THE INITIAL VALUE PROBLEM FOR THE SCHRÖDINGER EQUATION INVOLVING THE HENSTOCK–KURZWEIL INTEGRAL

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Abstract. Let \( L \) be the one-dimensional Schrödinger operator defined by \( Ly = -y'' + qy \). We investigate the existence of a solution to the initial value problem for the differential equation \((L-\lambda)y = g\), when \( q \) and \( g \) are Henstock–Kurzweil integrable functions on \([a, b]\). Results presented in this article are generalizations of classical results for the Lebesgue integral.

1. Introduction

Let \( q \) be a real valued function defined on \([a, b]\) and let \( L \) be the one-dimensional Schrödinger operator defined by \( Ly = -y'' + qy \). It is well known that if \( q, g \) are Lebesgue integrable functions on \([a, b]\), then there exists a unique solution \( f, f' \in AC([a, b]) \) of the differential equation \((L-\lambda)y = g\) satisfying the initial condition \( f(c) = \alpha \) and \( f'(c) = \beta \), where \( c \in [a, b] \), \( \lambda, \alpha, \beta \in \mathbb{C} \) and \( AC([a, b]) \) denotes the space of all absolutely continuous functions on \([a, b]\). See, for example, [5]. In this paper, we generalize this result when \( q, g \) are Henstock–Kurzweil integrable functions on \([a, b]\).

2. Preliminaries

In this section, the definition of the Henstock–Kurzweil integral and their main properties needed in this paper are presented.

Definition 2.1. Let \( f : [a, b] \to \mathbb{C} \) be a function. We say that \( f \) is Henstock–Kurzweil (shortly, HK-) integrable on \([a, b]\), if there is an \( A \in \mathbb{C} \) such that, for each \( \epsilon > 0 \), there exists a function \( \gamma_\epsilon : [a, b] \to (0, \infty) \) (named a gauge) for which

\[
\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| < \epsilon,
\]

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for any partition $P = \{(\left[x_{i-1}, x_i\right], t_i)\}_{i=1}^n$ such that $t_i \in [x_{i-1}, x_i]$ and $[x_{i-1}, x_i] \subseteq [t_i - \gamma(t_i), t_i + \gamma(t_i)]$ for all $i = 1, 2, \ldots, n$. The number $A$ is called the integral of $f$ over $[a, b]$ and it is denoted by $\int_a^b f$.

The set of all Henstock–Kurzweil integrable functions on $[a, b]$ is denoted by $HK([a, b])$. This set is a vector space and contains the union of $L([a, b])$, the space of Lebesgue integrable functions on $[a, b]$, and the Cauchy-Lebesgue integrable functions (i.e., improper Lebesgue integrals). It is well known that if $f \in HK([a, b])$ then not necessarily $|f| \in HK([a, b])$. If both $f$ and $|f|$ are HK-integrable on $[a, b]$, we say that $f$ is Henstock–Kurzweil absolutely integrable on $[a, b]$. The space of Henstock–Kurzweil absolutely integrable functions on $[a, b]$ coincides with the space $L([a, b])$.

For each $f \in HK([a, b])$ and $I \subseteq [a, b]$ the Alexiewicz seminorm of $f$ on $I$ is defined as

$$\|f\|_I = \sup_{J \subseteq I} \left| \int_J f \right|,$$

where the supremum is taken over all intervals $J$ contained in $I$.

**Definition 2.2.** Let $\varphi : [c, d] \to \mathbb{C}$ be a function. The variation of $\varphi$ on the interval $[c, d]$ is defined as

$$V_{[c,d]}\varphi = \sup \left\{ \sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| \mid \{x_i\}_{i=0}^n \text{ is a partition of } [c, d] \right\}.$$

We say that the function $\varphi$ is of bounded variation on $[c, d]$ if $V_{[c,d]}\varphi < \infty$. The space of all bounded variation functions on $[c, d]$ is denoted by $BV([c, d])$.

The next theorem shows that absolutely integrable functions are precisely those integrable functions whose indefinite integrals have bounded variation.

