REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 58, No. 2, 2017, Pages 331–342
Published online: June 22, 2017

AN EXTENSION OF BEST L^2 LOCAL APPROXIMATION

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Abstract. We introduce two classes of functions, one containing the class of L^2 differentiable functions, and another containing the class of L^2 lateral differentiable functions. For functions in these new classes we prove existence of best local approximation at several points. Moreover, we get results about the asymptotic behavior of the derivatives of the net of best approximations, which are unknown even for L^2 differentiable functions.

1. Introduction

Let $x_1, x_2, \ldots, x_k$ be k points in $\mathbb{R}$ and let $a > 0$ be such that the intervals $I_{a,i} := [x_i - a, x_i + a]$, $1 \leq i \leq k$, are pairwise disjoint. We denote by $\mathcal{L}$ the space of equivalence classes of Lebesgue measurable real functions defined on $I_a := \bigcup_{i=1}^{k} I_{a,i}$. For each Lebesgue measurable set $A \subset I_a$, with $|A| > 0$, we consider the semi-norms on $\mathcal{L}$,

$$\|f\|_A := \left(|A|^{-1} \int_A |f(x)|^2 \, dx \right)^{1/2},$$

where $|A|$ denotes the measure of the set A.

If $0 < \epsilon \leq a$, we use the notations $I_{-\epsilon,i} = [x_i - \epsilon, x_i]$, $I_{+\epsilon,i} = [x_i, x_i + \epsilon]$, $I_{\epsilon,i} = [x_i - \epsilon, x_i + \epsilon]$, and we write $\|f\|_{\pm\epsilon,i} = \|f\|_{I_{\pm\epsilon,i}}$, $\|f\|_{\epsilon,i} = \|f\|_{I_{\epsilon,i}}$. So, $\|f\|_\epsilon^2 := \|f\|_{I_\epsilon}^2 = k^{-1} \sum_{i=1}^{k} \|f\|_{\epsilon,i}^2$.

In all this work, s is an arbitrary non negative integer. We denote by Π^s the linear space of polynomials of degree at most s.

Henceforward, we fix $n \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N}$ such that $n + 1 = kq$.

If $f \in L^2(I_\epsilon)$, it is well known that there exists a unique best $\|\cdot\|_\epsilon$-approximation of f from Π^n, say P_ϵ, i.e., $P_\epsilon \in \Pi^n$ satisfies

$$\|f - P_\epsilon\|_\epsilon \leq \|f - P\|_\epsilon, \quad P \in \Pi^n.$$

2010 *Mathematics Subject Classification.* Primary 41A50; Secondary 41A10.
Key words and phrases. Best local approximation, L^2 differentiability, algebraic polynomials.
This work was supported by Universidad Nacional de Río Cuarto and CONICET.

Moreover, it is characterized by

$$\int_{I_\epsilon} (f - P_\epsilon)(x)P(x)\,dx = 0, \quad P \in \Pi^n. \tag{1.1}$$

If $\lim_{\epsilon\to 0} P_\epsilon$ exists, say P_0, it is called the *best local approximation of f on $\{x_i : 1 \leq i \leq k\}$, from Π^n.*

We recall that a function $f \in L^2(I_{a,i})$ is L^2 *left (right) differentiable of order s at x_i* if there exists $L_i(R_i) \in \Pi^s$ such that

$$\|f - L_i\|_{-\epsilon,i} = o(\epsilon^s) \quad (\|f - R_i\|_{+\epsilon,i} = o(\epsilon^s)). \tag{1.2}$$

The class of these functions is denoted by $t^2_{-s}(x_i)$ $(t^2_{+s}(x_i))$. If $L_i = R_i$, we say that f is L^2 *differentiable of order s at x_i* and we write $f \in t^2_s(x_i)$. This concept was introduced by Calderón and Zygmund in [1]. It is well known that there exists at most one polynomial satisfying (1.2) (see [6]).

The problem of best local approximation was formally introduced and studied in a paper by Chui, Shisha and Smith [2]. However, the initiation of this could be dated back to results of J. L. Walsh [7], who proved that the Taylor polynomial of an analytic function f over a domain is the limit of the net of best polynomial approximations of a given degree, by shrinking the domain to a single point. In [6] this problem was considered when f is L^2 differentiable of order $q-1$ at x_i, $1 \leq i \leq k$. In [4] the authors proved the existence of the best local approximation under weaker conditions, more precisely they assumed existence of L^2 lateral derivatives of order $q-1$ and L^2 differentiability of order $q-2$ at each point.

On the other hand, for $k = 1$, the authors in [3] considered a function f differentiable at 0 in the ordinary sense of order s, for some $s \leq n$, and they proved that $\lim_{\epsilon\to 0}(f - P_\epsilon)^{(j)}(0) = 0$, $0 \leq j \leq s$. In particular, if P is a cluster point of the net $\{P_\epsilon\}$, as $\epsilon \to 0$, then P interpolates the derivatives of f up to order s at 0. In [5] the authors obtained this property of interpolation for functions that satisify a more general condition than L^2 differentiability, called C^2 condition.

The main purpose of this paper is to extend the results established in [3] and [4], assuming the C^2 condition and C^2 lateral condition when we consider approximation on k points in $\mathbb{R}$ from Π^n, for the case $n+1$ multiple of k.

Definition 1.1. A function $f \in L^2(I_{a,i})$ satisfies the C^2 condition of order s at x_i, if there exists $Q_i \in \Pi^s$ such that

$$\int_{I_{\epsilon,i}} (f - Q_i)(x)(x - x_i)^j\,dx = o(\epsilon^{s+j+1}), \quad 0 \leq j \leq s, \text{ as } \epsilon \to 0. \tag{1.3}$$

Analogously, f satisfies the left (right) C^2 condition of order s at x_i, if there exists $L_i(R_i) \in \Pi^s$ satisfying (1.3) with $I_{-\epsilon,i}$ $(I_{+\epsilon,i})$ instead of $I_{\epsilon,i}$.

