METALLIC SHAPED HYPERSURFACES IN LORENTZIAN SPACE FORMS

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Abstract. We show that metallic shaped hypersurfaces in Lorentzian space forms are isoparametric and obtain their full classification.

1. Introduction

Let \( p \) and \( q \) be two positive integers. Consider the quadratic equation

\[ x^2 - px - q = 0. \]

The positive solution of this equation is

\[ \sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2} \]

and called a member of the metallic means family (briefly MMF) \([5]\). These numbers are called \((p,q)\)-metallic numbers \([5]\). For special values of \( p \) and \( q \) de Spinadel defined in \([6]\) the following metallic means:

i) For \( p = q = 1 \) we obtain \( \sigma_{G} = \frac{1 + \sqrt{5}}{2} \), which is the golden mean,

ii) For \( p = 2 \) and \( q = 1 \) we obtain \( \sigma_{Ag} = 1 + \sqrt{2} \), which is the silver mean,

iii) For \( p = 3 \) and \( q = 1 \) we obtain \( \sigma_{Br} = \frac{3 + \sqrt{13}}{2} \), which is the bronze mean,

iv) For \( p = 1 \) and \( q = 2 \) we obtain \( \sigma_{Cu} = 2 \), which is the copper mean,

v) For \( p = 1 \) and \( q = 3 \) we obtain \( \sigma_{Ni} = \frac{1 + \sqrt{13}}{2} \), which is the nickel mean.

Hence the metallic means family is a generalization of the golden mean. It is well-known that the golden mean is used widely in mathematics, natural sciences, music, art, etc. The MMF have been used in describing fractal geometry, quasiperiodic dynamics (for more details see \([9]\) and the references therein). Furthermore, El Naschie \([7]\) obtained the relationships between the Hausdorff dimension of higher order Cantor sets and the golden mean or silver mean.

In \([9]\), Hrețcanu and Crasmareanu defined the metallic structure on a manifold \( M \) as a \((1,1)\)-tensor field \( J \) on \( M \) satisfying the equation

\[ J^2 = pJ + qI, \]

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where $I$ is the Kronecker tensor field of $M$ and $p, q$ are positive integers. If $p = q = 1$, one obtains a golden structure on a manifold $M$, which was defined and studied in [4] and [8].

In [11], the present authors defined the metallic shaped hypersurfaces and they obtained the full classification of the metallic shaped hypersurfaces in real space forms. A hypersurface $M$ is called a metallic shaped hypersurface if the shape operator $A$ of $M$ is a metallic structure, i.e., $A^2 = pA + qI$, where $I$ is the identity on the tangent bundle of $M$ and $p, q$ are positive integers. If $p = q = 1$, one obtains a golden shaped hypersurface, defined by Crasmareanu, Hrețcanu and Munteanu in [3]. The full classification of golden shaped hypersurfaces in real space forms was given in [3].

In [13], Yang and Fu studied the golden shaped hypersurfaces in Lorentzian space forms and gave the full classification of this type of hypersurfaces. In the present study, as a generalization of [13], we consider the metallic shaped hypersurfaces in Lorentzian space forms and obtain the classification of this type of hypersurfaces. Using Mathematica [12], we draw some pictures (see Figures 1 and 2).

2. Main results

Let $M$ be a hypersurface of the Lorentzian space form $M^{n+1}_1(c)$ and for a certain normal vector field $N$, let $A = A_N$ be the shape operator. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the principal curvatures of $M$. If $M$ has constant principal curvatures and its shape operator is diagonalized then it is called an isoparametric hypersurface [2].

Definition 2.1 ([11]). $M$ is called a metallic shaped hypersurface if the shape operator $A$ is a metallic structure. Hence $A$ satisfies

$$A^2 = pA + qI,$$

where $I$ is the identity on the tangent bundle of $M$, $p$ and $q$ are positive integers. If $p = q = 1$, then we obtain a golden shaped hypersurface (see [3] and [13]). If $p = 2$ and $q = 1$, then the hypersurface is called silver shaped; if $p = 3$ and $q = 1$, then it is called bronze shaped; if $p = 1$ and $q = 2$, then it is called copper shaped; if $p = 1$ and $q = 3$, then it is called nickel shaped.