**Theorem 2.3 (Multiplier Theorem, [1, Theorem 10.12]).** Let $f \in HK([a, b])$. Then $|f|$ is HK-integrable if and only if the indefinite integral $F(x) = \int_a^x f$ has bounded variation on $[a, b]$. In this case

$$V_{[a,b]}F = \int_a^b |f|.$$
Theorem 2.5 (Lemma 24). If $g$ is a real valued function such that $g \in HK([a,b])$ and $f \in BV([a,b])$, then
\[
\left| \int_a^b fg \right| \leq \inf_{t \in [a,b]} |f(t)| \int_a^b g(t) \, dt + \|g\|_{[a,b]} V_{[a,b]} f.
\]

Theorem 2.6 (Corollary 3.2). If $g$ is a real valued function such that $g \in HK([a,b])$ and $(f_n)$ is a sequence in $BV([a,b])$ such that $V_{[a,b]} f_n \leq M$ for all $n \in \mathbb{N}$, and $g_n \to g$ pointwise on $[a,b]$, then
\[
\lim_{n \to \infty} \int_a^b fg_n = \int_a^b fg.
\]

It is well known that the Lebesgue integral may be characterized by the fact that the indefinite integral is absolutely continuous. A similar characterization is possible with the Henstock–Kurzweil integral.

Definition 2.7. Let $F : [a,b] \to \mathbb{C}$. We say that $F$ is absolutely continuous in the restricted sense on a set $E \subseteq [a,b]$ ($F \in AC_s(E)$), if for every $\epsilon > 0$ there exists $\eta_\epsilon > 0$ such that if $\{[u_i, v_i]\}_{i=1}^s$ is a collection of nonoverlapping intervals with endpoints in $E$ and such that $\sum_{i=1}^s (v_i - u_i) < \eta_\epsilon$, then
\[
\sum_{i=1}^s \sup \{|F(x) - F(y)| : x, y \in [u_i, v_i]\} < \epsilon.
\]

Moreover, $F$ is said to be generalized absolutely continuous in the restricted sense on $[a,b]$ ($F \in ACG_s([a,b])$), if $F$ is continuous on $[a,b]$ and there is a countable collection $(E_n)_{n=1}^\infty$ of sets in $[a,b]$ with $[a,b] = \bigcup_{n=1}^\infty E_n$ and $F \in AC_s(E_n)$ for all $n \in \mathbb{N}$.

Theorem 2.8 (Fundamental Theorem of Calculus, [2]). Let $f, F : [a,b] \to \mathbb{C}$ be functions and $c \in [a,b]$.

(1) $f \in HK([a,b])$ and $F(x) = \int_c^x f$ for all $x \in [a,b]$ if and only if $F \in ACG_s([a,b])$, $F(c) = 0$, and $F' = f$ almost everywhere on $[a,b]$.

If $f \in HK([a,b])$ and $f$ is continuous at $x \in [a,b]$ then $\frac{d}{dx} \int_c^x f = f(x)$.

(2) $F \in ACG_s([a,b])$ if and only if $F'$ exists almost everywhere on $[a,b]$, and $\int_c^x F' = F(x) - F(c)$ for all $x \in [a,b]$.

3. The existence and uniqueness theorem

In this section $q$ is a real valued function such that $q \in HK([a,b])$ and $L$ is the Schrödinger operator defined as
\[
Ly = -y'' + qy.
\]
Lemma 3.1. Let \( c \in [a, b], \lambda, \alpha, \beta \in \mathbb{C} \) and let \( A = \left( \begin{array}{cc} 0 & 1 \\ q - \lambda & 0 \end{array} \right) \). If \( g \in HK([a, b]) \), then there exists a unique solution \( f, f' \in ACG_*([a, b]) \) of the initial value problem

\[
(L - \lambda)y = g \quad \text{a.e.}
\]

\[
y(c) = \alpha
\]

\[
y'(c) = \beta
\]

if and only if there exists a unique solution \( u \in C([a, b], \mathbb{C}^2) \) of the equation

\[
u(x) = \int_c^x A(s)u(s) \, ds + w,
\]

where \( w : [a, b] \to \mathbb{C}^2 \) is defined as \( w(x) = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) - \int_c^x \left( \begin{array}{c} 0 \\ g(s) \end{array} \right) \, ds \).