We say that $f \in c^2_s(x_i)$ if it satisfies (1.3). We denote with $c^2_{-s}(x_i)$ $(c^2_{+s}(x_i))$ the class of functions which satisfy the left (right) C^2 condition of order s at x_i. We also denote $c^2_{\pm s}(x_i) := c^2_{+s}(x_i) \cap c^2_{-s}(x_i)$. It is easy to see that $t^2_s(x_i) \subseteq t^2_{-s}(x_i)(t^2_{+s}(x_i))$; however $c^2_s(x_i) \nsubseteq c^2_{-s}(x_i)(c^2_{+s}(x_i))$ as Example 2.8 shows.

In [5] it was proved that if $f \in c_s^2(x_i)$, there exists a unique polynomial $Q_i \in \Pi^s$ satisfying (1.3). Moreover, $t_s^2(x_i) \subsetneq c_s^2(x_i)$. The polynomial in Π^s that satisfies $\|f-Q_i\|_{\epsilon,i} = o(\epsilon^s)$ also satisfies (1.3). Analogously $t_{-s}^2(x_i) \subsetneq c_{-s}^2(x_i)$ and $t_{+s}^2(x_i) \subsetneq c_{+s}^2(x_i)$.

In Section 2, given a function in $c_s^2(x_i)$ (or $c_{\pm s}^2(x_i)), 1 \le i \le k$, under suitable conditions, we study the asymptotic behavior of the derivatives of P_ϵ, as $\epsilon \to 0$, and we establish a relation between the derivatives of P_ϵ up to order s with its derivatives of greater order (see Theorem 2.3 and Theorem 2.9). This type of result is unknown, even for L^2 differentiable functions. We also prove that if P is a cluster point of the net $\{P_\epsilon\}$, then P interpolates the derivatives of Q_i (or $\frac{Li+R_i}{2}$) up to order s at x_i. Finally, we get existence of the best local approximation on k points, for two new classes of functions.

2. The main results

Henceforward we will use the following notation. For $Q \in \Pi^n$, we denote by $T_{i,s}(Q)$ its Taylor polynomial at x_i of degree s. We write $T_{i,-1}(Q) = 0$ and we recall that $q \in \mathbb{N}$ satisfies $n+1 = kq$.

To establish the results we need two auxiliary lemmas.

Lemma 2.1. *Let $l \in \mathbb{Z}$. If one of the following cases holds*

a) $q = 1$, $l = -1$, and $f \in L^2(I_a)$,

b) $k = 1$, $q \ge 2$, $0 \le l \le q-2$, and $f \in L^2(I_a)$,

c) $k > 1$, $q \ge 2$, $0 \le l \le q-2$, and $f \in t_l^2(x_i)$, $1 \le i \le k$,

then

$$\int_{I_{\epsilon,i}} (f - P_\epsilon)(x)(x-x_i)^u \, dx = o(\epsilon^{2l+3}), \quad 0 \le u \le l+1,\ 1 \le i \le k.$$

Proof. We begin proving c). For each $0 \le u \le l+1$ and $1 \le i \le k$, we consider the polynomial $R_{u,i} \in \Pi^{(l+2)k-1}$ such that $R_{u,i}^{(j)}(x_z) = \delta_{(u,i),(j,z)}$, $0 \le j \le l+1$, $1 \le z \le k$, where δ is the Kronecker delta function. From (1.1) we have

$$\begin{aligned}
\int_{I_{\epsilon,i}} (f-P_\epsilon)(x) R_{u,i}(x)\,dx &= -\sum_{z=1,z\neq i}^{k} \int_{I_{\epsilon,z}} (f-P_\epsilon)(x) R_{u,i}(x)\,dx \\
&= -\sum_{z=1,z\neq i}^{k} \int_{I_{\epsilon,z}} (f-P_\epsilon)(x) \sum_{r=0}^{(l+2)k-1} \frac{R_{u,i}^{(r)}}{r!}(x_z)(x-x_z)^r\,dx \\
&= -\sum_{z=1,z\neq i}^{k} \int_{I_{\epsilon,z}} (f-P_\epsilon)(x) \sum_{r=l+2}^{(l+2)k-1} \frac{R_{u,i}^{(r)}}{r!}(x_z)(x-x_z)^r\,dx \\
&= -\sum_{z=1,z\neq i}^{k} \sum_{r=l+2}^{(l+2)k-1} \frac{R_{u,i}^{(r)}}{r!}(x_z) \int_{I_{\epsilon,z}} (f-P_\epsilon)(x)(x-x_z)^r\,dx.
\end{aligned} \tag{2.1}$$

Since $f \in t_l^2(x_i)$ there exists $Q_i \in \Pi^l$ such that $\|f-Q_i\|_{\epsilon,i} = o(\epsilon^l)$, $1 \le i \le k$. Let $S \in \Pi^{(l+2)k-1}$ defined by $S^{(j)}(x_i) = Q_i^{(j)}(x_i)$, $0 \le j \le l+1$, $1 \le i \le k$. Now,

