —which is a slight modification of a result proved by Block, Guckenheimer, Misiurewicz and Young ([5])— and elementary methods we compute the topological entropy of some periodic orbits, and using matrix (linear) algebra we are able to prove the continuity of the topological entropy at the periodic orbit $\theta = 1001$ in the Milnor–Thurston world (the continuity was proved in [11] but...
here we prove it by elementary methods). To do so, we consider a surjective two
to one linear map defined in two subintervals of $[0, 1]$ which preserve orientation at
the first interval and reverse orientation at the second one ($T : I_0 \cup I_1 \to I = [0, 1]$).
The maximal invariant set of this map can be modeled whit a succession of “zeros”
and “ones”, i.e., full-shift of two symbols. The dynamics of the restriction of the
linear map to this maximal invariant set endowed with the order and the topology
induced by the interval is called the Milnor-Thurston world (MTW) and it is a
representation of the shift of two symbols with the order defined by Milnor and
Thurston in [14].

While in [11] the continuity of the topological entropy is proved by using the
Hausdorff dimension, the novelty of the present paper is related with the tech-
niques implemented: we build a graph associated with the maximal invariant set
concerning to the orbit $\theta = 1001$, that describes the dynamics of the system.
Then we associate to this graph an incidence matrix; we compute its characteristic
polynomial and we compute the largest real eigenvalue. Finally, the entropy $h_0$
associated with the sub-shift generated by the periodic orbit corresponds to the
log of the largest real eigenvalue of the matrix ([8, 5]). To prove the continuity we
compute the entropy at a family of periodic orbits $h_n := (1001)^n 1$, and we prove
that $h_n \to h_0$ as $n \to \infty$.

1. Preliminaries

1.1. The Milnor–Thurston world. Let $T : I_0 \cup I_1 \to I = [0, 1]$ be a linear
map defined as follows: $T|_{I_0}$ preserves orientation and $T|_{I_1}$ reverses orientation,
and both restrictions are surjective. For this map we can consider the set
$\Lambda_T = \bigcap_{k=0}^{\infty} T^{-k}(I_0 \cup I_1)$, and for any $x \in \Lambda_T$ we have:

$$x \in \Lambda_T \iff x \in T^{-k}(I_0 \cup I_1), \quad \forall k \in \mathbb{N}$$

or, equivalently, $T^k(x) \in (I_0 \cup I_1), \forall k \in \mathbb{N}$. Figure 1 represents this map.

Clearly, we can associate to each $x \in \Lambda_T$ a succession of 0’s and 1’s using the
itinerary function $I_T : \Lambda_T \to \Sigma_2, I_T(x) = (I_T(x)(i)), \text{ where } I_T(x)(i) = j \iff
T^i(x) \in I_j$.

In $\Lambda_{T}$ we consider the Euclidean topology and we use the bijection $I_{T}$ to induce a topology in $\Sigma_{2}$. A set $A \subset \Sigma_{2}$ is open $\Leftrightarrow I_{T}^{-1}(A)$ is open in $\Lambda_{T}$. In the same way, we can define an order induced by the map $I_{T}: \theta \leq_{T} \alpha$ in $\Sigma_{2} \Leftrightarrow I_{T}^{-1}(\theta) \leq I_{T}^{-1}(\alpha)$ in $\mathbb{R}$. For being bijective and by inducing the topology and the order (in $\Sigma_{2}$) the map $I_{T}$ is an homeomorphism which preserves the order. Using this property and the fact that the shift function $\sigma : \Sigma_{2} \rightarrow \Sigma_{2}$ satisfies $\sigma(\theta_{0} \theta_{1} \theta_{2} \ldots) = \theta_{1} \theta_{2} \ldots$, it is easy to check that the diagram in Figure 2 is commutative.

$$\begin{array}{ccc}
\Lambda & \xrightarrow{T} & \Lambda \\
\downarrow I_{\Sigma} & & \downarrow I_{\Sigma} \\
\Sigma_{2} & \xrightarrow{\sigma} & \Sigma_{2}
\end{array}$$

Figure 2.

Now we can formally define the Milnor–Thurston world. A sequence $\alpha \in \Sigma_{0} = \{\theta \in \Sigma_{2}; \theta_{0} = 0\}$ is called minimal if $\sigma^{i}(\alpha) \geq_{T} \alpha$ for all $i \in \mathbb{N}_{0}$. A sequence $\beta \in \Sigma_{1} = \{\theta \in \Sigma_{2}; \theta_{0} = 1\}$ is called maximal if $\sigma^{i}(\beta) \leq_{T} \beta$ for all $i \in \mathbb{N}_{0}$. Let $\text{Min}$ (resp. $\text{Max}$) denote the set of minimal (resp. maximal) sequences. For $\alpha \in \Sigma_{0}$ and $\beta \in \Sigma_{1}$ let $\Sigma[\alpha, \beta] = \{\theta \in \Sigma_{2}; \alpha \leq_{T} \sigma^{i}(\theta) \leq_{T} \beta \text{ for any } i \in \mathbb{N}_{0}\} = \bigcap_{i=0}^{\infty} \sigma^{-j}([\alpha, \beta])$, where $[\alpha, \beta] = \{\theta \in \Sigma_{2}; \alpha \leq_{T} \theta \leq_{T} \beta\}$. Finally, the Milnor–Thurston world (MTW) is the set of pairs $(\alpha, \beta) \in \text{Min} \times \text{Max}$ such that $\{\alpha, \beta\} \subset \Sigma[\alpha, \beta]$ (see [12, 11]).

**Remark 1.1.** $\Sigma[\alpha, \beta]$ is a sub-shift of $\Sigma_{2}$.

**1.2. Definitions of topological entropy.**

**Definition 1.2.** Let $M \neq \emptyset$ and $\tau$ a topology such that $(M, \tau)$ is a compact space. Let $\mathcal{A} \subset \tau$ be a covering by open sets of $M$. We define the entropy of the covering $\mathcal{A}$ as $H(\mathcal{A}) = \log(N(\mathcal{A}))$, where $N(\mathcal{A}) = \min\{\text{Card}(\mathcal{A}') : \mathcal{A}' \subset \mathcal{A} \text{ is a finite subcovering of } M\}$.

**Definition 1.3.** Let $\mathcal{A}, \mathcal{B}$ be two coverings of the space $M$. We define $\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \subset \mathcal{A}, B \subset \mathcal{B}\}$.

Now, let $\phi : M \rightarrow M$ be a continuous map, and $\phi^{-1}(\mathcal{A}) = \{\phi^{-1}(U) ; U \in \mathcal{A}\}$, the inverse image covering.

**Definition 1.4.** The topological entropy of $\phi$ with respect to the covering $\mathcal{A}$ is $h(\phi, \mathcal{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{k=0}^{n-1} \phi^{-k}(\mathcal{A}))$.

**Definition 1.5.** The topological entropy of $\phi$ is $h_{\text{top}}(\phi) = \sup_{\mathcal{A}} \{h(\phi, \mathcal{A})\}$, where the sup is taken over all the open coverings $\mathcal{A}$ of $M$.