Proof. Let \( f, f' \in ACG_*([a, b]) \), be a solution of the initial value problem (3.1). Since \( f' \in C([a, b]) \), it follows that \( f(x) = \int_a^x f'(s) \, ds + f(a) \) and hence from Theorem 2.3 \( f \) is of bounded variation on \([a, b]\). This implies, by Theorem 2.4, that \( qf \in HK([a, b]) \). We set \( u = \left( \begin{array}{c} f' \\ f'' \end{array} \right) \), then \( u \in C([a, b], \mathbb{C}^2) \) and \( Au = \left( \begin{array}{c} qf - \lambda f \\ f'' \end{array} \right) \in HK([a, b]) \). Therefore for all \( x \in [a, b] \),

\[
\int_c^x A(s)u(s) \, ds + w(x) = \left( \begin{array}{c} \int_c^x f'(s) \, ds + \alpha \\ \int_c^x [qf(s) - \lambda f(s)] \, ds + \beta - \int_c^x g(s) \, ds \end{array} \right)
\]

\[
= \left( \begin{array}{c} \int_c^x f'(s) \, ds + f(c) \\ \int_c^x [f''(s) + g(s)] \, ds + f'(c) - \int_c^x g(s) \, ds \end{array} \right)
\]

\[
= \left( \begin{array}{c} f(c) + \int_c^x f'(s) \, ds \\ \int_c^x f''(s) \, ds + f'(c) \end{array} \right)
\]

\[
= \left( \begin{array}{c} f(x) \\ f'(x) \end{array} \right),
\]

where the last equality is due to Theorem 2.8 (2) because \( f, f' \in ACG_*([a, b]) \). Therefore \( u \) satisfies the equation (3.2).

Conversely, let \( u = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) \in C([a, b], \mathbb{C}^2) \) be a solution of equation (3.2); then for every \( x \in [a, b], \)

\[
u_1(x) = \int_c^x u_2(s) \, ds + \alpha,
\]

\[
u_2(x) = \int_c^x [(q(s) - \lambda)u_1(s) - g(s)] \, ds + \beta.
\]
Therefore \( u'_1 = u_2 \) on \([a, b]\) and, from Theorem 2.8 (1), \( u_2 \) is ACG on \([a, b]\) and \( u'_2 = (q - \lambda)u_1 - g \) almost everywhere on \([a, b]\). This implies that \( u_1, u'_1 \in ACG_2([a, b]) \) and \((L - \lambda)u_1 = g \) almost everywhere on \([a, b]\).

The uniqueness in any of the situations follows from the above. \( \square \)

**Theorem 3.2.** Let \( c \in [a, b], \lambda, \alpha, \beta \in \mathbb{C} \). If \( g \in HK([a, b]) \), then there exists a unique solution \( f \), \( f' \in ACG_2([a, b]) \) of the initial-value problem

\[
(L - \lambda)y = g \quad a.e.
\]

\[
y(c) = \alpha
\]

\[
y'(c) = \beta.
\]

**Proof.** Let \((y_n), (z_n)\) be two sequences of functions defined on \([a, b]\) as \( y_0(x) = \alpha, z_0(x) = \beta \), and for all \( n \in \mathbb{N} \),

\[
y_n(x) = \alpha + \int_c^x z_{n-1}(s) \, ds \quad \text{and} \quad z_n(x) = \beta - \int_c^x g(s) \, ds + \int_c^x (q(s) - \lambda) y_{n-1}(s) \, ds.
\]

Clearly \( y_0, z_0, y_1 \) and \( z_1 \) are well defined. Suppose that \( y_2, z_2, \ldots, y_n, z_n \) exist. Since \( z_{n-1} \in L([a, b]) \), from Theorem 2.3 \( y_n \) is of bounded variation on \([a, b]\) and hence, by Theorem 2.4 \((q - \lambda) y_n \in HK([a, b])\). Then \( z_{n+1} \) exists. Also observe that clearly \( y_{n+1} \) exists. Therefore \((y_n)\) and \((z_n)\) are well defined.

We claim that for every \( x \in [a, b] \) with \( x > c \),

\[
\int_c^x |z_n(s) - z_{n-1}(s)| \, ds
\]

\[
\leq \begin{cases} 
\|q - \lambda\|_n \alpha - g\|_{[a,b]} \frac{(x - c)^{k+1}}{(k + 1)!} \|q - \lambda\|_k, & \text{if } n = 2k + 1; \\
|\beta| \frac{(x - c)^{k+1}}{(k + 1)!} \|q - \lambda\|_k, & \text{if } n = 2k.
\end{cases}
\]

(3.3)