$\|f-S\|_{\epsilon,i} \leq \|f-Q_i\|_{\epsilon,i} + \|Q_i - S\|_{\epsilon,i} = o(\epsilon^l)$, so $\|f-S\|_\epsilon^2 = k^{-1}\sum_{i=1}^{k}\|f-S\|_{\epsilon,i}^2 = o(\epsilon^{2l})$, i.e., $\|f-S\|_\epsilon = o(\epsilon^l)$. Using Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\int_{I_{\epsilon,z}} (f-P_\epsilon)(x)(x-x_z)^r\,dx\right| \\
&\qquad \leq \int_{I_\epsilon} |(f-P_\epsilon)(x)(x-x_z)^r|\,dx \leq \epsilon^{r+1}\|f-P_\epsilon\|_\epsilon \\
&\qquad \leq \epsilon^{r+1}\|f-S\|_\epsilon = o(\epsilon^{2l+3}), \quad l+2 \leq r \leq (l+2)k-1,\ 1 \leq z \leq k.
\end{aligned} \tag{2.2}$$

From (2.1) and (2.2) we get

$$\int_{I_{\epsilon,i}} (f-P_\epsilon)(x)R_{u,i}(x)\,dx = o(\epsilon^{2l+3}). \tag{2.3}$$

On the other hand,

$$\begin{aligned}
&\int_{I_{\epsilon,i}} (f-P_\epsilon)(x)R_{u,i}(x)\,dx \\
&\qquad = \int_{I_{\epsilon,i}} (f-P_\epsilon)(x)\sum_{r=0}^{(l+2)k-1}\frac{R_{u,i}^{(r)}}{r!}(x_i)(x-x_i)^r\,dx \\
&\qquad = \int_{I_{\epsilon,i}} \frac{1}{u!}(f-P_\epsilon)(x)(x-x_i)^u\,dx \\
&\qquad\quad + \int_{I_{\epsilon,i}} (f-P_\epsilon)(x)\sum_{r=l+2}^{(l+2)k-1}\frac{R_{u,i}^{(r)}}{r!}(x_i)(x-x_i)^r\,dx \\
&\qquad = \int_{I_{\epsilon,i}} \frac{1}{u!}(f-P_\epsilon)(x)(x-x_i)^u\,dx + o(\epsilon^{2l+3}),
\end{aligned} \tag{2.4}$$

where the last equality follows from (2.2). Finally, the lemma is a consequence of (2.3) and (2.4).

a) If $k=1$ it is a direct consequence of (1.1). Suppose $k>1$ and consider $S=0$. Then $\epsilon^{r+1}\|f\|_\epsilon \leq (\frac{a}{\epsilon})^{\frac{1}{2}}\epsilon^{r+1}\|f\|_a = o(\epsilon)$, $1 \leq r \leq k-1$, so (2.2) is true. Now, the proof follows analogously to that of c).

b) This is a direct consequence of (1.1). □

Lemma 2.2. *Let $r_w \in \mathbb{N}\cup\{0\}$, $1 \leq w \leq m$, be such that $r_1 < r_2 < \cdots < r_m$. Given $1 \leq i \leq k$, let $\{U_\epsilon\}$ be a net of polynomials in the linear space generated by $\{(x-x_i)^{r_w} : 1 \leq w \leq m\}$. If*

$$\int_{I_{\epsilon,i}} U_\epsilon(x)(x-x_i)^{r_w}\,dx = o(\epsilon^{r_m+r_w+1}), \quad 1 \leq w \leq m, \tag{2.5}$$

then $U_\epsilon \to 0$ as $\epsilon \to 0$.

Proof. We write $U_\epsilon(x) = \sum_{s=1}^{m} a_{r_w}(\epsilon)(x-x_i)^{r_w}$. Multiplying both sides of (2.5) by $a_{r_w}(\epsilon)$, and summing over w from 1 to m, we get

$$\int_{I_{\epsilon,i}} U_\epsilon(x)^2\,dx = o(\epsilon^{r_m+1})\sum_{w=1}^{m} a_{r_w}(\epsilon)\epsilon^{r_w} = o(\epsilon^{r_m+1})\sum_{w=1}^{m} |a_{r_w}(\epsilon)|\epsilon^{r_w}. \tag{2.6}$$

Let Δ be the space of polynomials generated by $\{x^{r_1},\ldots,x^{r_m}\}$. We consider the norm ρ on Δ defined by

$$\rho(S) = \sum_{w=1}^{m} |d_{r_w}|, \quad S(x) = \sum_{w=1}^{m} d_{r_w} x^{r_w},$$

and $\|\cdot\|_{[-1,1]}$ the usual norm in $L^2([-1,1])$. Let $S^{[\epsilon]}(x) := S(x_i+\epsilon x)$. By the equivalence of the above norms on Δ, there exists $K>0$ such that

$$\sum_{w=1}^{m} |a_{r_w}(\epsilon)|\epsilon^{r_w} = \rho(U_\epsilon^{[\epsilon]}) \le K\|U_\epsilon^{[\epsilon]}\|_{[-1,1]}. \tag{2.7}$$

Therefore by (2.6) we have that $\|U_\epsilon^{[\epsilon]}\|^2_{[-1,1]} = o(\epsilon^{r_m})\|U_\epsilon^{[\epsilon]}\|_{[-1,1]}$. So, $\|U_\epsilon^{[\epsilon]}\|_{[-1,1]} = o(\epsilon^{r_m})$. In consequence, from (2.7) we obtain $a_{r_w}(\epsilon) = o(\epsilon^{r_m - r_w})$, $1\le w\le m$, i.e., $U_\epsilon \to 0$ as $\epsilon\to 0$. □

If $q\ge 2$ and $l\in\mathbb{N}\cup\{0\}$, $l\le q-2$, we consider

$$b_{\epsilon,i,l} = \begin{cases} 1+\max\{|P_\epsilon^{(v)}(x_i)| : l+2\le v\le n\} & \text{if } l<q-2,\\ 1 & \text{if } l=q-2.\end{cases} \tag{2.8}$$

For $q=1$ and $l=-1$, we put $b_{\epsilon,i,l}=1$.