Finally, in our case, we define the entropy map $H$:

**Definition 1.6.** $H : \text{MTW} \rightarrow [0, \ln(2)]$, $H(\alpha, \beta) = h_{\text{top}}(\sigma|_{\Lambda_{T}[\alpha, \beta]})$. 

1.3. Some notation and results. Let \( I \subset \mathbb{R} \) be an interval, \( f : I \to \mathbb{R} \) a continuous map, and \( A = \{I_1, I_2, \ldots, I_s\} \) a partition of \( I \) such that \( \bigcup_{i=1}^s I_i = I \), with \( i \neq j \) and each pairwise intersection is at most one point.

**Definition 1.7.** We say that \( I - f \) covers \( n \) times \( J \) if there are subintervals \( K_1, K_2, \ldots, K_n \) on \( I \) (with disjoint interior) such that \( f(K_i) = J, i = 1, 2, \ldots, n \).

**Definition 1.8 (A-graph).** Associated to the partition \( A \) of the interval \( I \) we define a \( A \)-graph oriented with vertices \( I_1, I_2, \ldots, I_s \) such that if \( I_i f \) covers \( n \) times \( I_j \) then there are \( n \) arrows from \( I_i \) to \( I_j \).

**Definition 1.9 (Incidence matrix).** Associated to the \( A \)-graph, we define the incidence matrix \( M = (m_{ij})_{s \times s} \) where \( m_{ij} \) is the number of arrows from \( I_i \) to \( I_j \).

Now we define the entropy associated to the \( A \)-graph.

**Definition 1.10 (Perron’s eigenvalue).** The entropy of the \( A \)-graph is defined as \( \log(r(M)) \), where \( r(M) = \max\{|\lambda| : \lambda \text{ eigenvalue of } M\} \).

**Definition 1.11.** Let \( M = (m_{ij}) \) be an \( n \times n \) matrix. Associated to a finite sequence \( p = (p_j)_{j=0}^k \) of elements from \( \{1, 2, \ldots, n\} \) we define its **width** as \( w(p) = \prod_{j=1}^k m_{p_{j-1}p_j} \). We will say that \( p \) is a **path** in \( M \) if \( w(p) \neq 0 \). In this case, the number \( k = l(p) \) is called the **length** of the path \( p \). A **loop** is a path \( p = (p_0, p_1, \ldots, p_k) \) for which \( p_i \neq p_{i+1}, i = 0, 1, \ldots, k-1 \) and \( p_k = p_0 \). A **rome** in \( M \) is a subset \( R \subset \{1, 2, \ldots, n\} \) for which there is no loop outside of \( R \), i.e., there is no loop \( p = (p_0, \ldots, p_k) \) with \( \{p_0, p_1, \ldots, p_k\} \cap R = \emptyset \). Given a rome \( R \) and a path \( p = (p_0, \ldots, p_k) \), we will say the path \( p \) is **simple** regarding \( R \) if \( \{p_0, p_k\} \subset R \) and \( \{p_1, \ldots, p_{k-1}\} \cap R = \emptyset \). Finally, given a rome \( R = \{r_1, \ldots, r_k\} \) with \( r_i \neq r_j \) for \( i \neq j \) the matrix \( A_R \) is defined by \( A_R = A_R(x) = (a_{ij})_{i,j=1}^k = (a_{ij}(x))_{i,j=1}^k \), where \( a_{ij}(x) = \sum_p w(p)x^{-l(p)} \) and the sum is over all the simple paths that originate in \( r_i \) and end at \( r_j \).

2. The main result

The following proposition has an important role in proving the main result of this article, because we use it to compute the characteristic polynomial for a given matrix \( M \).

**Proposition 2.1.** Given a matrix \( M = (m_{ij})_{n \times n} \) and a rome \( R = \{I_{r_0}, I_{r_1}, \ldots, I_{r_k}\} \) (where \( r_i \neq r_j \) if \( i \neq j \)) over \( M \), then

\[
p_M(x) = \det(M - xI_n) = (-1)^{n-k}x^n \det(A_R(x) - I_k).
\]

For the proof see [8].

Using this result and Perron’s definition [1.10] we prove in the present paper the following theorem.
Theorem 2.2. Let $h_{\text{top}}$ be the topological entropy and $\sigma$ the shift map. Then $h_{\text{top}}(\theta) = H(\sigma(\theta), \theta) = h_{\text{top}}(\sigma|_{\Lambda_\theta})$ is continuous at $\theta = 1001$, where

$$\Lambda_\theta = \bigcap_{n=0}^\infty \sigma^{-n}([\sigma(\theta), \theta]).$$

In this paper we follow the line of research started in [4].

3. Proof of the main result

To prove the main result we will prove that the map $h_{\text{top}}$ is continuous from the right and from the left at the value $\theta = 1001$.

Lemma 3.1. $h_{\text{top}}$ is continuous from the right at $\theta = 1001$.

Proof. In the MTW we have $1001 < 1000$, so first we will prove $h_{\text{top}}(1001) = h_{\text{top}}(1000)$. To do that we will compute the topological entropy for both periodic orbits.

In order to do that we can use the following methodology: Given a maximal periodical orbit $\theta \in \Sigma_2$ of period $n$, we have its maximal invariant set $\Lambda_\theta = \bigcap_{n=0}^\infty \sigma^{-n}([\sigma(\theta), \theta])$ (which is a finite sub shift; see Figure 3). Initially we produce a partition of the interval $[\sigma(\theta), \theta]$ by using the induced order of the iterations $\sigma^i(\theta)$, $i \in \{0, 1, 2, \ldots, n\}$.

![Figure 3. Graph of $\Lambda_\theta$.](image)

In the interior of this partition we can find the sequences $0\theta$ and $1\theta (= \sigma^{n-1}(\theta))$. We include both values as part of the partition (one of these two values is part of the orbit of $\theta$) and we compute the associated $A$-graph. With this we have associated the incidence matrix $M = (m_{ij})_{n \times n}$. Finally, we choose a rome to compute the polynomial matrix $A_R(x)$ and we calculate the maximal real root of the characteristic polynomial.
The topological entropy at $\theta = 1001$:

First, we apply the shift map to the periodic orbit:

0: $\theta = 1001$ \hspace{1cm} (3.1)

1: $\sigma(\theta) = 0011$ \hspace{1cm} (3.2)

2: $\sigma^2(\theta) = 0110$ \hspace{1cm} (3.3)

3: $\sigma^3(\theta) = 1100$ \hspace{1cm} (3.4)

From (3.1) and (3.4) we have $\theta > \sigma^3(\theta)$; (3.2) and (3.3) imply $\sigma^2(\theta) > \sigma(\theta)$; finally, $\sigma^3(\theta) > \sigma^2(\theta)$, then we have $\theta > \sigma^3(\theta) > \sigma^2(\theta) > \sigma(\theta)$.

