A NOTE ON WEIGHTED INEQUALITIES FOR A ONE-SIDED MAXIMAL OPERATOR IN $\mathbb{R}^n$

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Abstract. We introduce a new dyadic one-sided maximal operator $M_{d}^{+}$ in $\mathbb{R}^n$ that allows us to obtain good weights for the $L^p$-boundedness of a one-sided maximal operator $N^{+}$ in $\mathbb{R}^n$, which is equivalent to the classical one-sided Hardy–Littlewood maximal operator in the case $n = 1$, but not in the case $n > 1$. In order to do this, we characterize the good pairs of weights for the weak and strong type inequalities for $M_{d}^{+}$ and we use a Fefferman–Stein type inequality which gives that, in a certain sense, $M_{d}^{+}$ controls $N^{+}$.

1. Introduction

For $f$ locally integrable on $\mathbb{R}$, the one-sided Hardy–Littlewood maximal functions are

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(y)| \, dy$$

and

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)| \, dy.$$  

These operators are interesting because they control some one-sided operators such as singular integrals with kernels supported in $(-\infty, 0)$ or $(0, \infty)$. Sawyer characterized the $L^p$-weighted inequalities for $M^+$ in [10] (see other proofs in [1], [7], [8] and [5]).

In [5] we introduced an appropriate dyadic one-sided maximal operator $M_{d}^+$ in $\mathbb{R}$, which allows us to give a new proof of Sawyer’s results, proving a Feffermann–Stein type inequality in means as in the classical case.

The natural generalization of $M^+$ in $\mathbb{R}^n$ is the following: given $x = (x_1, x_2, \ldots, x_n)$ we define

$$M^{+\cdots+} f(x) = \sup_{h>0} \frac{1}{h^n} \int_{Q_x(h)} |f(y)| \, dy,$$

where $Q_x(h) = [x_1, x_1+h) \times [x_2, x_2+h) \times \cdots \times [x_n, x_n+h)$. In $\mathbb{R}^n$ we have two one-sided operators. In $\mathbb{R}^n$ we obviously have $2^n$ one-sided operators that we do not

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write explicitly. Quite surprisingly, the weighted inequalities for $M^{+\cdots+}$ in $\mathbb{R}^n$ have not been characterized. The characterization of the weak type $(p, p)$ inequality in $\mathbb{R}^2$ has been obtained in [3]. It is not difficult to prove the following:

**Theorem 1.1.** Let $1 \leq p < \infty$ and let $u, v$ be nonnegative measurable functions. If $M^{+\cdots+}$ is of weak type $(p, p)$ with respect to $(u, v)$, that is, there exists $C > 0$ such that for any $\lambda > 0$ and $f \in L^p(v)$

$$
\int_{\{x \in \mathbb{R}^n : M^{+\cdots+} f(x) > \lambda\}} u(x) \, dx \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx,
$$

then the pair of weights $(u, v)$ satisfies $A^+_p$, that is, there exists $C > 0$ such that for all $h > 0$,

$$
\frac{1}{h^n} \left( \int_{Q^+_x(h)} u \right)^{1/p} \left( \int_{Q_x(h)} v^{1-p'} \right)^{1/p'} \leq C, \quad p > 1,
$$

$$
\frac{1}{h^n} \int_{Q^+_x(h)} u \leq C v(x), \quad a.e. \ x = (x_1, \ldots, x_n), \quad p = 1,
$$

where $Q^+_x(h) = [x_1 - h, x_1) \times \cdots \times [x_n - h, x_n)$.

In [3] it was proved that the previous conditions are equivalent in the case $n = 2$. But, apparently, the arguments used are valid only for $n = 2$. Consequently, the problem of characterizing the weighted $L^p$ inequalities for $M^{+\cdots+}$ remains open in dimension greater than one.