It is known that if $M$ is a Lorentzian hypersurface in $M^{n+1}_1(c)$, then the normal vector is spacelike. In [10], M. A. Magid showed that the shape operator is one of the following forms:

$$A = \begin{bmatrix}
a_1 & 0 & 0 & \ldots & 0 \\
0 & a_2 & 0 & \ldots & 0 \\
0 & 0 & a_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & a_n
\end{bmatrix},$$

(1)
First we give the following proposition:

**Proposition 2.1.** Let $M$ be a metallic shaped hypersurface in a Lorentzian space form $M^{n+1}_1(c)$. Then the shape operator can be diagonalized and the principal curvatures are $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ and $p - \sigma_{p,q} = \frac{p-\sqrt{p^2+4q}}{2}$, which means that the hypersurface is isoparametric.

**Proof.** Assume that $M$ is a spacelike hypersurface in the Lorentzian space form $M^{n+1}_1(c)$. Hence the normal vector is timelike and it is known that the shape operator can be diagonalized by choosing the orthogonal frame field on $M$. Since $M$ is a metallic shaped hypersurface, the relation $A^2 = pA + qI$ gives us that the principal curvatures of the spacelike metallic shaped hypersurface are $\sigma_{p,q}$ and $p - \sigma_{p,q}$. Hence the hypersurface is isoparametric.

Now we consider the above four forms of the shape operators given by M. A. Magid in [10].

If the shape operator is of the form (1) and $A^2 = pA + qI$, then the principal curvatures of the hypersurface are $\sigma_{p,q}$ and $p - \sigma_{p,q}$.

If the shape operator is of the form (2), then $a_0^2 = pa_0 + q$, $2a_0 = p$, $a_1^2 = pa_1 + q$, $\ldots$, $a_{n-2}^2 = pa_{n-2} + q$. So we get $q = -\frac{p^2}{4} < 0$, which is impossible because of the definition of $q$.

If the shape operator is of the form (3), then $a_0^2 = pa_0 + q$, $-1 = 0$, which is a contradiction.

If the shape operator is of the form (4), then $a_0^2 - b_0^2 = pa_0 + q$, $2a_0b_0 = pb_0$, $a_1^2 = pa_1 + q$, $\ldots$, $a_{n-2}^2 = pa_{n-2} + q$. So it follows that $b_0 = 0$ and $a_i = \sigma_{p,q}$ or $p - \sigma_{p,q}$. This proves the proposition. □
In [1], Abe, Koike, and Yamaguchi showed that the isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ have the following cases:

1. $\mathbb{H}^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = 0 \} \subset \mathbb{R}^n$, $A = [0]$,
2. $\mathbb{H}^n(c) = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}, c < 0 \} \subset \mathbb{R}^{n+1}$, $A = \pm \sqrt{-c} I$,
3. $\mathbb{H}^r \times \mathbb{H}^{n-r} = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}, c < 0 \} \subset \mathbb{R}^{n+1}$, $A = \pm \left( 0, \pm \sqrt{-c} I_{n-r} \right)$,
4. $\mathbb{R}^{n+1}_1 = \{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0 \} \subset \mathbb{R}^{n+1}$, $A = [0]$,
5. $\mathbb{S}^n(c) = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}, c > 0 \} \subset \mathbb{R}^{n+1}$, $A = \pm \sqrt{c} I$,
6. $\mathbb{R}^{r} \times \mathbb{S}^{n-r} = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}, c > 0 \} \subset \mathbb{R}^{n+1}$, $A = \pm \left( 0, \pm \sqrt{c} I_{n-r} \right)$.

Here (R1)–(R3) are spacelike hypersurfaces and (R4)–(R6) are Lorentzian hypersurfaces in $\mathbb{R}^{n+1}$.