Defining $0\theta = 01001$ and $1\theta = 11001$ we have $1\theta = 11001 = 1100110011001 \ldots = 1100 = \sigma^3(\theta)$; also, $0\theta > \sigma^2(\theta)$. Expanding the inequality: $\theta > 1\theta = \sigma^3(\theta) > 0\theta > \sigma^2(\theta) > \sigma(\theta)$. Figure 4 shows the relationship between the iterations of the periodical orbit and the associated $A$-graph. Its incidence matrix is

$$
\begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
$$

![Graph of the shift map associated to 1001]

**Figure 4.** Topological entropy at $\theta = 1001$.

We consider a rome with a unique element, $I_1$, so the closed paths are $1 \to 1, 1 \to 2 \to 3 \to 1$ and $1 \to 3 \to 1$, so the $A_R(x)$ matrix is given by $x^{-1} + x^{-2} + x^{-3}$. Applying Proposition 2.1 we have:

$$(-1)^{3-1} x^3 \det(A_R(x) - I) = x^3(x^{-1} + x^{-2} + x^{-3} - 1) = 1 + x + x^2 - x^3.$$

Finally, we have $p(1001)(x) = 1 + x + x^2 - x^3$, whose largest real root is 1.8392867. We conclude that $h_{\text{top}}(1001) \approx \log(1.8392867)$.

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1 From now on, the vector $(x, y, z, u, v, w)$ corresponds to $x \to y \to z \to u \to v \to w$. 

The topological entropy at $\theta = 1000$:

For the iterations of the sequence 1000 we have:

0: $\theta = 1000$ (3.5)
1: $\sigma(\theta) = 0001$ (3.6)
2: $\sigma^2(\theta) = 0010$ (3.7)
3: $\sigma^3(\theta) = 0100$ (3.8)

Defining $0\theta = 01000$ and $1\theta = 11000$ we have $0\theta = 01000 = 0100010001000 \ldots = 0100 = \sigma^3(\theta)$, also $0\theta > \sigma^2(\theta)$. Expanding the inequalities: $\theta > 1\theta > 0\theta = \sigma^3(\theta) > \sigma^2(\theta) > \sigma(\theta)$. The graph associated to the shift map associated to 1000 is the same as the previous one. This implies that we obtain the same value for the topological entropy.

Hence, we have $h_{\text{top}}(1001) = h_{\text{top}}(1000)$, so for any subsequence $\eta_n \subset \Sigma_2$ such that $1001 \leq \eta_n \leq 1000$ we have $h_{\text{top}}(1001) \leq h_{\text{top}}(\eta_n) \leq h_{\text{top}}(1000)$. Applying the limit over $n$ we conclude:

$$\lim_{n \to \infty} h_{\text{top}}(\eta_n) = h_{\text{top}}(1001) = h_{\text{top}}(1000).$$

\[\square\]

Remark 3.2. To complete the proof of Proposition 2.2 we have to prove the continuity of topological entropy at 1001 from the left. This is —essentially— the main difficulty of this paper.

Lemma 3.3. $h_{\text{top}}$ is continuous from the left at 1001.

To prove this lemma we will use a subsequence $\theta_n$ that converges from the left to 1001, then we calculate $h_{\text{top}}$ for $\theta_n$, with $n = 1, 2, 3, 4, 5$, to find a pattern on the subsequence of the roots of the characteristic polynomials. Finally, we will show that for any sequence between a subsequence $\theta_{k(n)}$ and $\theta_{k(n)+1}$ its topological entropy converges to the topological entropy of 1001. However, this procedure is not direct and we have to prove it by induction.

3.1. Induction procedure.

3.1.1. Step 1. Deduction of the general procedure. For $\theta_1 = 10011$ we can order its iterations like $0 > 3 > 4 = 1\theta_1 > 0\theta_1 > 2 > 1$. This allows us to define the following intervals:

$I_1 = [1\theta_1 = \sigma^4(\theta_1), \sigma^3(\theta_1)]$, $I_2 = [\sigma^3(\theta_1), \theta_1]$,

$I_3 = [\sigma(\theta_1), \sigma^2(\theta_1)]$, $I_4 = [\sigma^3(\theta_1), 0\theta_1]$.

By ordering them we get Figure 5.

![Figure 5. Intervals associated to $\theta_1$.](image-url)
Also, we can check that $1 \in I_1$ and obtain the $A$-graph of $\theta_1$ by the associated graph of $\sigma|_{\theta_1}$ (Figure 6).

![Diagram](image1.png)

**Figure 6.** Representation of the dynamics associated to $\theta_1$.

The restriction $\sigma|_{\theta_1}$ has the following dynamics: $I_1 \sigma$-cover $I_1$ and $I_2$; $I_2 \sigma$-cover $I_3$ and $I_4$; $I_3 \sigma$-cover $I_4$ and $I_1$; and $I_4 \sigma$-cover $I_2$ (Figure 7).

![Diagram](image2.png)

**Figure 7.** Representation of the $\sigma$-covering associated to $\theta_1$.

We summarize the loops and paths associated to the rime $R = \{1, 4\}$:

- $a_{11} : I_1 \rightarrow I_1 : (1, 1)$ and $(1, 2, 3, 1)$,
- $a_{14} : I_1 \rightarrow I_4 : (1, 2, 4)$ and $(1, 2, 3, 4)$,
- $a_{41} : I_4 \rightarrow I_1 : (4, 2, 3, 1)$,
- $a_{44} : I_4 \rightarrow I_4 : (4, 2, 4)$ and $(4, 2, 3, 4)$.

Finally, the transition matrix $A_R(x)$ is

$$
\begin{pmatrix}
  x^{-1} + x^{-3} & x^{-2} + x^{-3} \\
  x^{-3} & x^{-2} + x^{-3}
\end{pmatrix}.
$$

Then

$$
p_1(x) = (-1)^{4-2} x^4 \det(A_R(x) - I) = -xp(1001)(x) + 1,
$$

whose maximal real root is $\lambda = 1.7220838$. So we have that $h_{\text{top}} \approx \log(1.7220838)$.
For $\theta = (1001)^2 = 100110011$ we can order its iterations like $0 > 4 > 7 > 3 > 8 = 1\theta > 0\theta > 6 > 2 > 5 > 1$. This order allows us to define the following intervals:

\begin{align*}
    I_1 &= [\sigma^3(\theta), \sigma^7(\theta)], \\
    I_2 &= [\sigma^7(\theta), \sigma^4(\theta)], \\
    I_3 &= [\sigma^5(\theta), \sigma^3(\theta)], \\
    I_4 &= [\sigma^6(\theta), 0\theta], \\
    I_5 &= [\sigma^4(\theta), \theta], \\
    I_6 &= [\sigma(\theta), \sigma^5(\theta)], \\
    I_7 &= [\sigma^2(\theta), \sigma^6(\theta)], \\
    I_8 &= [1\theta = \sigma^8(\theta), \sigma^3(\theta)].
\end{align*}

By ordering them we get Figure 8.

Also, we can check that $1 \in I_1$ and obtain the $A$-graph of $\sigma|\theta_2$ (Figure 9).