The next theorem is one of our main results.

Theorem 2.3. *Let $l\in\mathbb{Z}$. If one of the following cases holds:*

a) $q=1$, $l=-1$, and $f\in c_0^2(x_i)$, $1\le i\le k$,
b) $k=1$, $q\ge 2$, $0\le l\le q-2$, and $f\in c_{l+1}^2(x_1)$,
c) $q\ge 2$, $0\le l\le q-2$, $k>1$, and $f\in c_{l+1}^2(x_i)\cap t_l^2(x_i)$, $1\le i\le k$,

then

$$T_{i,l+1}(P_\epsilon) - Q_i = o(1)b_{\epsilon,i,l}, \quad 1\le i\le k, \tag{2.9}$$

uniformly on compact sets, where $Q_i\in\Pi^{l+1}$ was introduced in Definition 1.1. In particular, if P is a cluster point of the net $\{P_\epsilon\}$, as $\epsilon\to 0$, then $T_{i,l+1}(P)=Q_i$, $1\le i\le k$.

Proof. Under the hypotheses a), b) or c), we have $f\in c_{l+1}^2(x_i)$, $1\le i\le k$. So

$$\int_{I_{\epsilon,i}} (f-Q_i)(x)(x-x_i)^j\,dx = o(\epsilon^{l+j+2}), \quad 0\le j\le l+1. \tag{2.10}$$

By Lemma 2.1, we get

$$\int_{I_{\epsilon,i}} (f-P_\epsilon)(x)(x-x_i)^j\,dx = o(\epsilon^{2l+3}), \quad 0\le j\le l+1. \tag{2.11}$$

From (2.10) and (2.11) we obtain

$$\int_{I_{\epsilon,i}} (P_\epsilon - Q_i)(x)(x-x_i)^j\, dx = o(\epsilon^{l+j+2}), \quad 0 \le j \le l+1. \tag{2.12}$$

It immediately follows from the definition of $b_{\epsilon,i,l}$ that the set

$$A_{l,i,\delta} := \{b_{\epsilon,i,l}^{-1} P_\epsilon^{(v)}(x_i) : l+2 \le v \le n,\ 0 < \epsilon < \delta\},$$

is bounded for $0 \le l < q-2$ and $1 \le i \le k$. If $k=1$, $A_{q-2,i,\delta}$ is empty and therefore bounded by convention. We will see next that $A_{q-2,i,\delta}$ is also bounded for $k>1$ and δ sufficiently small. Indeed, assume that the net $\{P_\epsilon\}$ is not uniformly bounded as $\epsilon \to 0$ and let $H \in \Pi^n$ be the unique polynomial satisfying $H^{(v)}(x_u) = Q_u^{(v)}(x_u)$, $0 \le v \le q-1$, $1 \le u \le k$. Then there exists a sequence $\epsilon_m \to 0$ such that $\|P_{\epsilon_m} - H\|_{\infty,I_a} \to \infty$. From (2.12) we get

$$\int_{I_{\epsilon,i}} (P_\epsilon - H)(x)(x-x_i)^j\, dx = o(\epsilon^{q+j}), \quad 0 \le j \le q-1. \tag{2.13}$$

Let $D_{\epsilon_m} := \frac{P_{\epsilon_m} - H}{\|P_{\epsilon_m} - H\|_{\infty,I_a}}$. Since $\{D_{\epsilon_m}\}$ is uniformly bounded, (2.13) implies

$$\begin{aligned}\int_{I_{\epsilon_m,i}} T_{i,q-1}(D_{\epsilon_m})(x)(x-x_i)^j\, dx &= o({\epsilon_m}^{j+q}) - \sum_{v=q}^{n} \frac{D_{\epsilon_m}^{(v)}(x_i)}{v!} \int_{I_{\epsilon_m,i}} (x-x_i)^{j+v}\, dx \\ &= o({\epsilon_m}^{j+q}) + O({\epsilon_m}^{j+q+1}) = o({\epsilon_m}^{j+q}), \quad 0 \le j \le q-1.\end{aligned} \tag{2.14}$$

By Lemma 2.2 we have that $T_{i,q-1}(D_{\epsilon_m}) \to 0$, $1 \le i \le k$, as $\epsilon_m \to 0$. Let ρ be the norm on Π^n defined by $\rho(P) := \max\limits_{1\le i\le k} \max\limits_{0 \le v \le q-1} |P^{(v)}(x_i)|$. Then $\rho(D_{\epsilon_m}) \to 0$, a contradiction since $\|D_{\epsilon_m}\|_{\infty,I_a} = 1$. So, the net $\{P_\epsilon\}$ is uniformly bounded.

From (2.12) we obtain

$$\int_{I_{\epsilon,i}} T_{i,l+1}\left(\frac{P_\epsilon - Q_i}{b_{\epsilon,i,l}}\right)(x)(x-x_i)^j\, dx = o(\epsilon^{l+j+2}) - \sum_{v=l+2}^{n} \frac{P_\epsilon^{(v)}(x_i)}{v! b_{\epsilon,i,l}} \int_{I_{\epsilon,i}} (x-x_i)^{j+v}\, dx, \tag{2.15}$$

where the sum is omitted if $k=1$ and $l=q-2$. Since $A_{l,i,\delta}$ is bounded for $0 \le l \le q-2$ and δ sufficiently small, from (2.15) we get

$$\int_{I_{\epsilon,i}} T_{i,l+1}\left(\frac{P_\epsilon - Q_i}{b_{\epsilon,i,l}}\right)(x)(x-x_i)^j\, dx = o(\epsilon^{l+j+2}), \quad 0 \le j \le l+1. \tag{2.16}$$

Now, Lemma 2.2 with $r_w = w-1$, $1 \le w \le l+2$, implies (2.9).