For the metallic shaped hypersurfaces in $\mathbb{R}^{n+1}$, we give the following theorem:

**Theorem 2.1.** The only metallic shaped hypersurfaces of Minkowski space $\mathbb{R}^{n+1}$ are:

1. $\mathbb{H}^n(c) = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \} \subset \mathbb{R}^{n+1}$, $A = \sqrt{-c} I$, where $c = -\sigma_{p,q}^2$,
2. $\mathbb{H}^n(c) = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \} \subset \mathbb{R}^{n+1}$, $A = -\sqrt{-c} I$, where $c = -(p - \sigma_{p,q})^2$,
3. $\mathbb{S}^n_1(c) = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \} \subset \mathbb{R}^{n+1}$, $A = \sqrt{c} I$, where $c = \sigma_{p,q}^2$,
4. $\mathbb{S}^n_1(c) = \{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \} \subset \mathbb{R}^{n+1}$, $A = -\sqrt{c} I$, where $c = (p - \sigma_{p,q})^2$.

**Proof.** We know from Proposition 2.1 that a metallic shaped isoparametric hypersurface of a Lorentzian space form has non-zero constant principal curvatures $\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$ and $p - \sigma_{p,q} = \frac{p - \sqrt{p^2 + 4q}}{2}$. Because of this reason the cases (R1), (R3), (R4) and (R6) are impossible.

First we consider the case (R2). If the eigenvalue of the shape operator $A$ is $\sigma_{p,q}$ then $\sqrt{-c} = \sigma_{p,q}$, $c < 0$. So the spacelike metallic shaped hypersurface is

$$\mathbb{H}^n(c) = \left\{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\} \subset \mathbb{R}^{n+1}, \quad A = \sqrt{-c} I,$$

where

$$c = -\sigma_{p,q}^2 = -\frac{p^2 + p \sqrt{p^2 + 4q} + 4q}{2}.$$

If the eigenvalue of the shape operator $A$ is $p - \sigma_{p,q}$ then $\sqrt{-c} = p - \sigma_{p,q}$, $c < 0$. Hence the spacelike metallic shaped hypersurface is

$$\mathbb{H}^n(c) = \left\{ x \in \mathbb{R}^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\} \subset \mathbb{R}^{n+1}, \quad A = -\sqrt{-c} I,$$

$\square$
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Figure 1. The golden shaped hypersurface $H^2 \left( \frac{-3+\sqrt{5}}{2} \right) \subset \mathbb{R}^3_1$.

where

$$c = -(p - \sigma_{p,q})^2 = \frac{-p^2 + p\sqrt{p^2 + 4q} - 2q}{2}.$$ 

Now we consider the case (R5). If the eigenvalue of the shape operator $A$ is $\sigma_{p,q}$ then $\sqrt{c} = \sigma_{p,q}$, $c > 0$. So the Lorentzian metallic shaped hypersurface is

$$S_1^n(c) = \left\{ x \in \mathbb{R}^{n+1}_1 : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\}, \quad A = \sqrt{c}I,$$

where

$$c = \sigma_{p,q}^2 = \frac{p^2 + p\sqrt{p^2 + 4q} + 2q}{2}.$$ 

If the eigenvalue of the shape operator $A$ is $\sigma_{p,q}$ then $-\sqrt{c} = p - \sigma_{p,q}$, $c > 0$. Hence the Lorentzian metallic shaped hypersurface is

$$S_1^n(c) = \left\{ x \in \mathbb{R}^{n+1}_1 : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\}, \quad A = -\sqrt{c}I,$$

where

$$c = (p - \sigma_{p,q})^2 = \frac{p^2 - p\sqrt{p^2 + 4q} + 2q}{2}.$$ 

This proves the theorem.  □
Figure 2. The golden shaped hypersurface $S^2_{\frac{3-\sqrt{5}}{2}} \subset \mathbb{R}^3$.

The isoparametric hypersurfaces of de Sitter space $S^{n+1}_1(1)$ were given by Abe, Koike, and Yamaguchi in [1] as follows:

(S1) $\mathbb{R}^n = \{x \in S^{n+1}_1(1) \subset \mathbb{R}^{n+2}_1 : x_1 = x_{n+2} + t_0, t_0 > 0\}, A = \pm I,$

(S2) $S^n(c) = \{x \in S^{n+1}_1(1) \subset \mathbb{R}^{n+2}_1 : x_1 = \sqrt{\frac{1}{c} - 1}, 0 < c \leq 1\}, A = \pm \sqrt{1 - c}I,$

(S3) $H^n(c) = \{x \in S^{n+1}_1(1) \subset \mathbb{R}^{n+2}_1 : x_{n+2} = \sqrt{1 - \frac{1}{c}}, c < 0\}, A = \pm \sqrt{1 - c}I,$