The restriction $\sigma|\theta_2$ has the following dynamics: $I_1 \sigma$-cover $I_1$, $I_2$ and $I_8$; $I_2 \sigma$-cover $I_3$, $I_4$ and $I_7$; $I_3 \sigma$-cover $I_4$ and $I_8$; $I_4 \sigma$-cover $I_2$ and $I_5$; $I_5 \sigma$-cover $I_6$; $I_6 \sigma$-cover $I_7$; $I_7 \sigma$-cover $I_1$; and $I_8 \sigma$-cover $I_5$ (Figure 10).

From this representation we can obtain the loops and paths of the rome $R = \{1, 4\}$:

- $a_{11} = (1, 1); (1, 2, 7, 1); (1, 8, 5, 6, 7, 1); (1, 2, 3, 8, 5, 6, 7, 1),$
- $a_{14} = (1, 2, 4)$ and $(1, 2, 3, 4),$
- $a_{41} = (4, 2, 7, 1); (4, 5, 6, 7, 1); (4, 2, 3, 8, 5, 6, 7, 1),$
- $a_{44} = (4, 2, 4)$ and $(4, 2, 3, 4).$
Finally, the transition matrix $A_R(x)$ is:

$$
\begin{pmatrix}
    x^{-1} + x^{-3} + x^{-5} + x^{-7} & x^{-2} + x^{-3} \\
    x^{-3} + x^{-4} + x^{-7} & x^{-2} + x^{-3}
\end{pmatrix}.
$$

Then,

$$
p_2(x) = (-1)^8 x^8 \det(A_R(x) - I) = x^4 p_1(x) - (x^3 + x^2 + x - 1).
$$

For the next steps (3, 4 and 5) we just show the main stage of each iteration.

**Intervals associated to $\theta_i$, $i = 3, 4, 5$ (Figures 11–13):**

**Figure 10.** Representation of the $\sigma$-covering associated to $\theta_2$.

**Figure 11.** Intervals associated to $\theta_3$.

**Figure 12.** Intervals associated to $\theta_4$.

**Figure 13.** Intervals associated to $\theta_5$. 
Representation of the $A$-graphs and shift maps associated to $\theta_i$, $i = 3, 4, 5$ (Figures 14–16):

(a) $A$-graph associated to $\theta_3$

(b) shift map associated to $\theta_3$

Figure 14. Representation of the dynamics associated to $\theta_3$. 
(a) $A$-graph associated to $\theta_4$

(b) shift map associated to $\theta_4$

**Figure 15.** Representation of $\theta_4$. 
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(a) $A$-graph associated to $\theta_5$

(b) shift map associated to $\theta_5$

Figure 16. Representation of $\theta_5$. 
σ-coverings associated to \( \theta_i, \ i = 3, 4, 5 \) (Figures 17–19):

\[
\begin{align*}
&I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6 \rightarrow I_7 \rightarrow I_8 \rightarrow I_9 \rightarrow I_{10} \rightarrow I_{11} \rightarrow I_{12} \rightarrow I_{13} \rightarrow I_{14} \rightarrow I_{15} \rightarrow I_{16} \\
&\text{Figures 17. } \sigma\text{-covering associated to } \theta_3.
\end{align*}
\]

\[
\begin{align*}
&I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6 \rightarrow I_7 \rightarrow I_8 \rightarrow I_9 \rightarrow I_{10} \rightarrow I_{11} \rightarrow I_{12} \rightarrow I_{13} \rightarrow I_{14} \rightarrow I_{15} \rightarrow I_{16}
\end{align*}
\]

\[
\begin{align*}
&\text{Figures 18. } \sigma\text{-covering associated to } \theta_4.
\end{align*}
\]

\[
\begin{align*}
&I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6 \rightarrow I_7 \rightarrow I_8 \rightarrow I_9 \rightarrow I_{10} \rightarrow I_{11} \rightarrow I_{12} \rightarrow I_{13} \rightarrow I_{14} \rightarrow I_{15} \rightarrow I_{16} \\
&\text{Figures 19. } \sigma\text{-covering associated to } \theta_5.
\end{align*}
\]
\(A_R(x)\) matrices associated to \(\theta_i, \, i = 3, 4, 5:\)

For \(\theta_3:\)
\[
\begin{pmatrix}
x^{-1} + x^{-3} + x^{-5} + x^{-7} + x^{-9} + x^{-11} & x^{-2} + x^{-3} \\
x^{-3} + x^{-4} + x^{-7} + x^{-8} + x^{-11} & x^{-2} + x^{-3}
\end{pmatrix},
\]
\[p_3(x) = (-1)^{12-2}x^{12} \det(A_R(x) - I) = x^4p_2(x) - (x^3 + x^2 + x - 1).
\]

For \(\theta_4:\)
\[
\begin{pmatrix}
x^{-1} + x^{-3} + x^{-5} + x^{-7} + x^{-9} + x^{-11} + x^{-13} + x^{-15} & x^{-2} + x^{-3} \\
x^{-3} + x^{-4} + x^{-7} + x^{-8} + x^{-11} + x^{-12} + x^{-15} & x^{-2} + x^{-3}
\end{pmatrix},
\]
\[p_4(x) = (-1)^{16-2}x^{16} \det(A_R(x) - I) = x^4p_3(x) - (x^3 + x^2 + x - 1).
\]

For \(\theta_5:\)
\[
\begin{pmatrix}
x^{-1} + x^{-3} + x^{-5} + x^{-7} + x^{-9} + x^{-11} + x^{-13} + x^{-15} + x^{-17} + x^{-19} & x^{-2} + x^{-3} \\
x^{-3} + x^{-4} + x^{-7} + x^{-8} + x^{-11} + x^{-12} + x^{-15} + x^{-16} + x^{-19} & x^{-2} + x^{-3}
\end{pmatrix},
\]
\[p_5(x) = (-1)^{20-2}x^{20} \det(A_R(x) - I) = x^4p_5(x) - (x^3 + x^2 + x - 1).
\]

3.1.2. Step 2. Generalization. Considering the previous constructions, we can assume (inductively) the following order for \(\theta = \theta_k = (1001)^k 1\) and its iterations: \(\theta > 4 > 8 > 12 > 16 > 20 > \cdots > 4(k-1) > 4k-1 > 4(k-1)-1 > 4(k-2) - 1 > \cdots > 23 > 19 > 15 > 11 > 7 > 3 > 4k = 1\theta_k > 0\theta_k > 4k-2 > 4(k-1) - 2 > 4(k-2) - 2 > \cdots > 22 > 18 > 14 > 10 > 6 > 2 > 4k - 3 > 4(k-1) - 3 > 4(k-2) - 3 > \cdots > 21 > 17 > 13 > 9 > 5 > 1.