Finally, if $P = \lim\limits_{\epsilon_m \to 0} P_{\epsilon_m}$ then $b_{\epsilon_m,i,l}$ are bounded. So, from (2.9) it follows that $T_{i,l+1}(P) = Q_i$, $1 \le i \le k$. □

Remark 2.4. We observe that the case $k > 1$ in Theorem 2.3 is not a formal extension of the case $k=1$. In fact, we need to apply Lemma 2.1, and to prove that the set $A_{q-2,i,\delta}$ is bounded for δ sufficiently small.

Remark 2.5. Let $P_{+\epsilon}$ $(P_{-\epsilon})$ be the best approximation to f from Π^n on $\bigcup_{i=1}^{k} I_{+\epsilon,i}$ $(\bigcup_{i=1}^{k} I_{-\epsilon,i})$, and let $b_{+\epsilon,i,l}$ $(b_{-\epsilon,i,l})$ be defined as in (2.8) with $P_{+\epsilon}$ $(P_{-\epsilon})$ instead of P_ϵ. We observe that Theorem 2.3 remains valid if we consider $P_{+\epsilon}$ $(P_{-\epsilon})$, and the condition over f is satisfied on the right (left) at each point.

The following corollary is an immediate consequence of Theorem 2.3 with $l = q-2$.

Corollary 2.6. *If $f \in c^2_{q-1}(x_i) \cap t^2_{q-2}(x_i)$, $1 \le i \le k$, then there exists the best local approximation of f on $\{x_i : 1 \le i \le k\}$, from Π^n. It is the unique polynomial $H \in \Pi^n$ satisfying $H^{(j)}(x_i) = Q_i^{(j)}(x_i)$, $0 \le j \le q-1$, $1 \le i \le k$, where $Q_i \in \Pi^{q-1}$ satisfies Definition 1.1. In the case $q = 1$ or $k = 1$ we only assume $f \in c^2_{q-1}(x_i), 1 \le i \le k$.*

Remark 2.7. We observe that $c^2_s(x_i)$ is not, in general, a subset of $c^2_{\pm s}(x_i)$. In fact, next we give an example of a function $f \in c^2_0(0)$ such that $f \notin c^2_{\pm 0}(0)$.

Example 2.8. Let $\{h_m\}$ and $\{\epsilon_m\}$ be two sequences of nonnegative real numbers, such that $h_m \downarrow 0$, $\epsilon_m \downarrow 0$, $\epsilon_{m+1} < h_m < \epsilon_m$, $\frac{\epsilon_m}{h_m} \to \infty$, and $\frac{h_m}{\epsilon_{m+1}} \to \infty$. For example, we can take $\epsilon_m = 2^{-2^{2m}}$ and $h_m = 2^{-2^{2m+1}}$. We define $f : (-\epsilon_1, \epsilon_1) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \cup_{m=1}^{\infty} [h_m, \epsilon_m), \\ 0 & \text{if } x \in \cup_{m=1}^{\infty} [\epsilon_{m+1}, h_m), \end{cases}$$

$f(0) = 0$ and $f(x) = -f(-x)$, $x \in (-\epsilon_1, 0)$. We have

$$\|f - P_{+\epsilon_m}\|^2_{+\epsilon_m} \le \|f - 1\|^2_{+\epsilon_m} = \int_0^{h_m} (f(x) - 1)^2 \frac{dx}{\epsilon_m} \le \frac{h_m}{\epsilon_m} \to 0, \tag{2.17}$$

$$\|f - P_{+h_m}\|^2_{+h_m} \le \|f\|^2_{+h_m} = \frac{\epsilon_{m+1}}{h_m} \int_0^{\epsilon_{m+1}} f(x)^2 \frac{dx}{\epsilon_{m+1}} \le \frac{\epsilon_{m+1}}{h_m} \to 0. \tag{2.18}$$

From (2.17) we get $\|P_{+\epsilon_m} - 1\|_{+\epsilon_m} \le \|P_{+\epsilon_m} - f\|_{+\epsilon_m} + \|f - 1\|_{+\epsilon_m} \to 0$. Now, Pólya's inequality implies that $P_{+\epsilon_m} \to 1$. Analogously, from (2.18) we obtain that $P_{+h_m} \to 0$. On the other hand, we know that if $f \in c^2_{+0}(0)$ then $\lim_{\epsilon \to 0} P_{+\epsilon}$ exists (see Remark 2.5), so $f \notin c^2_{+0}(0)$. Analogously, $f \notin c^2_{-0}(0)$. As f is odd, clearly $f \in c^2_0(0)$.

Let $\mathcal{P}_i(h)$ $(\mathcal{I}_i(h))$ be the even (odd) part of the function h with respect to x_i, i.e.,

$$\mathcal{P}_i(h)(x) := \frac{h(x) + h(2x_i - x)}{2} \quad \left(\mathcal{I}_i(h)(x) := \frac{h(x) - h(2x_i - x)}{2}\right).$$

It is easy to see that, $h = \mathcal{P}_i(h) + \mathcal{I}_i(h)$, $\mathcal{P}_i(h)(x_i + x) = \mathcal{P}_i(h)(x_i - x)$, and $\mathcal{I}_i(h)(x_i + x) = -\mathcal{I}_i(h)(x_i - x)$.