(S4) $H^r(c_1) \times H^{n-r}(c_2) = \{x \in S^{n+1}_1(1) \subset \mathbb{R}^{n+2} : \sum_{i=1}^{r+2} x_i^2 = \frac{1}{c_1}, -x_i^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}\},$

\[ \left(\frac{1}{c_1} + \frac{1}{c_2} = 1, c_1 > 0, c_2 < 0\right), A = \pm \left(\sqrt{1 - c_1}I_r \oplus \sqrt{1 - c_2}I_{n-r}\right),\]

(S5) $S^n_1(c) = \{x \in S^{n+1}_1(1) \subset \mathbb{R}^{n+2}_1 : x_{n+2} = \sqrt{1 - \frac{1}{c}}, c \geq 1\}, A = \pm \sqrt{c - 1}I,$

(S6) $S^r(c_1) \times S^{n-r}_1(c_2) = \{x \in S^{n+1}_1(1) \subset \mathbb{R}^{n+2} : \sum_{i=2}^{r+2} x_i^2 = \frac{1}{c_1}, -x_i^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}\},$

\[ \left(\frac{1}{c_1} + \frac{1}{c_2} = 1, c_1 > 0, c_2 > 0\right), A = \pm \left(\sqrt{c_1 - 1}I_r \oplus (-\sqrt{c_2 - 1}I_{n-r})\right).\]

Here (S1)–(S4) are spacelike hypersurfaces and (S5), (S6) are Lorentzian hypersurfaces of $S^{n+1}_1(1)$.

**Theorem 2.2.** The only metallic shaped hypersurfaces of de Sitter space $S^{n+1}_1(1)$ are:

1) $\mathbb{R}^n = \{x \in S^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_1 = x_{n+2} + t_0, t_0 > 0\}, A = -I, q = p + 1.$
2) $S^n(c) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_1 = \sqrt{\frac{1}{c} - 1}, 0 < c < 1 \right\}$, $A = -\sqrt{1 - c}I$, where $c = 1 - (p - \sigma_{p,q})^2$, $q - 1 < p$.

3) $\mathbb{H}^n(c) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_n = \sqrt{1 - \frac{1}{c}}, 0 < c \right\}$, $A = \sqrt{1 - c}I$, where $c = 1 - \sigma_{p,q}^2$.

4) $\mathbb{H}^n(c) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_n = \sqrt{1 - \frac{1}{c}}, c < 0 \right\}$, $A = -\sqrt{1 - c}I$, where $c = 1 - (p - \sigma_{p,q})^2$ and $q > 1 + p$.

5) $S^n_1(c) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_n = \sqrt{1 - \frac{1}{c}}, A = \sqrt{c - I}I, \right\}$, where $c = 1 + \sigma_{p,q}^2$.

6) $S^n_1(c) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_n = \sqrt{1 - \frac{1}{c}}, A = -\sqrt{c - I}I, \right\}$, where $c = 1 + (p - \sigma_{p,q})^2$.

7) $S^r(c_1) \times S^{n-r}(c_2) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_1 = \frac{1}{c_1}, x_2 = \frac{1}{c_2}, \right\}$, $A = \frac{1}{c_1} + \frac{1}{c_2} = 1, A = (\sqrt{c_1 - I}I_1 \oplus (\sqrt{c_2 - I}I_{n-r})), \right\}$, where $c_1 = 1 + \sigma_{p,1}^2$ and $c_2 = 1 + (p - \sigma_{p,1})^2$.

8) $S^r(c_1) \times S^{n-r}(c_2) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_1 = \frac{1}{c_1}, x_2 = \frac{1}{c_2}, \right\}$, $A = (-\sqrt{c_1 - I}I_1 \oplus (\sqrt{c_2 - I}I_{n-r})), \right\}$, where $c_1 = 1 + (p - \sigma_{p,1})^2$ and $c_2 = 1 + \sigma_{p,1}^2$.

Proof. By the use of Proposition 2.1 since a metallic shaped isoparametric hypersurface of a Lorentzian space form has non-zero constant principal curvatures $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ and $p - \sigma_{p,q} = \frac{p-\sqrt{p^2+4q}}{2}$, the case (S4) is not possible.