This order of the iterations allows us to define the intervals
\[
\begin{align*}
I_1 &= [\sigma^{4k-5}(\theta_k), \sigma^{4k-1}(\theta_k)] \\
I_3 &= [\sigma^{4k-3}(\theta_k), \sigma^2(\theta_k)] \\
I_5 &= [\sigma^4(\theta_k), \theta_k] \\
I_7 &= [\sigma^2(\theta_k), \sigma^6(\theta_k)] \\
I_9 &= [\sigma^8(\theta_k), \sigma^4(\theta_k)] \\
I_{11} &= [\sigma^6(\theta_k), \sigma^1(0(\theta_k))] \\
I_{12} &= [\sigma^7(\theta_k), \sigma^{11}(\theta_k)]
\end{align*}
\]
\[\vdots\]
\[
\begin{align*}
I_{4k-2} &= [\sigma^{4k-7}(\theta_k), \sigma^{4k-3}(\theta_k)] \\
I_{4k-1} &= [\sigma^{4k-6}(\theta_k), \sigma^{4k-2}(\theta_k)] \\
I_{4k} &= [\theta_k = \sigma^{4k-1}(\theta_k), \sigma^3(\theta_k)],
\end{align*}
\]
whose representation, over the Milnor–Thurston world, is given by Figure 20.

![Figure 20. Order of the intervals associated to \(\theta_k\).](image-url)
Also, we can check that $1 \in I_1$ and that the associated graph of $\sigma|_{\theta_k}$ is as shown in Figure 21; its diagram of $\sigma$-covering is given by Figure 22.

**Figure 21.** Shift map associated to $\theta_k$.

**Figure 22.** Diagram of $\sigma$-covering associated to $\theta_k$. 

From that diagram we can deduce the loops and paths associated to the rime $R = \{1, 4\}$. With this we can obtain its associated $A_R(x)$ matrix

$$A_R^k(x) = \begin{pmatrix} a_{11}(x) & x^{-2} + x^{-3} \\ a_{21}(x) & x^{-2} + x^{-3} \end{pmatrix},$$

where

$$a_{11}(x) = x^{-1} + x^{-3} + \cdots + x^{-(4(k-1)-3)} + x^{-(4(k-1)-1)} + x^{-(4k-3)} + x^{-(4k-1)}$$

$$a_{21}(x) = x^{-3} + x^{-4} + \cdots + x^{-(4(k-1)-4)} + x^{-(4(k-1)-1)} + x^{-(4k-4)} + x^{-(4k-1)}.$$

Finally, for the characteristic polynomial associated to the matrix defined by the $A$-graph, we have:

$$p_k(x) = (-1)^{4k-2}x^{4k} \det(A_R^k(x) - I) = x^4p_{k-1}(x) - (x^3 + x^2 + x - 1).$$

### 3.1.3. Step 3. Proof for the case $n = k+1$.

Now, let us take $\theta_{k+1} = (1001)^{k+1}$. It is not hard to see that $\theta_{k+1}$ and its iterations satisfy: $(\sigma^j(\theta_{k+1}), j = 0, 1, 2, \ldots, 4k+4)$: $\theta > 4 > 8 > 12 > 16 > \cdots > 4(k-2) > 4(k-1) > 4k > 4k+3 > 4k-1 > 4(k-1) - 1 > \cdots > 19 > 15 > 11 > 7 > 3 > 4k = 10\theta_{k+1} > 0\theta_{k+1} > 4k+2 > 4k-2 > 4k-1 - 2 > \cdots > 22 > 18 > 14 > 10 > 6 > 2 > 4k+1 > 4k-3 > 4(k-1) - 3 > \cdots > 21 > 17 > 13 > 9 > 5 > 1$. (A proof of this fact will be given later in this section).

This order allows us to define the following intervals:

$I_1 = [\sigma^{4k-1}(\theta_{k+1}), \sigma^{4k+3}(\theta_{k+1})]$ \hspace{1cm} $I_2 = [\sigma^{4k+3}(\theta_{k+1}), \sigma^{4k}(\theta_{k+1})]$  

$I_3 = [\sigma^{4k+1}(\theta_{k+1}), \sigma^2(\theta_{k+1})]$ \hspace{1cm} $I_4 = [\sigma^{4k+2}(\theta_{k+1}), \sigma\theta_{k+1}]$  

$I_5 = [\sigma(\theta_{k+1}), \sigma(\theta_{k+1})]$ \hspace{1cm} $I_6 = [\sigma(\theta_{k+1}), \sigma^5(\theta_{k+1})]$  

$I_7 = [\sigma^2(\theta_{k+1}), \sigma^6(\theta_{k+1})]$ \hspace{1cm} $I_8 = [\sigma^3(\theta_{k+1}), \sigma^7(\theta_{k+1})]$  

$I_9 = [\sigma^4(\theta_{k+1}), \sigma^4(\theta_{k+1})]$ \hspace{1cm} $I_{10} = [\sigma^5(\theta_{k+1}), \sigma^9(\theta_{k+1})]$  

$I_{4k+1} = [\sigma^{4k-4}(\theta_{k+1}), \sigma^{4k}(\theta_{k+1})]$ \hspace{1cm} $I_{4k+2} = [\sigma^{4k-3}(\theta_{k+1}), \sigma^{4k+1}(\theta_{k+1})]$  

$I_{4k+3} = [\sigma^{4k-2}(\theta_{k+1}), \sigma^{4k+2}(\theta_{k+1})]$ \hspace{1cm} $I_{4k+4} = [1\theta_{k+1} = \sigma^{4k+3}(\theta_{k+1}), \sigma^3(\theta_{k+1})]$,

whose representation, over the Milnor–Thurston world, is given by Figure 23.

![Figure 23. Order of the intervals associated to $\theta_{k+1}$.](image)

Also, we can check that $1 \in I_1$ and that the graph associated to $\sigma|_{\theta_{k+1}}$ is as shown in Figure 24, its diagram of $\sigma$-covering is given by Figure 25.
Figure 24. Shift map associated to $\theta_{k+1}$.

Figure 25. Diagram of $\sigma$-covering associated to $\theta_{k+1}$. 
We can obtain the loops and paths associated to the rome \( R = \{1, 4\} \) and their associated \( A_R(x) \) matrix,

\[
A_R^k(x) = \begin{pmatrix}
a_{11}(x) & x^{-2} + x^{-3} \\
\quad a_{21}(x) & x^{-2} + x^{-3}
\end{pmatrix},
\]

where

\[
a_{11}(x) = x^{-1} + x^{-3} + \cdots + x^{-(4k-3)} + x^{-(4k-1)} + x^{-(4k+1)} + x^{-(4k+3)}
\]

\[
a_{21}(x) = x^{-3} + x^{-4} + \cdots + x^{-(4k-4)} + x^{-(4k-1)} + x^{-4k} + x^{-(4k+3)}.
\]

Then, if we write

\[
A_R^k(x) = \begin{pmatrix}
a_{11}^k & a_{14}^k \\
\quad a_{41}^k & a_{44}^k
\end{pmatrix}, \quad A_R^{k+1}(x) = \begin{pmatrix}
a_{11}^{k+1} & a_{14}^{k+1} \\
\quad a_{41}^{k+1} & a_{44}^{k+1}
\end{pmatrix},
\]

we can see that

\[
a_{k+1}^{11} = a_{k}^{11} + x^{-(4k+1)} + x^{-(4k+3)}, \quad a_{k+1}^{14} = a_{k}^{14},
\]

\[
a_{k+1}^{41} = a_{k}^{41} + x^{-(4k+3)} + x^{-4k}, \quad a_{k+1}^{44} = a_{k}^{44}.
\]