We prove this only for \( n \) odd, by induction on \( k \). For \( k = 0 \) we have that

\[
\int_c^x |z_1(s) - z_0(s)| \, ds = \int_c^x \left| \int_c^s [(q(t) - \lambda) \alpha - g(t)] \, dt \right| \, ds
\]

\[
\leq \int_c^x \|q - \lambda\|_n \alpha - g\|_{[a,b]} \, ds
\]

\[
= \|q - \lambda\|_n \alpha - g\|_{[a,b]}(x - c)
\]

for all \( x > c \). Now, suppose that for every \( x > c \),

\[
\int_c^x |z_{2k+1}(s) - z_{2k}(s)| \, ds \leq \|q - \lambda\|_n \alpha - g\|_{[a,b]} \frac{(x - c)^{k+1}}{(k + 1)!} \|q - \lambda\|_k.
\]

Let \( x > c \), observe that

\[
\int_c^x |z_{2(k+1)+1}(s) - z_{2(k+1)}(s)| \, ds = \int_c^x \left| \int_c^s (q(t) - \lambda) [y_{2(k+1)}(t) - y_{2k+1}(t)] \, dt \right| \, ds.
\]

Now, since \( y_{2(k+1)} - y_{2k+1} \) is of bounded variation on \([c, s]\) and

\[
V_{[c, s]}(y_{2(k+1)} - y_{2k+1}) \leq \int_c^s |z_{2k+1}(t) - z_{2k}(t)| \, dt,
\]

it follows by Theorem 2.5 that

\[
\int_c^x \left| \int_c^x (g(t) - \lambda)(y_{2(k+1)}(t) - y_{2k+1}(t)) \, dt \right| \, ds
\]

\[
\leq \int_c^x \left| \inf_{t \in [c, s]} |y_{2(k+1)}(t) - y_{2k+1}(t)| + V_{[c, s]}(y_{2(k+1)} - y_{2k+1}) \right| q - \lambda \|_{[a, b]} \, ds
\]

\[
\leq \int_c^x [\|y_{2(k+1)}(c) - y_{2k+1}(c)\| + V_{[c, s]}(y_{2(k+1)} - y_{2k+1})] \| q - \lambda \|_{[a, b]} \, ds
\]

\[
= \int_c^x V_{[c, s]}(y_{2(k+1)} - y_{2k+1}) \, ds \| q - \lambda \|_{[a, b]}
\]

\[
\leq \int_c^x \left| \int_c^s |z_{2k+1}(t) - z_{2k}(t)| \, dt \right| \, ds \| q - \lambda \|_{[a, b]}
\]

\[
\leq \int_c^x \| (q - \lambda)\alpha - g\|_{[a, b]} \frac{(s - c)^{k+1}}{(k + 1)!} \| q - \lambda \|_{[a, b]} \, ds \| q - \lambda \|_{[a, b]}
\]

\[
\leq \| (q - \lambda)\alpha - g\|_{[a, b]} \frac{(x - c)^{k+2}}{(k + 2)!} \| q - \lambda \|_{[a, b]}^{k+1}.
\]

Thus (3.3) follows by induction. Similarly we can prove that for every \( x \in [a, b] \) with \( x < c \),

\[
\int_c^x |z_n(s) - z_{n-1}(s)| \, ds
\]

\[
\leq \begin{cases} 
\| (q - \lambda)\alpha - g\|_{[a, b]} \frac{(c - x)^{k+1}}{(k + 1)!} \| q - \lambda \|_{[a, b]}^k, & \text{if } n = 2k + 1; \\
|\beta| \frac{(c - x)^{k+1}}{(k + 1)!} \| q - \lambda \|_{[a, b]}^k, & \text{if } n = 2k.
\end{cases}
\]

(3.4)

For each \( k \in \mathbb{N} \) define \( l_k = y_{2k+1} - y_{2k} \). Take \( k \in \mathbb{N} \) and \( x \in [a, b] \). If \( x < c \), then

\[
|l_k(x)| = |y_{2k+1}(x) - y_{2k}(x)|
\]

\[
= \left| \int_c^x [z_{2k}(s) - z_{2k-1}(s)] \, ds \right|
\]

\[
\leq \int_c^x |z_{2k}(s) - z_{2k-1}(s)| \, ds
\]

\[
\leq |\beta| \frac{(c - x)^{k+1}}{(k + 1)!} \| q - \lambda \|_{[a, b]}^k.
\]

Therefore

\[
|l_k(x)| \leq \frac{|\beta|}{\| q - \lambda \|_{[a, b]}^k} \frac{(b - a)^{k+1}}{(k + 1)!} \| q - \lambda \|_{[a, b]}^{k+1}.
\]