First we consider the case (S1). If the eigenvalue of the shape operator $A$ is $\sigma_{p,q}$ then $1 = \sigma_{p,q}$. This gives us $p + q = 1$. Since $p$ and $q$ are positive integers, this is not possible. Now assume that the eigenvalue of the shape operator $A$ is $p - \sigma_{p,q}$. Then $-1 = p - \sigma_{p,q}$. This gives us $p + 1 = q$. Hence the spacelike metallic shaped hypersurface is

$$\mathbb{R}^n = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_1 = x_{n+2} + t_0, t_0 > 0 \right\}, A = -I, q = p + 1.$$ 

Now we consider the case (S2). If the eigenvalue of the shape operator $A$ is $\sigma_{p,q}$ then $\sqrt{1 - c} = \sigma_{p,q}$. Hence $c = \frac{2 - p^2 - p\sqrt{p^2 + 4q} - 2q}{2}$. But for (S2), since $0 < c \leq 1$, $p$ and $q$ are positive integers, and this is not possible. If the eigenvalue of the shape operator $A$ is $p - \sigma_{p,q}$, then $-\sqrt{1 - c} = p - \sigma_{p,q}$. So we have $q - 1 < p$. Hence the spacelike metallic shaped hypersurface is

$$S^n(c) = \left\{ x \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : x_1 = \sqrt{\frac{1}{c} - 1}, 0 < c < 1 \right\}, A = -\sqrt{1 - c}I,$$

where

$$c = 1 - (p - \sigma_{p,q})^2 = \frac{2 - p^2 + p\sqrt{p^2 + 4q} - 2q}{2}.$$
Now we consider the case (S3). If the eigenvalue of the shape operator $A$ is $\sigma_{p,q}$ then $\sqrt{1 - c} = \sigma_{p,q}$. Hence $c = 1 - \sigma_{p,q}^2 = \frac{2p^2 - p\sqrt{p^2 + 4q + 2}}{2}$. Since $c < 0$ for (S3) the spacelike hypersurface is

$$\mathbb{H}^n(c) = \left\{ x \in S_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}, c < 0 \right\}, A = \sqrt{1 - c} I,$$

where $c = 1 - \sigma_{p,q}^2$. If the eigenvalue of the shape operator $A$ is $p - \sigma_{p,q}$ then $-\sqrt{1 - c} = p - \sigma_{p,q}$. This gives us $c = 1 - (p - \sigma_{p,q})^2 = \frac{2p^2 - p\sqrt{p^2 + 4q + 2}}{2}$. Since $c < 0$, we obtain $q > 1 + p$. Then the spacelike hypersurface is

$$\mathbb{H}^n(c) = \left\{ x \in S_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}, c < 0 \right\}, A = -\sqrt{1 - c} I,$$

where $c = 1 - (p - \sigma_{p,q})^2$ and $q > 1 + p$.

Now we consider the case (S5). If the eigenvalue of the shape operator $A$ is $\sigma_{p,q}$ then $\sqrt{c - 1} = \sigma_{p,q}$. This gives us $c = 1 + \sigma_{p,q}^2 = \frac{2p^2 + p\sqrt{p^2 + 4q + 2}}{2}$. For all positive integers $p, q$ we have $c > 1$. Hence the Lorentzian hypersurface of $S_1^{n+1}(1)$ is

$$S_1^n(c) = \left\{ x \in S_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}, A = \sqrt{c - 1} I, \right\},$$

where

$$c = 1 + \sigma_{p,q}^2.$$

If the eigenvalue of the shape operator $A$ is $p - \sigma_{p,q}$ then $-\sqrt{c - 1} = p - \sigma_{p,q}$. This gives us $c = 1 + (p - \sigma_{p,q})^2 = \frac{2p^2 + p\sqrt{p^2 + 4q + 2}}{2}$. For all positive integers $p, q$, we have $c > 1$. Hence the Lorentzian hypersurface of $S_1^{n+1}(1)$ is

$$S_1^n(c) = \left\{ x \in S_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}, A = -\sqrt{c - 1} I, \right\},$$