Then,

\[
p_{k+1}(x) = (-1)^{4(k+1)-2} x^{4(k+1)} \det(A_R^{k+1}(x) - I)
\]

\[
= x^{4k+4} \begin{vmatrix}
a_{11}^{k+1} - 1 & a_{14}^{k+1} \\
\quad a_{41}^{k+1} & a_{44}^{k+1} - 1
\end{vmatrix}
\]

\[
= x^{4k+4} \begin{vmatrix}
a_{11}^{k} + x^{-(4k+1)} + x^{-(4k+3)} - 1 & a_{14}^{k} \\
\quad a_{41}^{k+1} + x^{-(4k+3)} + x^{-4k} & a_{44}^{k+1} - 1
\end{vmatrix}
\]

\[
= x^{4(k+1)} \left[ ((a_{11}^{k} - 1) + x^{-(4k+1)} + x^{-(4k+3)}) (a_{44}^{k} - 1)
\right.

\[
- (a_{41}^{k} + x^{-(4k+3)} + x^{-4k}) (a_{14}^{k})]
\]

\[
= x^{4(k+1)} \left[ (a_{11}^{k} - 1)(a_{44}^{k} - 1) - a_{41}^{k} a_{14}^{k}
\right.

\[
+ (a_{44}^{k} - 1)(x^{-(4k+1)} + x^{-(4k+3)}) - a_{14}^{k}(x^{-(4k+3)} + x^{-4k}) - a_{14}^{k} x^{-4k}
\right]
\]

Since \( a_{44}^{k} = a_{14}^{k} \) we have:

\[
= x^4 \cdot p_k(x) + x^{4k+4} [a_{44}^{k} x^{-(4k+1)} - x^{-(4k+1)} - x^{-(4k+3)} - a_{14}^{k} x^{-4k}]
\]

\[
= x^4 \cdot p_k(x) + x^{4k+4} [x^{-(4k+3)} + x^{-(4k+4)} - x^{-(4k+1)}
\]

\[
- x^{-(4k+3)} - x^{-(4k+2)} - x^{-(4k+3)}]
\]

\[
= x^4 \cdot p_k(x) + x^{4k+4} [x^{-(4k+4)} - x^{-(4k+3)} - x^{-(4k+2)} - x^{-(4k+1)}]
\]

\[
= x^4 \cdot p_k(x) + (1 - x - x^2 - x^3)
\]

\[
= x^4 \cdot p_k(x) - (x^3 + x^2 + x - 1).
\]

Therefore \( p_{k+1}(x) = x^4 \cdot p_k(x) - (x^3 + x^2 + x - 1) \).
So, let us prove that iterations of the sequence $\theta_{k+1}$ appear in the given order.

For that we consider:

1  \equiv  \sigma(\theta_{k+1}) = \sigma(1(0011)^{k+1}) = (0011)^{k+1} \\
2  \equiv  \sigma(1) = \sigma(0011...001100111) = \sigma(0(0110)^k0111) = (0110)^k01110 \\
3  \equiv  \sigma(2) = \sigma(0(1100)^k1110) = (1100)^k11100 \\
4  \equiv  \sigma(3) = \sigma(1(1001)^k1100) = (1001)^k1(1001) \\
5  \equiv  \sigma(4) = \sigma(1(0011)^k(1001)) = (0011)^k(1001) \\
6  \equiv  \sigma(5) = \sigma(0(0110)^{k-1}011100111) = (0110)^{k-1}0111(1001)10 \\
7  \equiv  \sigma(6) = \sigma(0(1100)^{k-1}11110100) = (1100)^{k-1}1111(1001)100 \\
8  \equiv  \sigma(7) = \sigma(1(1001)^{k-1}11011100) = (1001)^{k-1}1(1001)^2 \\
9  \equiv  \sigma(8) = \sigma(1(0011)^{k-1}(1001)^2) = (0011)^{k-1}(1001)^21 \\
10 \equiv \sigma(9) = \sigma(0(0110)^{k-2}0111(1001)^21) = (0110)^{k-2}0111(1001)^210 \\
11 \equiv \sigma(10) = \sigma(0(1100)^{k-2}111101001) = (1100)^{k-2}1111(1001)^2100 \\
12 \equiv \sigma(11) = \sigma(1(1001)^{k-2}110111000) = (1001)^{k-2}1(1001)^3 \\
13 \equiv \sigma(12) = \sigma(1(0011)^{k-2}(1001)^3) = (0011)^{k-2}(1001)^31 \\
14 \equiv \sigma(13) = \sigma(0(0110)^{k-3}0111(1001)^31) = (0110)^{k-3}0111(1001)^310 \\
15 \equiv \sigma(14) = \sigma(0(1100)^{k-3}1111(1001)^310) = (1100)^{k-3}1111(1001)^3100 \\
16 \equiv \sigma(15) = \sigma(1(1001)^{k-3}111(1001)^3100) = (1001)^{k-3}111(1001)^4 \\

So, 0 = (1001)^{k+1}; 4 = (1001)^k1(1001); 8 = (1001)^{k-1}11(1001)^2; \\
12 = (1001)^{k-2}1(1001)^3; 16 = (1001)^{k-3}1(1001)^4. \\
\implies 4j - 4 = (1001)^{k-(j-2)}1(1001)^j-1, \quad j \geq 1.

On the other hand, 1 = (0011)^{k+1}; 5 = (0011)^k(1001)1; 9 = (0011)^{k-1}(1001)^2; \\
13 = (0011)^{k-2}(1001)^31 \\
\implies 4j - 3 = (0011)^{k-(j-2)}(1001)^j-1, \quad j \geq 1.

Also, 2 = (0110)^k01110; 6 = (0110)^{k-1}0111(1001)^10; 10 = (0110)^{k-2}0111(1001)^210; \\
14 = (0110)^{k-3}0111(1001)^310 \\
\implies 4j - 2 = (0110)^{k-(j-1)}0111(1001)^j-10, \quad j \geq 1.