Theorem 2.9. *If one of the following cases holds:*

a) $q = 1$, $l = -1$, *and* $f \in c^2_{\pm 0}(x_i)$, $1 \le i \le k$,

b) $k = 1$, $q \ge 2$, $0 \le l \le q-2$, $f \in c^2_{\pm(l+1)}(x_1)$ *and* $T_{1,l}(L_1) = T_{1,l}(R_1)$,

c) $q \ge 2$, $0 \le l \le q-2$, *and* $f \in c^2_{\pm(l+1)}(x_i) \cap t^2_l(x_i)$, $1 \le i \le k$,

then

$$T_{i,l+1}(P_\epsilon) - \frac{L_i + R_i}{2} = o(1) b_{\epsilon,i,l}, \quad 1 \le i \le k, \tag{2.19}$$

uniformly on compact sets, where $L_i, R_i \in \Pi^{l+1}$ *were introduced in Definition 1.1. In particular, if* P *is a cluster point of the net* $\{P_\epsilon\}$, *as* $\epsilon \to 0$, *then* $T_{i,l+1}(P) = \frac{L_i+R_i}{2}$, $1 \le i \le k$.

Proof. Since $f \in c^2_{\pm(l+1)}(x_i)$, $1 \le i \le k$, we have

i) $\int_{I_{+\epsilon,i}} (f - R_i)(x)(x - x_i)^j \, dx = o(\epsilon^{l+j+2})$, $0 \le j \le l+1$,

ii) $\int_{I_{-\epsilon,i}} (f - L_i)(x)(x - x_i)^j \, dx = o(\epsilon^{l+j+2})$, $0 \le j \le l+1$.

By means of a change of variable from i) and ii) we get

iii) $\int_{I_{-\epsilon,i}} (f - R_i)(2x_i - x)(x_i - x)^j \, dx = o(\epsilon^{l+j+2})$, $0 \le j \le l+1$,

iv) $\int_{I_{+\epsilon,i}} (f - L_i)(2x_i - x)(x_i - x)^j \, dx = o(\epsilon^{l+j+2})$, $0 \le j \le l+1$.

From i) it follows that

$$\begin{aligned}\int_{I_{+\epsilon,i}} (f - T_{i,l}(R_i))(x)(x - x_i)^j \, dx &= o(\epsilon^{l+j+2}) + \int_{I_{+\epsilon,i}} \frac{R_i^{(l+1)}(x_i)}{(l+1)!}(x - x_i)^{j+l+1} \, dx \\ &= o(\epsilon^{l+j+2}) + O(\epsilon^{l+j+2}) = o(\epsilon^{l+j+1}), \quad 0 \le j \le l.\end{aligned}$$

Thus, $f \in c^2_{+l}(x_i)$. Analogously, from ii) it follows that $f \in c^2_{-l}(x_i)$. In addition, as we have observed in the Introduction, $t^2_l(x_i) \subset t^2_{-l}(x_i) \subset c^2_{-l}(x_i)$ and $t^2_l(x_i) \subset t^2_{+l}(x_i) \subset c^2_{+s}(x_i)$. If c) holds, by uniqueness of the polynomial in Π^l that satisfies (1.3), we have that $T_{i,l}(R_i) = T_{i,l}(L_i)$, $1 \le i \le k$. Under the hypothesis a) or b) we also get $T_{1,l}(R_1) = T_{1,l}(L_1)$. We write $S_i = T_{i,l}(R_i)$, $R_i(x) = S_i(x) + c_{1,i}(x - x_i)^{l+1}$ and $L_i(x) = S_i(x) + c_{2,i}(x - x_i)^{l+1}$.

From i) and iv) we obtain

$$\begin{aligned}\int_{I_{+\epsilon,i}} [f(x) + f(2x_i - x) - (S_i(x) + c_{1,i}(x - x_i)^{l+1} \\ + S_i(2x_i - x) + c_{2,i}(x_i - x)^{l+1})](x - x_i)^j \, dx \\ = o(\epsilon^{l+j+2}), \quad 0 \le j \le l+1.\end{aligned} \tag{2.20}$$

In a similar way, from ii) and iii) we get

$$\begin{aligned}\int_{I_{-\epsilon,i}} [f(x) + f(2x_i - x) - (S_i(x) + c_{2,i}(x - x_i)^{l+1} \\ + S_i(2x_i - x) + c_{1,i}(x_i - x)^{l+1})](x - x_i)^j \, dx \\ = o(\epsilon^{l+j+2}), \quad 0 \le j \le l+1.\end{aligned} \tag{2.21}$$

Now, we assume l odd. From (2.20) and (2.21) we get

$$\int_{I_{\epsilon,i}} [f(x)+f(2x_i-x)-(R_i(x)+L_i(2x_i-x))](x-x_i)^j\,dx = o(\epsilon^{l+j+2}), \quad 0\le j\le l+1. \tag{2.22}$$

By (2.22) we obtain

$$\int_{I_{\epsilon,i}} \mathcal{P}_i\left(f-\frac{R_i+L_i}{2}\right)(x)(x-x_i)^j\,dx = o(\epsilon^{l+j+2}), \quad 0\le j\le l+1. \tag{2.23}$$

Since $f\in t_l^2(x_i)$, Lemma 2.1 implies

$$\int_{I_{\epsilon,i}} \mathcal{P}_i\left(f-P_\epsilon\right)(x)(x-x_i)^j\,dx = o(\epsilon^{l+j+2}), \quad j=0,2,\dots,l+1. \tag{2.24}$$