where

$$c = 1 + (p - \sigma_{p,q})^2.$$

Now we consider the case (S6). If $\sqrt{c_1 - 1} = \sigma_{p,q}$ and $-\sqrt{c_2 - 1} = p - \sigma_{p,q}$, then $c_1 = 1 + \sigma_{p,q}^2 = \frac{2p^2 + p\sqrt{p^2 + 4q + 2}}{2}$ and $c_2 = 1 + (p - \sigma_{p,q})^2 = \frac{2p^2 - p\sqrt{p^2 + 4q + 2}}{2}$. Since $\frac{1}{c_1} + \frac{1}{c_2} = 1$ we get $q = 1$. Hence the Lorentzian hypersurface of $S_1^{n+1}(1)$ is

$$S'(c_1) \times S_1^{n-r}(c_2) = \left\{ x \in S_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : \sum_{i=2}^{r+2} x_i^2 = \frac{1}{c_1}, -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}, \left( \frac{1}{c_1} + \frac{1}{c_2} = 1, A = (\sqrt{c_1 - 1} I_r \oplus (-\sqrt{c_2 - 1} I_{n-r})), \right) \right\},$$

where $c_1 = 1 + \sigma_{p,1}^2$ and $c_2 = 1 + (p - \sigma_{p,1})^2$. If $-\sqrt{c_1 - 1} = p - \sigma_{p,q}$ and $\sqrt{c_2 - 1} = \sigma_{p,q}$, then $c_1 = 1 + (p - \sigma_{p,q})^2$ and $c_2 = 1 + \sigma_{p,q}^2$. Since $\frac{1}{c_1} + \frac{1}{c_2} = 1$ we get $q = 1$. Hence the Lorentzian hypersurface of $S_1^{n+1}(1)$ is
Theorem 2.3. The only metallic shaped hypersurfaces of anti-de Sitter space $\mathbb{H}_1^{n+1}(1)$ are:

1) $\mathbb{H}^n(c) = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = \sqrt{\frac{1}{c} + 1}, A = -\sqrt{-1 - c} I \}$, where $c = -1 + (p - \sigma_{p,1})^2$.

2) $\mathbb{H}^n(c) = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = \sqrt{\frac{1}{c} + 1}, A = -\sqrt{-1 - c} I \}$, where $c = -1 + (p - \sigma_{p,1})^2$.

3) $\mathbb{H}^r(c_1) \times \mathbb{H}^{n-r}(c_2) = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : -x_1^2 + \sum_{i=3}^{n+2} x_i^2 = \frac{1}{c_1}, -x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2} \}$, where $c_1 = -(1 + \sigma_{p,1}^2)$, $c_2 = -[1 + (p - \sigma_{p,1})^2]$.

Abe, Koike, and Yamaguchi classified isoparametric hypersurfaces of $\mathbb{H}_1^{n+1}(-1)$ as follows (I):

(H1) $\mathbb{H}^n(c) = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = \sqrt{\frac{1}{c} + 1}, c \leq -1 \}$, $A = \pm \sqrt{-1 - c} I$,

(H2) $\mathbb{H}^r(c_1) \times \mathbb{H}^{n-r}(c_2) = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : -x_1^2 + \sum_{i=3}^{n+2} x_i^2 = \frac{1}{c_1}, -x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2} \}$, $A = \pm \sqrt{-1 - c_1 I_r} \oplus \sqrt{-1 - c_2 I_{n-r}}$.

(H3) $\mathbb{R}_1^n = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = x_{n+2} + t_0, t_0 > 0 \}$, $A = \pm I$.

(H4) $\mathbb{S}_1^n = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = \sqrt{\frac{1}{c} + 1}, c > 0 \}$, $A = \pm \sqrt{1 + c} I$.

(H5) $\mathbb{H}_1^{n+1}(1) = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_{n+2} = \sqrt{-1 - c}, -1 \leq c < 0 \}$, $A = \pm \sqrt{1 + c} I$.

(H6) $\mathbb{S}_1^n(c_1) \times \mathbb{H}^{n-r}(c_2) = \{ x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : -x_1^2 + \sum_{i=3}^{n+2} x_i^2 = \frac{1}{c_1}, -x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2} \}$, $A = \pm \sqrt{1 + c_1 I_r} \oplus \sqrt{1 + c_2 I_{n-r}}$.