We have tried to extend the results of [3] to higher dimensions. We have not achieved our main goal, that is, characterizing the good weights for $M^{+\cdots+}$, but we have got new interesting results for the following one-sided maximal operator in $\mathbb{R}^n$ which has appeared in several previous works (see [2], [4] and [1]): Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let us define

$$
N^{+\cdots+} f(x) = \sup_{h > 0} \frac{1}{h^n} \int_{Q^+_x(h)} |f(y)| \, dy,
$$

where $Q^+_x(h) = [x_1 + h, x_1 + 2h) \times [x_2 + h, x_2 + 2h) \times \cdots \times [x_n + h, x_n + 2h)$. For $n = 1$, $M^+$ and $N^+$ are equivalent. It is clear that $N^{+\cdots+} f(x) \leq 2^n M^{+\cdots+} f(x)$ but there is no constant $C > 0$ satisfying $M^{+\cdots+} f(x) \leq CN^{+\cdots+} f(x)$ for $n > 1$. In [9] Ombrosi proved that $(u, v) \in A^+_p$ implies the weak type inequality [1] for $N^{+\cdots+}$ (instead of $M^{+\cdots+}$). In [4] Lerner and Ombrosi proved that in the case of equal weights, $u = v$, and $n = 2$, the $A^+_p$ condition implies that $N^{++}$ is bounded from $L^p(u)$ into $L^p(u)$. In [2] Berkovits extends this result for $n \geq 3$. In this note we generalize the result in [4] to any dimension giving a proof completely different from the one proposed by Berkovits, and we also give a sufficient condition in a pair of weights $(u, v)$ for the boundedness of $N^{+\cdots+}$ from $L^p(v)$ into $L^p(u)$.
2. ONE-SIDED DYADIC MAXIMAL FUNCTION IN $\mathbb{R}^n$

We are going to consider a one-sided dyadic maximal function in $\mathbb{R}^n$ that generalizes the one defined in [5]. In order to do this, let us establish the following notation: if $Q = [x_1,x_1 + h] \times \cdots \times [x_n,x_n + h]$ is a cube, denote $Q^- = [x_1,x_1 + h/2] \times \cdots \times [x_n,x_n + h/2]$ and $Q^+ = [x_1 + h/2,x_1 + h] \times \cdots \times [x_n + h/2,x_n + h]$. Let $A^+_x = \{ Q \text{ dyadic : } x \in Q^- \}$. The one-sided dyadic maximal function is defined by

$$M^+_d f(x) = \sup_{Q \in A^+_x} \frac{1}{|Q^+|} \int_{Q^+} |f(y)| dy.$$ 

Observe the difference between this operator and the corresponding one in [9]. Our operator satisfies that $M^{+\cdots+}_d f \leq C M_d f$, where $M_d$ is the classical dyadic maximal operator $M_d f(x) = \sup_{Q \text{ dyadic : } x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| dy$, while the one-sided dyadic operator in [9] does not satisfy this condition.

It is interesting to note the following properties. Let $Q$ and $R$ be dyadic cubes.

(i) If $Q^+ \subset R^+$ then $Q^- \subset R^- \cup R^+$.
(ii) If $Q^- \cap R^+ \neq \emptyset$ and $Q^+ \cap R^+ \neq \emptyset$ then $R = Q$.
(iii) If $Q^+ \cap R^+ \neq \emptyset$ and $Q^+ \cap R^- \neq \emptyset$ then $R \subset Q^+$.

These properties allow us to get a Fefferman–Stein type inequality as in [5]. Let $h = (h_1,\ldots,h_n) \in \mathbb{R}^n$. Denote $\tau_h f(x) = \tau_h f(x_1,\ldots,x_n) = f(x_1-h_1,\ldots,x_n-h_n)$.

**Proposition 2.1.** There exists a constant $C > 0$ such that

$$N^{+\cdots+}_k f(x) \leq \frac{C}{(2k+4)^n} \int_{(0,2k+4) \times \cdots \times (0,2k+4)} (\tau_{-t} \circ M^+_d \circ \tau_t) f(x) dt,$$

for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $f$ bounded with compact support, where $N^{+\cdots+}_k$ is the truncated operator of $N^{+\cdots+}$ taking supremum on $0 < h < 2k$.

The proof of this result follows the same pattern as in the case $n = 1$ and it is left to the interested reader.