This inequality also holds when \( x \geq c \).
Now, since
\[ \sum_{k=1}^{\infty} \frac{|\beta|}{\|q - \lambda\|_{[a,b]}} \frac{(b-a)^{k+1} \|q - \lambda\|^{k+1}_{[a,b]}}{(k+1)!} \leq \frac{|\beta|}{\|q - \lambda\|_{[a,b]}} e^{(b-a)\|q - \lambda\|_{[a,b]}}, \]
it follows that \( \sum_{k=1}^{\infty} l_k \) converges uniformly on \([a,b]\). Again using the equations (3.3) and (3.4), we obtain that if \( h_k = y_{2k} - y_{2k-1} \) then \( \sum_{k=1}^{\infty} h_k \) converges uniformly on \([a,b]\). Therefore \( y_0 + \sum_{n=1}^{\infty} [y_n - y_{n-1}] = y_0 + [y_1 - y_0] + \sum_{k=1}^{\infty} l_k + \sum_{k=1}^{\infty} h_k \) converges uniformly on \([a,b]\). Thus its sequences of partial sums \( s_n \) converges uniformly to a limit function \( y \) on \([a,b]\).

In other words, the sequence \((y_n)\) converges uniformly to \( y \) on \([a,b]\).

On the other hand, from the inequalities
\[ |z_{2k+1}(x) - z_{2k}(x)| \leq \|q - \lambda\| \alpha - g\|_{[a,b]} \frac{(b-a)^k \|q - \lambda\|_k}{k!} \]
and
\[ |z_{2k}(x) - z_{2k-1}(x)| \leq |\beta| \frac{(b-a)^k \|q - \lambda\|_k}{k!}, \]
it follows that \( z_0 + \sum_{k=0}^{\infty} [z_n - z_{n-1}] \) converges uniformly on \([a,b]\). If \( z \) denotes its sum then \( z_n \) converges uniformly to \( z \) on \([a,b]\).

Consequently, for each \( x \in [a,b] \),
\[
y(x) = \lim_{n \to \infty} y_n(x) = \lim_{n \to \infty} \left[ \alpha + \int_c^x z_{n-1}(s) \, ds \right] \\
= \alpha + \lim_{n \to \infty} \int_c^x z_{n-1}(s) \, ds \\
= \alpha + \int_c^x \lim_{n \to \infty} z_{n-1}(s) \, ds \\
= \alpha + \int_c^x z(s) \, ds.
\]

It is not possible to apply the above idea to the sequence \((y_n)\) because \((q - \lambda)y_{n-1}\) might not converge uniformly to \((q - \lambda)y\). However, we can use Theorem 2.6 in order to have a similar result to the above. Let \( x \in [a,b] \) with \( x > c \). We first show that \((y_n)\) is of uniformly bounded variation on \([c,x]\).
For every \( n \in \mathbb{N} \),

\[
V_{[c,x]}y_n = V_{[c,x]} \left[ y_1 + \sum_{k=1}^{n-1} [y_{k+1} - y_k] \right]
\]

\[
\leq V_{[c,x]}y_1 + \sum_{k=1}^{n-1} V_{[c,x]}[y_{k+1} - y_k]
\]

\[
\leq V_{[c,x]}y_1 + \sum_{k=1}^{n-1} \int_c^x |z_k(s) - z_{k-1}(s)| \, ds
\]

\[
\leq V_{[c,x]}y_1 + \sum_{k=1}^{\infty} \int_c^x |z_k(s) - z_{k-1}(s)| \, ds
\]

\[
= V_{[c,x]}y_1 + \int_c^x |z_1(s) - z_0(s)| \, ds
\]

\[
+ \sum_{k=1}^{\infty} \int_c^x |z_{2k+1}(s) - z_{2k}(s)| \, ds + \sum_{k=1}^{\infty} \int_c^x |z_{2k}(s) - z_{2k-1}(s)| \, ds
\]

\[
\leq \left[ \frac{\|q - \lambda\alpha - q\|_{[a,b]}}{\|q - \lambda\|_{[a,b]}} + \frac{|\beta|}{\|q - \lambda\|_{[a,b]}} \right] \sum_{k=0}^{\infty} \frac{(b - a)^{k+1}\|q - \lambda\|_{[a,b]}}{(k + 1)!}
\]

\[
= \frac{\|(q - \lambda)\alpha - q\|_{[a,b]} + |\beta|}{\|q - \lambda\|_{[a,b]}} e^{(b-a)\|q - \lambda\|}.
\]