Finally, 3 = (1001)^k11100; 7 = (1100)^{k-1}11(1001)^100; 11 = (1100)^{k-2}11(1001)^2100; \\
15 = (1100)^{k-3}11(1001)^3100 \\
\implies 4j - 1 = (1100)^{k-(j-1)}11(1001)^j-1100, \quad j \geq 1.
Taking \( j = k \) we have
\[
\begin{align*}
4k - 4 &= (1001)21(1001)^{k-1} \\
4k - 3 &= (0011)2(1001)^{k-1} \\
4k - 2 &= (0110)011(1001)^{k-1}10 \\
4k - 1 &= (1100)11(1001)^{k-1}100.
\end{align*}
\]

Iterating the previous part by \( \sigma^4 \), we get:
\[
\begin{align*}
4(k + 1) - 4 &= (1001)1(1001)^k \\
4(k + 1) - 3 &= (0011)(1001)^k1 \\
4(k + 1) - 2 &= 011(1001)^k10 \\
4(k + 1) - 1 &= 11(1001)^k100.
\end{align*}
\]

So, from the expressions of the iterations of \( \theta_{k+1} = (1001)^{k+1} \) we conclude that
\[
\theta > 4 > 8 > 12 > 16 > \cdots > 4(k - 1) > 4k > 4k + 3 > 4k - 1 > 4(k - 1) - 1 > \cdots > \\
\cdots > 19 > 15 > 11 > 7 > 3 > 4k = 16k > 0 \theta_k > 4k + 2 > 4k - 2 > 4(k - 1) - 2 > \\
\cdots > 22 > 18 > 14 > 10 > 6 > 2 > 4k + 1 > 4k - 3 > 4(k - 1) - 3 > \cdots > 21 > \\
17 > 13 > 9 > 5 > 1. \text{ Which completes the inductive procedure and we have the} \text{ characteristic polynomials.}
\]

Let us now show the existence of the largest real root of the polynomials
\( p((1001)^k1)(x) \), \( k = 1, 2, 3, \ldots \); later we will see that this sequence of roots is increasing and tends to the largest root of the polynomial \( p(1001)(x) \).

Let \( \overline{x} = 1.8393 \ldots \) be the largest real root of the polynomial \( p(1001)(x) \). We have:
\[
p((1001)1)(x) = -xp(1001)(x) + 1 \Rightarrow p((1001)1)(\overline{x}) = 1 > 0
\]
and
\[
p((1001)1)(1) = -p(1001)(1) + 1 = -(-1 + 1 + 1 + 1) + 1 < 0.
\]
So, \( \exists x_1 \in [1, \overline{x}] \) such that \( p((1001)1)(x_1) = 0 \). We have \( 1 < x_1 < \overline{x} \).

From \( p((1001)1)(x) = -x(1 + x + x^2 - x^3) + 1 = x^4 - x^3 - x^2 - x + 1 \), we obtain for \( x \geq \overline{x} \),
\[
p((1001)1)(x) \geq p((1001)1)(\overline{x}) = 1, \text{ i.e., } x_1 \text{ is the largest real root of } p((1001)1)(x).
\]
For \( k = 2 \),
\[
p((1001)^21)(x) = x^4p((1001)1)(x) - (x^3 + x^2 + x - 1)
= x^4(-xp(1001)(x) + 1) - (x^3 + x^2 + x - 1)
\implies p((1001)^21)(\overline{x}) = \overline{x}^4(1 - (\overline{x}^3 + \overline{x}^2 + \overline{x} - 1)
= -\overline{x}(\overline{x}^3 + \overline{x}^2 + \overline{x} + 1) + 1.
\]

Therefore,
\[
p((1001)^21)(\overline{x}) = 1 > 0.
\]
Also, \( p((1001)^21)(x_1) = -(x_1^3 + x_1^2 + x_1 - 1) < 0 \). Therefore \( \exists x_2 \in ]x_1, \overline{x} [ \) such that \( p((1001)^21)(x_2) = 0 \). We have \( 1 < x_1 < x_2 < \overline{x} \).

Now,

\[
p((1001)^21)(x) = x^4(x^4 - x^3 - x^2 - x + 1) - (x^3 + x^2 + x - 1)
\]

\[
= x^5(x^3 - x^2 - x - 1) + x(x^3 - x^2 - x - 1) + 1.
\]

Therefore, if \( x \geq \overline{x} \), \( p((1001)^21)(x) \geq p((1001)^21)(\overline{x}) = 1 \), i.e., \( x_2 \) is the largest real root of \( p((1001)^21)(x) \).

For \( k = 3 \),

\[
p((1001)^31)(x) = x^4p((1001)^21)(x) - (x^3 + x^2 + x - 1)
\Rightarrow p((1001)^31)(\overline{x}) = \overline{x}^4 - (\overline{x}^3 + \overline{x}^2 + \overline{x} - 1)
= -\overline{x} (-\overline{x}^3 + \overline{x}^2 + \overline{x} + 1) + 1 = 1.
\]

Therefore, \( p((1001)^31)(\overline{x}) > 1 \).

Also,

\[
p((1001)^31)(x_2) = -(x_2^3 + x_2^2 + x_2 - 1) < 0.
\]

Then, \( \exists x_3 \in ]x_2, \overline{x} [ \) such that \( p((1001)^31)(x_3) = 0 \). We have \( 1 < x_1 < x_2 < x_3 < \overline{x} \).

Now,

\[
p((1001)^31)(x) = x^4(x^4 + 1)(x^3 + x^2 + x - 1) + 1) - (x^3 + x^2 + x - 1).
\]

Therefore, if \( x \geq \overline{x} \), \( p((1001)^31)(x) \geq p((1001)^31)(\overline{x}) = 1 \), i.e., \( x_3 \) is the largest real root of \( p((1001)^31)(x) \).

Let us now assume that \( \exists x_k \in ]x_{k-1}, \overline{x} [ \) such that \( p((1001)^k1)(x_k) = 0 \). Also, that we have \( 1 < x_1 < x_2 < x_3 < \cdots < x_k < \overline{x} \).

For \( k + 1 \) we have

\[
p((1001)^{k+1}1)(x) = x^4p((1001)^k1)(x) - (x^3 + x^2 + x - 1)
\Rightarrow p((1001)^{k+1}1)(\overline{x}) = \overline{x}^4 - (\overline{x}^3 + \overline{x}^2 + \overline{x} - 1)
= -\overline{x} (-\overline{x}^3 + \overline{x}^2 + \overline{x} + 1) + 1 = 1.
\]

Therefore \( p((1001)^{k+1}1)(\overline{x}) > 1 \).

Also,

\[
p((1001)^{k+1}1)(x_k) = -(x_k^3 + x_k^2 + x_k - 1) < 0.
\]

Therefore \( \exists x_{k+1} \in ]x_k, \overline{x} [ \) such that \( p((1001)^{k+1}1)(x_{k+1}) = 0 \). We have \( 1 < x_1 < x_2 < x_3 < \cdots < x_k < x_{k+1} < \overline{x} \).

In the same way, if \( x \geq \overline{x} \), \( p((1001)^{k+1}1)(x) \geq p((1001)^{k+1}1)(\overline{x}) = 1 \), i.e., \( x_{k+1} \) is the largest real root of \( p((1001)^{k+1}1)(x) \).

With these arguments, we have shown that there is a sequence of maximal real roots, monotone increasing and bounded above by \( \overline{x} \).