As we said before, Proposition 2.1 will allow us to get some results for $N^{+\cdots+}$ by studying the good weights for the dyadic operator $M^{+\cdots+}_d$.

**Definition 2.2.** We shall say that a pair of weights $(u,v)$ belongs to the class $A^+_{p,d}$ if there exists a constant $C > 0$ such that for all dyadic cubes $Q$,

$$\frac{1}{|Q^-|} \left( \int_{Q^-} u \right)^{1/p} \left( \int_{Q^+} v^{1-p'} \right)^{1/p'} \leq C, \quad p > 1, \quad (A^+_{p,d})$$

$$\frac{1}{|Q^-|} \int_{Q^-} u \leq C v(x), \quad \text{a.e. } x \in Q^+, \quad p = 1. \quad (A^+_{1,d})$$

In the case $u = v$ we simply write $u \in A^+_{p,d}$ instead of $(u,u) \in A^+_{p,d}$.

In the same way that for $n = 1$ we have the following characterization.

**Theorem 2.3.** Let $1 < p < \infty$. Then the following assertions are equivalent.

1. $M^{+\cdots+}_d$ is bounded from $L^p(u)$ into $L^p(u)$.
(2) $u \in A^+_{p,d}$.

Theorem 2.4 and Proposition 2.1 provide us with a sufficient condition in a weight $u$ for the boundedness of $N^{+\cdots+}$ in $L^p(u)$, $1 < p < \infty$. This extends Ombrosi’s result to dimension $n \geq 3$ with a proof different from the one proposed by Berkovits. In [3] it was proved that this condition is necessary for the boundedness of $M^{+\cdots+}$ in $L^p(u)$.

**Theorem 2.4.** Let $1 < p < \infty$. If $u$ is a weight satisfying
\[
\sup_x \sup_{h>0} \frac{1}{h^n} \left( \int_{Q^-_x(h)} u \right)^{1/p} \left( \int_{Q^+_x(h)} u^{1-p'} \right)^{1/p'} < \infty, \quad \text{(A}^+_p)\]
then $N^{+\cdots+}$ is bounded from $L^p(u)$ into $L^p(u)$.

**Proof.** Observe first that for $t = (t_1, \ldots, t_n)$, $t_1, \ldots, t_n > 0$, the weights $\tau_t u$ satisfy $(A^+_{p,d})$ uniformly on $t$. Then, by Jensen’s and Fubini’s theorems
\[
\int_{\mathbb{R}^n} (N_k^{+\cdots+} f(x))^p u(x) \, dx
\leq C \int_{\mathbb{R}^n} \left( \frac{1}{(2k+4)^n} \int_{(0,2k+4]^n} (\tau_t \circ M_d^{+\cdots+} \circ \tau_t) f(x) \, dt \right)^p u(x) \, dx
\leq C \int_{\mathbb{R}^n} \left( \frac{1}{(2k+4)^n} \int_{(0,2k+4]^n} (M_d^{+\cdots+}(t \tau_t f)(x+t))^p u(x) \, dx \right) \, dt
= C \int_{\mathbb{R}^n} \left( \frac{1}{(2k+4)^n} \int_{(0,2k+4]^n} (M_d^{+\cdots+}(t \tau_t f)(x))^p \tau_t u(x) \, dx \right) \, dt
\leq C \int_{\mathbb{R}^n} \left( |f(x)|^p \tau_t u(x) \, dx \right) \, dt
= C \int_{\mathbb{R}^n} |f(x)|^p u(x) \, dx.
\]

Letting $k$ tend to infinity we get the desired result. \hfill \Box

For different weights we have the next results.

**Theorem 2.5.** Let $1 \leq p < \infty$. Then the following conditions are equivalent:

1. $M_d^{+\cdots+}$ is of weak type $(p,p)$ with respect to $(u,v)$.
2. $(u,v) \in A^+_{p,d}.$

**Theorem 2.6.** Let $1 < p < \infty$. Let $u$ and $v$ be nonnegative measurable functions and $\sigma = v^{1-p'}$. The following assertions are equivalent.