As we have observed in the proof of Theorem 2.3, $A_{l,i,\delta}$ is bounded for $0\le l\le q-2$ and δ sufficiently small. In consequence, from (2.24) we have

$$\begin{aligned}\int_{I_{\epsilon,i}} &\frac{1}{b_{\epsilon,i,l}}\mathcal{P}_i(f-T_{i,l+1}(P_\epsilon))(x)(x-x_i)^j\,dx\\ &= \sum_{v=l+2}^{n}\frac{P_\epsilon^{(v)}(x_i)}{v!b_{\epsilon,i,l}}\int_{I_{\epsilon,i}}\mathcal{P}_i\left((x-x_i)^v\right)(x-x_i)^j\,dx + o(\epsilon^{l+j+2})\\ &= o(\epsilon^{l+j+2}), \quad j=0,2,\dots,l+1.\end{aligned} \tag{2.25}$$

By (2.23) and (2.25), we get

$$\begin{aligned}\int_{I_{\epsilon,i}} \frac{1}{b_{\epsilon,i,l}}\mathcal{P}_i&\left(T_{i,l+1}(P_\epsilon)-\frac{R_i+L_i}{2}\right)(x)(x-x_i)^j\,dx\\ &= o(\epsilon^{l+j+2}), \quad j=0,2,\dots,l+1.\end{aligned} \tag{2.26}$$

Using Lemma 2.2 with $r_w=2(w-1)$, $1\le w\le \frac{l+1}{2}+1$, we get

$$\mathcal{P}_i\left(T_{i,l+1}(P_\epsilon)-\frac{R_i+L_i}{2}\right) = o(1)b_{\epsilon,i,l}, \quad 1\le i\le k. \tag{2.27}$$

Now, we assume l even. Adding to both sides of (2.21) the expression

$$-2\int_{I_{-\epsilon,i}} (c_{1,i}(x-x_i)^{l+1}+c_{2,i}(x_i-x)^{l+1})(x-x_i)^j\,dx, \quad 0\le j\le l+1,$$

we obtain

$$\begin{aligned}\int_{I_{-\epsilon,i}} &[(f(x)+f(2x_i-x))-(S_i(x)+c_{1,i}(x-x_i)^{l+1}+S_i(2x_i-x)\\ &+c_{2,i}(x_i-x)^{l+1})](x-x_i)^j\,dx = O(\epsilon^{l+j+2}), \quad 0\le j\le l+1.\end{aligned} \tag{2.28}$$

As the integrand in (2.20) and (2.28) is the same, we have

$$\begin{aligned}\int_{I_{\epsilon,i}} &[(f(x)+f(2x_i-x))-(R_i(x)+L_i(2x_i-x))](x-x_i)^j\,dx\\ &= o(\epsilon^{l+j+1}), \quad 0\le j\le l+1.\end{aligned} \tag{2.29}$$

It is easy to see that $\int_{I_{\epsilon,i}}(R_i(x)+L_i(2x_i-x))(x-x_i)^j\,dx = \int_{I_{\epsilon,i}}(L_i(x)+R_i(2x_i-x))(x-x_i)^j\,dx$, for j even. In consequence, from (2.29) we obtain

$$\begin{aligned}
&\int_{I_{\epsilon,i}} \mathcal{P}_i\left(f-\frac{R_i+L_i}{2}\right)(x)(x-x_i)^j\,dx \\
&\quad= \int_{I_{\epsilon,i}} \left(\mathcal{P}_i(f)-\frac{R_i(x)+L_i(x)+R_i(2x_i-x)+L_i(2x_i-x)}{4}\right)(x-x_i)^j\,dx \\
&\quad= o(\epsilon^{l+j+1}), \quad j=0,2,\dots,l.
\end{aligned} \tag{2.30}$$

Now, we proceed as in (2.25). As $f\in t_l^2(x_i)$ and $A_{l,i,\delta}$, $0\le l\le q-2$, is bounded for δ sufficiently small, Lemma 2.1 implies

$$\int_{I_{\epsilon,i}} \frac{1}{b_{\epsilon,i,l}}\mathcal{P}_i\left(f-T_{i,l+1}(P_\epsilon)\right)(x)(x-x_i)^j\,dx = o(\epsilon^{l+j+2}), \quad j=0,2,\dots,l. \tag{2.31}$$

By (2.30) and (2.31), we obtain

$$\int_{I_{\epsilon,i}} \frac{1}{b_{\epsilon,i,l}}\mathcal{P}_i\left(T_{i,l+1}(P_\epsilon)-\frac{R_i+L_i}{2}\right)(x)(x-x_i)^j\,dx = o(\epsilon^{l+j+1}), \quad j=0,2,\dots,l. \tag{2.32}$$

Since l is even, we have that $b_{\epsilon,i,l}^{-1}\mathcal{P}_i\left(T_{i,l+1}(P_\epsilon)-\frac{R_i+L_i}{2}\right)\in\Pi^l$. In consequence, using Lemma 2.2 with $r_w=2(w-1)$, $1\le w\le \frac{l}{2}+1$, we get (2.27).

In a similar way to the previous proof, considering j odd, we can prove that

$$\mathcal{I}_i\left(T_{i,l+1}(P_\epsilon)-\frac{R_i+L_i}{2}\right)=o(1)b_{\epsilon,i,l}, \quad 1\le i\le k, \tag{2.33}$$

for arbitrary l. From (2.27) and (2.33) we obtain (2.19).

Finally, let P be a cluster point of the net $\{P_\epsilon\}$, as $\epsilon\to 0$. Then there exists a sequence $\epsilon_m\to 0$ such that $P=\lim\limits_{\epsilon_m\to 0}P_{\epsilon_m}$. Thus, $b_{\epsilon_m,i,l}$ is bounded. From (2.19) it follows that $T_{i,l+1}(P)=\frac{R_i+L_i}{2}$, $1\le i\le k$. $\square$

The following corollary is an immediate consequence of Theorem 2.9 with $l=q-2$.