Therefore, by Theorem 2.6 it follows that

\[
\lim_{n \to \infty} \int_c^x (q(s) - \lambda)y_{n-1}(s) = \int_c^x (q(s) - \lambda)y(s) \, ds.
\]

Thus

\[
z(x) = \lim_{n \to \infty} z_n(x) = \lim_{n \to \infty} \left[ \beta - \int_c^x g(s) \, ds + \int_c^x (q(s) - \lambda)y_{n-1}(s) \, ds \right]
\]

\[
= \beta - \int_c^x g(s) \, ds + \lim_{n \to \infty} \int_c^x (q(s) - \lambda)y_{n-1}(s) \, ds
\]

\[
= \beta - \int_c^x g(s) \, ds + \int_c^x (q(s) - \lambda)y(s) \, ds.
\]

A similar reasoning shows that the equality (3.5) holds when \( x < c \).

Consequently, \( u = \begin{pmatrix} y \\ z \end{pmatrix} \) is a solution of equation (3.2). We shall now prove that this solution is unique. Assume that \( \begin{pmatrix} \overline{y} \\ \overline{z} \end{pmatrix} \) is another solution of equation (3.2). Therefore

\[
\overline{y}(x) = \alpha + \int_c^x \overline{z}(s) \, ds
\]
and
\[ \overline{z}(x) = \beta - \int_c^x g(s) \, ds + \int_c^x [q(s) - \lambda] \overline{y}(s) \, ds \]
for all \( x \in [a, b] \). Observe that for every \( k \in \mathbb{N} \),
\[ \int_c^x |z_{2k}(s) - \overline{z}(s)| \, ds \leq \| (q - \lambda) \overline{y} - g \|_{[a,b]} \frac{(x-c)^{k+1}}{(k+1)!} \|q - \lambda\|^k, \text{ if } c < x, \]
and
\[ \int_x^c |z_{2k}(s) - \overline{z}(s)| \, ds \leq \| (q - \lambda) \overline{y} - g \|_{[a,b]} \frac{(c-x)^{k+1}}{(k+1)!} \|q - \lambda\|^k, \text{ if } x < c. \]
From this it follows that
\[ |\overline{y}_{2k+1}(x) - \overline{y}(x)| \leq \frac{\|(q - \lambda) \overline{y} - g\|_{[a,b]} \|(b-a)^{k+1} \|q - \lambda\|^{k+1}}{(k+1)!} \]
and
\[ |z_{2(k+1)}(x) - \overline{z}(x)| \leq \frac{\|(q - \lambda) \overline{y} - g\|_{[a,b]} \|(b-a)^{k+1} \|q - \lambda\|^{k+1}}{(k+1)!} \]
for all \( x \in [a, b] \). Thus \( y_{2k+1}(x) \to \overline{y}(x) \) and \( z_{2(k+1)}(x) \to \overline{z}(x) \), but \( y_{2k+1}(x) \to y(x) \) and \( z_{2(k+1)}(x) \to z(x) \). Therefore \( \overline{y}(x) = y(x) \) and \( \overline{z}(x) = z(x) \). \( \square \)

4. Example

In this section, we give an example for the application of Theorem 3.2.

**Example 4.1.** Let \( q \) be a function defined on \([0, 1]\) as
\[ q(x) = \begin{cases} \frac{2\pi}{x} \sin \left( \frac{\pi}{x^2} \right), & \text{if } x \in (0, 1]; \\ 0, & \text{if } x = 0, \end{cases} \]
and let \( g : [0, 1] \to \mathbb{R} \) be defined by
\[ g(x) = \begin{cases} (-1)^{k+1} \frac{2^k}{k}, & \text{for } x \in [c_{k-1}, c_k), \; k \in \mathbb{N}; \\ 0, & \text{for } x = 1, \end{cases} \]
where \( c_k = 1 - \frac{1}{2^k}, \; k = 0, 1, 2, \ldots \) Then \( q \) and \( g \) are unbounded HK-integrable functions on \([0, 1]\). Therefore, by Theorem 3.2, the initial value problem
\[ -y'' + q(x)y - 2y = g(x) \quad \text{a.e.} \]
\[ y\left(\frac{1}{2}\right) = 0 \]
\[ y'\left(\frac{1}{2}\right) = 1 \]
has a solution.

The functions \( q \) and \( g \) are not Lebesgue integrable on \([0, 1]\). Hence, this example is not covered by any result using the Lebesgue integral. Thus, Theorem 3.2 is more general than the classical result of existence and uniqueness given at the introduction.
REFERENCES


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