1. $M_d^{+\cdots+}$ is bounded from $L^p(v)$ into $L^p(u)$.
2. There exists $C > 0$ such that
\[
\int_{Q^- \cup Q^+} (M_d^{+\cdots+}(\sigma \chi_{Q^+})(x))^p u(x) \, dx \leq C \int_{Q^+} \sigma(x) \, dx < \infty, \quad \text{(S}^+_{p,d})\]
for all dyadic cubes $Q$ with $\int_{Q^-} u > 0$. 

We also omit the proof of these results because it follows again the steps of the corresponding results in [5]. We only want to point out that the $n$ dimensional version of Lemma 2.1 in [9] holds, with the dyadic cubes $Q$.

Using again Proposition 2.1 and Theorem 2.6 we obtain the following.

**Theorem 2.7.** Let $1 < p < \infty$. If $u, v$ are two weights satisfying that there exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and all $h > 0$,

$$\int_{Q_{x,h}^- \cup Q_{x,h}^+} (M^{++\ldots}(\sigma\chi_{Q_{x,h}}))^p u \leq C \int_{Q_{x,h}} \sigma < \infty$$

whenever $\int_{Q_{x,h}}^- u > 0$, where $\sigma = v^{1-p'}$, then $N^{++\ldots}$ is bounded from $L^p(v)$ into $L^p(u)$.

**Proof.** The proof follows the same steps that the proof of Theorem 2.4 once we prove that $(\tau_t u, \tau_t v)$ satisfies $S^+_{p,d}$ uniformly on $t$. Then, we have to prove that there exists $C > 0$ such that for all dyadic cubes and all $t$,

$$\int_{Q^-\cup Q^+} (M^{++\ldots}(\tau_t\sigma\chi_{Q^+})(x))^p \tau_t u(x) \, dx \leq C \int_{Q^+} \tau_t \sigma(x) \, dx < \infty,$$

for all dyadic cubes $Q$ with $\int_{Q^-} \tau_t u > 0$.

By a change of variable,

$$\int_{Q^-\cup Q^+} (M^{++\ldots}(\tau_t\sigma\chi_{Q^+})(x))^p \tau_t u(x) \, dx$$

$$= \int_{(Q^- - t)\cup(Q^+ - t)} (M^{++\ldots}(\tau_t\sigma\chi_{Q^+})(x + t))^p u(x) \, dx.$$

Using again a change of variable we get

$$M^{++\ldots}(\tau_t\sigma\chi_{Q^+})(x + t) = \sup_{A_{x+t}^+} \frac{1}{|R^+|} \int_{R^+} \tau_t\sigma\chi_{Q^+}(s) \, ds$$

$$= \sup_{A_{x+t}^+} \frac{1}{|R^+|} \int_{(R^+ - t)\cap(Q^+ - t)} \sigma(z) \, dz$$

$$= \sup_{\{R \text{ dyadic}: x \in R^- - t\}} \frac{1}{|R^+ - t|} \int_{(R^+ - t)} \sigma(z)\chi_{Q^+ - t} \, dz$$

$$\leq CM^{++\ldots}(\sigma\chi_{Q^+ - t})(x).$$

Then, using $(S^+_p)$,

$$\int_{Q^-\cup Q^+} (M^{++\ldots}(\tau_t\sigma\chi_{Q^+}))^p \tau_t u \leq C \int_{(Q^- - t)\cup(Q^+ - t)} M^{++\ldots}(\sigma\chi_{Q^+ - t})$$

$$\leq C \int_{(Q^+ - t)} \sigma = C \int_{Q^+} \tau_t \sigma < \infty. \quad \square$$
Remark 2.8. In [6] we studied another one-sided dyadic maximal operator,
\[ \tilde{M}_d^{++} f(x) = \sup_{Q \in A^+} \frac{1}{|Q \setminus Q^-|} \int_{Q \setminus Q^-} |f(y)| \, dy, \]
that allowed us to obtain a sufficient condition for the boundedness of \( M^{++} \) (Th. 4.3). Unluckily, we have to say that the condition \((A)_p^+\) appearing in [6] is, actually, equivalent to the classical \( A_p \) Muckenhoupt condition.

References


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