Corollary 2.10. *If $f\in c^2_{\pm(q-1)}(x_i)\cap t^2_{q-2}(x_i)$, $1\le i\le k$, then there exists the best local approximation of f on $\{x_i : 1\le i\le k\}$, from Π^n. It is the unique polynomial $H\in\Pi^n$ satisfying $H^{(j)}(x_i)=(\frac{L_i+R_i}{2})^{(j)}(x_i)$, $0\le j\le q-1$, $1\le i\le k$, where $L_i,R_i\in\Pi^{q-1}$ satisfy Definition 1.1. In the case $q=1$ or $k=1$ we only assume $f\in c^2_{\pm(q-1)}(x_i)$, $1\le i\le k$.*

This result was proved in [4, Theorem 2.9], for $f\in t^2_{-(q-1)}(x_i)\cap t^2_{+(q-1)}(x_i)\cap t^2_{q-2}(x_i)$, $1\le i\le k$. We give an example to show that Corollary 2.10 extends that Theorem 2.9 from [4]. More precisely, if $q=2$, $k=1$, and $x_1=0$, we define a function $f\in c^2_{\pm1}(0)\cap t^2_0(0)$ which is not L^2 right differentiable of order 1 at 0.

Example 2.11. Let $\epsilon_m = \frac{1}{m}$, $m \in \mathbb{N}$, $h_m = \epsilon_m - \frac{1}{m^6}$. It is easy to see that $\epsilon_{m+1} < h_m < \epsilon_m$, $m \geq 2$. We consider the function $f : [\frac{-1}{2}, \frac{1}{2}] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} m & \text{if } x \in [h_m, \epsilon_m] \\ 0 & \text{if } x \notin \cup_{m \geq 2}[h_m, \epsilon_m], \end{cases}$$

$f(0) = 0$ and $f(x) = f(-x)$ if $x \in [-\frac{1}{2}, 0)$. Let $0 < \epsilon \leq \frac{1}{2}$. Suppose that there exists $m \in \mathbb{N}$ such that $\epsilon \in [h_m, \epsilon_m]$; then

$$\begin{aligned} 0 &\leq \epsilon^{-3} \int_0^{\epsilon} f(x)\,dx \leq h_m^{-3} \int_0^{\epsilon_m} f(x)\,dx = h_m^{-3} \sum_{j=m}^{\infty} \int_{h_j}^{\epsilon_j} f(x)\,dx \\ &= h_m^{-3} \sum_{j=m}^{\infty} \int_{h_j}^{\epsilon_j} j\,dx = h_m^{-3} \sum_{j=m}^{\infty} j(\epsilon_j - h_j) = h_m^{-3} \sum_{j=m}^{\infty} j^{-5} \asymp \frac{1}{m}. \end{aligned} \tag{2.34}$$

If $\epsilon \notin \cup_{j \geq 2}[h_j, \epsilon_j]$, let $m \in \mathbb{N}$ be such that $0 < h_m < \epsilon_m < \epsilon < h_{m-1}$. Then

$$0 \leq \epsilon^{-3} \int_0^{\epsilon} f(x)\,dx \leq \epsilon_m^{-3} \int_0^{\epsilon_m} f(x)\,dx \leq h_m^{-3} \int_0^{\epsilon_m} f(x)\,dx \asymp \frac{1}{m}. \tag{2.35}$$

Since $0 \leq \int_0^{\epsilon} f(x)x\,dx \leq \int_0^{\epsilon} f(x)\,dx$, and f is even, from (2.34) and (2.35) we have that $f \in c^2_{\pm 1}(0)$. On the other hand, f is not L^2 right differentiable of order 1 at 0. In fact, for $S \in \Pi^1$ there exists a $m_0 \in \mathbb{N}$ such that $|S(x)| \leq \frac{f(x)}{4}$, for all $x \in [h_j, \epsilon_j]$, $j \geq m_0$. So, $|f(x) - S(x)|^2 \geq f(x)(f(x) - 2|S(x)|) \geq \frac{f(x)^2}{2}$ for all $x \in [h_j, \epsilon_j]$, $j \geq m_0$. Therefore, for $m \geq m_0$ we have

$$\begin{aligned} \epsilon_m^{-3} \int_0^{\epsilon_m} |(f-S)(x)|^2\,dx &\geq \epsilon_m^{-3} \sum_{j=m}^{\infty} \int_{h_j}^{\epsilon_j} |(f-S)(x)|^2\,dx \\ &\geq \frac{1}{2}\epsilon_m^{-3} \sum_{j=m}^{\infty} \int_{h_j}^{\epsilon_j} |f(x)|^2\,dx = \frac{1}{2}\epsilon_m^{-3} \sum_{j=m}^{\infty} j^2(\epsilon_j - h_j) = \frac{1}{2}\epsilon_m^{-3} \sum_{j=m}^{\infty} j^{-4} \asymp \frac{1}{2}. \end{aligned}$$

Finally, we will prove that f is L^2 differentiable of order 0 at 0. If either $\epsilon \in [h_m, \epsilon_m]$ or $\epsilon \in [\epsilon_m, h_{m-1}]$, we obtain

$$\begin{aligned} 0 &\leq \frac{1}{\epsilon} \int_0^{\epsilon} f(x)^2\,dx \leq h_m^{-1} \int_0^{\epsilon_m} f(x)^2\,dx = h_m^{-1} \sum_{j=m}^{\infty} \int_{h_j}^{\epsilon_j} f(x)^2\,dx \\ &= h_m^{-1} \sum_{j=m}^{\infty} j^2(\epsilon_j - h_j) = h_m^{-1} \sum_{j=m}^{\infty} j^{-4} \asymp \frac{1}{m^2}. \end{aligned}$$

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Received: July 29, 2016
Accepted: May 15, 2017