This proves the theorem. □
4) $H^r(c_1) \times H^{n-r}(c_2) = \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1^2 + x_2^2 = \frac{1}{c_1} \right\}$, where $c_1 = -1, x_1 = x_2 = 0$.

5) $R^1_n = \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1 = x_{n+2} + t_0, t_0 > 0 \right\}$, $A = -I, q = p + 1$.

6) $S^0_n(c) = \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1 = \sqrt{\frac{1}{c} + 1} \right\}$, $A = \sqrt{1 + c}I$, where $c = \sigma^2_{p,q} - 1$.

7) $S^0_n(c) = \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1 = \sqrt{\frac{1}{c} + 1}, c > 0 \right\}$, $A = -\sqrt{1 + c}I$, where $c = (p - \sigma_{p,q})^2 - 1$ and $q > 1 + p$.

8) $H^1_1(c) = \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1 = \sqrt{-1 - \frac{1}{c}}, -1 \leq c < 0 \right\}$, $A = -\sqrt{1 + c}I$, where $c = (p - \sigma_{p,q})^2 - 1$ and $q < 1 + p$.

Proof. Here (H1) and (H2) are spacelike hypersurfaces and (H3)–(H6) are Lorentzian hypersurfaces of $H^{n+1}_1(-1)$.

Since the eigenvalues of the shape operator of the hypersurface in $H^{n+1}_1(-1)$ are $\sigma_{p,q} > 0$ and $p - \sigma_{p,q} < 0$, (H6) is not possible.

We consider the case (H1). If the eigenvalue of the shape operator is $\sqrt{-1 - c} = \sigma_{p,q}$, then we have $c = -(1 + \sigma_{p,q})^2 = -\frac{2 + p^2 + p\sqrt{p^2 + 4q + 2q}}{2}$. Hence the spacelike hypersurface of $H^{n+1}_1(-1)$ is

$$H^n(c) = \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1 = \sqrt{\frac{1}{c} + 1} \right\}, A = -\sqrt{1 - c}I,$$

where $c = -(1 + \sigma_{p,q})^2$. If the eigenvalue of the shape operator is $\sqrt{-1 - c} = p - \sigma_{p,q}$, then we have $c = -\left[1 + (p - \sigma_{p,q})^2\right] = -\frac{2 + p^2 + p\sqrt{p^2 + 4q + 2q}}{2}$. Hence the spacelike hypersurface of $H^{n+1}_1(-1)$ is

$$H^n(c) = \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1 = \sqrt{\frac{1}{c} + 1} \right\}, A = -\sqrt{1 - c}I,$$

where $c = -\left[1 + (p - \sigma_{p,q})^2\right]$.

Now we consider the case (H2). If the eigenvalue of the shape operator is $\sqrt{-1 - c_1} = \sigma_{p,q}$ and $\sqrt{-1 - c_2} = p - \sigma_{p,q}$, then we have $c_1 = -(1 + \sigma_{p,q})^2 = -\frac{2 + p^2 + p\sqrt{p^2 + 4q + 2q}}{2}, c_2 = -\left[1 + (p - \sigma_{p,q})^2\right] = -\frac{2 + p^2 + p\sqrt{p^2 + 4q + 2q}}{2}$ and $q = 1$.

Hence the spacelike hypersurface of $H^{n+1}_1(-1)$ is

$$H^r(c_1) \times H^{n-r}(c_2)$$

$$= \left\{ x \in H^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1^2 + \sum_{i=3}^{r+2} x_i^2 = \frac{1}{c_1}, x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2} \right\},$$

$$(\frac{1}{c_1} + \frac{1}{c_2} = -1), A = \left(-\sqrt{1 - c_1}I \oplus \left(-\sqrt{1 - c_2}\right)I_{n-r}\right),$$