**Affirmation 1.** \( \lim_{k \to \infty} x_k = \overline{x} \).
Proof. We have:

\[ p((1001)^{k+1})_1(x) \]
\[ = x^4 p((1001)^k)_1(x) - (x^3 + x^2 + x - 1) \]
\[ = x^4 [x^4 p((1001)^{k-1})_1(x) - (x^3 + x^2 + x - 1)] - (x^3 + x^2 + x - 1) \]
\[ = x^8 p((1001)^{k-1})_1(x) - (x^3 + x^2 + x - 1)(1 + x^4) \]
\[ = x^8 [x^4 p((1001)^{k-2})_1(x) - (x^3 + x^2 + x - 1)] - (x^3 + x^2 + x - 1)(1 + x^4) \]
\[ = x^{12} p((1001)^{k-2})_1(x) - (x^3 + x^2 + x - 1)(1 + x^4 + x^8) \]
\[ \vdots \]
\[ = x^{4k} p((1001)_1(x) - (x^3 + x^2 + x - 1) \sum_{i=0}^{k-1} x^{4i} \]
\[ = x^{4k} (-xp((1001)_1(x) + 1) - (x^3 + x^2 + x - 1) \sum_{i=0}^{k-1} x^{4i} \]
\[ = -x^{4k+1} p(1001)_1(x) + x^{4k} - \sum_{i=0}^{k-1} x^{4i+3} - \sum_{i=0}^{k-1} x^{4i+2} - \sum_{i=0}^{k-1} x^{4i+1} + \sum_{i=0}^{k-1} x^{4i}. \]

Defining \( Z_k(x) = x^{4k} - \sum_{i=0}^{k-1} x^{4i+3} - \sum_{i=0}^{k-1} x^{4i+2} - \sum_{i=0}^{k-1} x^{4i+1} + \sum_{i=0}^{k-1} x^{4i} \) we have

\[ Z_1(x) = x^4 + x^3 + x^2 + x - 1 = p((1001)_1(x) \]
\[ Z_2(x) = x^8 - x^3 - x^2 - x^2 - x^5 + 1 + x^4 \]
\[ = x^8 - x^7 - x^6 - x^5 + x^4 + x^3 + x^2 + x - 1 = p((1001)^2)_1(x) \]
\[ Z_3(x) = x^{12} - x^3 - x^2 - x^11 - x^2 - x^6 - x^{10} - x - x^5 - x^9 + 1 + x^4 + x^8 \]
\[ = x^{12} - x^11 - x^{10} - x^9 + x^8 - x^7 - x^6 - x^5 + x^4 + x^3 + x^2 + x - 1 \]
\[ = p((1001)^3)_1(x). \]

Inductively, we may conclude that \( Z_k(x) = p((1001)^k)_1(x) \). So,

\[ p((1001)^{k+1})_1(x) \]
\[ = -x^{4k+1} p(1001)_1(x) + p((1001)^k)_1(x) \]
\[ = -x^{4k+1} p(1001)_1(x) - x^{4(k-1)+1} p(1001)_1(x) + p((1001)^{k-1})_1(x) \]
\[ = -p(1001)_1(x) x^{4k+1} + x^{4(k-1)+1} + p((1001)^{k-1})_1(x) \]
\[ \vdots \]
\[ = -p(1001)_1(x) x^{4k+1} + x^{4(k-1)+1} + \ldots + x^{4j+1} + p((1001)^{k-j})_1(x) \]
\[ -xp(1001)(x) \sum_{i=0}^{j} x^{4i} + p((1001)^{k-j}1)(x) \]
\[ = -xp(1001)(x) \sum_{i=0}^{k-1} x^{4i} + p((1001)1)(x) \]
\[ = -xp(1001)(x) \sum_{i=0}^{k-1} x^{4i} - xp(1001)(x) + 1 \]
\[ = -xp(1001)(x) \left( \sum_{i=0}^{k-1} x^{4i} + 1 \right) + 1. \]

As for \( x = x_{k+1} \) we have \( p((1001)^{k+1}1)(x_{k+1}) = 0 \). Then

\[ x_{k+1}p(1001)(x_{k+1}) \left( \sum_{i=0}^{k-1} x^{4i} + 1 \right) = 1 \]
\[ \implies x_{k+1}p(1001)(x_{k+1}) = \frac{1}{\left( \sum_{i=0}^{k-1} x^{4i} + 1 \right)} \]
\[ \implies x_{k+1}p(1001)(x_{k+1}) = \frac{1}{\frac{x_{k+1}^{4k} + x_{k+1}^{4} - 2}{x_{k+1}^{4k} + x_{k+1}^{4} - 2}}. \]

As \( x_{k+1} > 1 \), letting \( k \to \infty \) we have \( (x_{k+1}^{4k} + x_{k+1}^{4} - 2) \to \infty \), then \( p(1001)(x_{k+1}) \to 0 \). However,

\[ \lim_{k \to \infty} p(1001)(x_{k+1}) = 0 \iff \lim_{k \to \infty} x_{k+1} = \xi. \]

Then, we have shown that this sequence of maximal real roots converges to \( \xi \). \( \square \)

Finally we are in a position to show the continuity of the topological entropy at \( \theta = 1001 \).

4. CONTINUITY OF THE TOPOLOGICAL ENTROPY AT \( \theta = 1001 \)

Let us see that

\[ h_{\text{top}}((1001)^n1) = \ln x_n. \]

Applying limits:

\[ \lim_{n \to \infty} h_{\text{top}}((1001)^n1) = \lim_{n \to \infty} \ln x_n = \ln(\xi) = h_{\text{top}}(1001) = h(\lim_{n \to \infty} (1001)^n1). \]

**Proposition 4.1.** The map \( h_{\text{top}} = h_{\text{top}}(\sigma|_{\Sigma(\theta, \theta)}) \) is continuous at \( \theta = 1001 \).
Proof. We have $h_{\text{top}}(1001) = h_{\text{top}}(1000)$, so for each $\theta$ such that $1001 < \theta < 1000$ we have

$$h_{\text{top}}(1001) \leq h_{\text{top}}(\theta) \leq h_{\text{top}}(1000).$$

Hence, $h_{\text{top}}(\theta) = h_{\text{top}}(1001)$. Thus, for any sequence $(\zeta_n)_{n \in \mathbb{N}} \subset \Sigma_1$ such that $\zeta_{n+1} < \zeta_n$ and $\lim_{n \to \infty} \zeta_n = 1001$ we have $h_{\text{top}}(\zeta_n) = h_{\text{top}}(1001)$ (as long as $1001 < \zeta_n < 1000$).

On the other hand, if $(\zeta_n)_{n \in \mathbb{N}} \subset \Sigma_1$ is a sequence such that $\zeta_n < \zeta_{n+1}$ and $\lim_{n \to \infty} \zeta_n = 1001$, then for each $n$ big enough, there exists $k(n)$ such that

$$(1001)^{k(n)+1} \leq \zeta_n \leq (1001)^{k(n)+1}$$

with $\lim_{n \to \infty} k(n) = \infty$. Then

$$x_{k(n)} = h_{\text{top}}((1001)^{k(n)+1}) \leq h_{\text{top}}(\zeta_n) \leq h_{\text{top}}((1001)^{k(n)+1}) = x_{k(n)+1}.$$ 

As

$$\lim_{n \to \infty} \zeta_n = \lim_{n \to \infty} (1001)^{k(n)+1} = 1001,$$

we conclude that $\lim_{n \to \infty} h_{\text{top}}(\zeta_n) = h_{\text{top}}(1001)$.

So, we have proved that $h_{\text{top}}(\theta)$ is continuous at $\theta = 1001$, as we announced in our main result. $\square$

References


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