where \( c_1 = -(1 + \sigma_{p,1}^2) \) and \( c_2 = -[1 + (p - \sigma_{p,1})^2] \). If the eigenvalue of the shape operator is \(-\sqrt{-1 - c_1} = p - \sigma_{p,q} \) and \( \sqrt{-1 - c_2} = \sigma_{p,q} \), then we have \( c_1 = -[1 + (p - \sigma_{p,q})^2] \), \( c_2 = -(1 + \sigma_{p,q}^2) \) and \( q = 1 \). Hence the spacelike hypersurface of \( \mathbb{H}^{n+1}_1(-1) \) is \( \mathbb{H}^r(c_1) \times \mathbb{H}^{n-r}(c_2) = \{ x \in \mathbb{H}^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : -x_1^2 + \sum_{i=3}^{r+2} x_i^2 = \frac{1}{c_1}, -x_2^2 + \frac{n+2}{c_2}, \( \frac{1}{c_1} + \frac{1}{c_2} = -1 \} \), \( A = (-\sqrt{-1 - c_1}I_r \oplus (-\sqrt{-1 - c_2}I_{n-r}) \), where \( c_1 = -[1 + (p - \sigma_{p,1})^2] \) and \( c_2 = -(1 + \sigma_{p,1}^2) \).

Now we consider the case (H3). If the eigenvalue of the shape operator is \( q > 1 + p \), then \( p \) and \( q \) are positive integers, this is not possible. If the eigenvalue of the shape operator is \( -1 = p - \sigma_{p,q} \), then \( p + 1 = q \). Hence the Lorentzian hypersurface of \( \mathbb{H}^{n+1}_1(-1) \) is

\[
\mathbb{R}^n = \{ x \in \mathbb{H}^{n+1}_1(-1) \subset \mathbb{R}^{n+2}_2 : x_1 = x_{n+2} + t_0, t_0 > 0 \}, \ A = -I, \ q = p + 1.
\]

Now we consider the case (H4). If the eigenvalue of the shape operator is \( \sqrt{1 + c} = \sigma_{p,q} \), then \( c = \sigma_{p,q}^2 - 1 = \frac{p^2 + p\sqrt{p^2 + 4q + 2q - 2}}{2} \). Hence the Lorentzian hypersurface of \( \mathbb{H}^{n+1}_1(-1) \) is

\[
S^n_1(c) = \{ x \in \mathbb{H}^{n+1}_1(-1) \subset \mathbb{R}^{2n+2}_2 : x_1 = \sqrt{\frac{1}{c} + 1}, c > 0 \}, \ A = \sqrt{1 + c}I,
\]
where \( c = \sigma_{p,q}^2 - 1 \). If the eigenvalue of the shape operator is \( -\sqrt{1 + c} = p - \sigma_{p,q} \), then \( c = (p - \sigma_{p,q})^2 - 1 = \frac{p^2 - p\sqrt{p^2 + 4q + 2q - 2}}{2} \). Since \( c > 0 \), this gives us \( q > 1 + p \). Hence the Lorentzian hypersurface of \( \mathbb{H}^{n+1}_1(-1) \) is

\[
S^n_1(c) = \{ x \in \mathbb{H}^{n+1}_1(-1) \subset \mathbb{R}^{2n+2}_2 : x_1 = \sqrt{\frac{1}{c} + 1}, c > 0 \}, \ A = -\sqrt{1 + c}I,
\]
where \( c = (p - \sigma_{p,q})^2 - 1 \) and \( q > 1 + p \).

Now we consider the case (H5). If the eigenvalue of the shape operator is \( \sqrt{1 + c} = \sigma_{p,q} \), then \( c = \frac{p^2 + p\sqrt{p^2 + 4q + 2q - 2}}{2} \). Since \( -1 \leq c < 0 \), this is impossible. If the eigenvalue of the shape operator is \( -\sqrt{1 + c} = p - \sigma_{p,q} \), then \( c = (p - \sigma_{p,q})^2 - 1 = \frac{p^2 - p\sqrt{p^2 + 4q + 2q - 2}}{2} \). Since \( -1 \leq c < 0 \), we obtain \( q < 1 + p \). Hence the Lorentzian hypersurface of \( \mathbb{H}^{n+1}_1(-1) \) is

\[
\mathbb{H}^{n+1}_1(c) = \{ x \in \mathbb{H}^{n+1}_1(-1) \subset \mathbb{R}^{2n+2}_2 : x_{n+2} = \sqrt{-1 - \frac{1}{c}}, -1 \leq c < 0 \}, \ A = -\sqrt{1 + c}I,
\]
where \( c = (p - \sigma_{p,q})^2 - 1 \) and \( q < 1 + p \). \( \square \)
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