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ON PARTIAL ORDERS IN PROPER $*$-RINGS

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ABSTRACT. We study orders in proper $*$-rings that are derived from the core-nilpotent decomposition. The notion of the C-N-star partial order and the S-star partial order is extended from $M_n(\mathbb{C})$, the set of all $n \times n$ complex matrices, to the set of all Drazin invertible elements in proper $*$-rings with identity. Properties of these orders are investigated and their characterizations are presented. For a proper $*$-ring $\mathcal{A}$ with identity, it is shown that on the set of all Drazin invertible elements $a \in \mathcal{A}$ where the core part of a is an EP element, the C-N-star partial order implies the star partial order.

1. INTRODUCTION

Let S be a semigroup. An involution * on S is called *proper* if $a^*a = a^*b = b^*a = b^*b$, where $a, b \in S$, implies $a = b$. If a semigroup S is equipped with a proper involution, then S is called *a proper $*$-semigroup*. Natural special cases of proper $*$-semigroups are all proper $*$-rings (in particular, $M_n(\mathbb{C})$, the ring of all $n \times n$ complex matrices), with "properness" defined via $aa^* = 0$ implying $a = 0$. Drazin introduced in [2] a partial order, now known as *the star partial order,* on proper $*$-semigroups. The definition follows. Let S be a proper $*$-semigroup. For $a, b \in S$, we write

$$a \leq^* b \quad \text{if} \quad a^*a = a^*b \quad \text{and} \quad aa^* = ba^*. \tag{1}$$

Recall that an element $a \in S$ is called regular when $a \in aSa$, and $*$-regular when there exists an element $a^\dagger \in S$ such that $aa^\dagger a = a$, $a^\dagger aa^\dagger = a^\dagger$, $(aa^\dagger)^* = aa^\dagger$, and $(a^\dagger a)^* = a^\dagger a$. The element $a^\dagger$, which is unique if it exists, is known as *the Moore-Penrose (generalized) inverse* of a. We say that an element $a \in S$ has *a Drazin inverse* $b \in S$ if

$$ab = ba, \qquad b = ab^2, \qquad a^k = a^{k+1}b \tag{2}$$

for some non-negative integer k (see [1]). If a has a Drazin inverse, then we say that a is *Drazin invertible* and the smallest non-negative integer k in (2) is called *the index* $i(a)$ of a. It is well known that there is at most one b such that (2) holds. The unique b, if it exists, will be denoted by a^D.

Let $\mathcal{A}$ be a ring with the (multiplicative) identity. We say that $a \in \mathcal{A}$ has *the group inverse* $a^\# \in \mathcal{A}$ if $x = a^\#$ satisfies the following equations: $axa = a$,

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$xax = x$, and $ax = xa$. Mitra introduced in [9] a partial order on the set of all $n \times n$ matrices over a field $\mathbb{F}$ which have the group inverse. This order, known as *the sharp partial order*, was generalized in [6] and independently in [13] to rings. The definition from [6] follows. Denote by $\mathcal{G}(\mathcal{A})$ the set of all elements in $\mathcal{A}$ which have the group inverse. For $a \in \mathcal{G}(\mathcal{A})$ and $b \in \mathcal{A}$, we write

$$a \leq^{\sharp} b \quad \text{if} \quad a^{\#}a = a^{\#}b \quad \text{and} \quad aa^{\#} = ba^{\#}. \tag{3}$$

In [6], the author proved that the sharp order $\leq^{\sharp}$ is indeed a partial order on $\mathcal{G}(\mathcal{A})$.

Denote now by $\mathcal{N}(\mathcal{A})$ the set of all nilpotent elements in $\mathcal{A}$. Koliha gave in [4] an equivalent definition of the Drazin inverse for rings with identity. Namely, for $a, b \in \mathcal{A}$, (2) is equivalent to

$$ab = ba, \qquad b = ab^2, \qquad a - a^2 b \in \mathcal{N}(\mathcal{A}). \tag{4}$$

Moreover, the index $i(a)$ of a is equal to the nilpotency index of $a - a^2 b$. Note that group invertibility is a special case of Drazin invertibility (see [3]). Namely, if $i(a) \leq 1$, then the Drazin inverse of a is exactly the group inverse of a.

Suppose $a \in \mathcal{A}$ has the Drazin inverse. It is known (see for example [15]) that then a may be written as

$$a = c + n \tag{5}$$

where $c, n \in \mathcal{A}$, c has the group inverse, $cn = nc = 0$, and n is nilpotent with index of nilpotency equal to $i(a)$. Then c is called *the core part* of a and n *the nilpotent part* of a. Note that $c^{\#}n = 0 = nc^{\#}$ and therefore $c^{\#}ac^{\#} = c^{\#}$, $ac^{\#} = c^{\#}a$, and $a - a^2 c^{\#} = n$. It follows (see (4)) that $a^D = c^{\#}$. Since the Drazin inverse of every element in $\mathcal{A}$ is unique if it exists, we may conclude that c and n from (5) are unique. In fact,

$$c = a^2 a^D \qquad \text{and} \qquad n = a - a^2 a^D. \tag{6}$$

We refer to $c + n$ as *the core-nilpotent decomposition* of a.

It is known (see for example [10, Theorem 2.4.26]) that every matrix $A \in M_n(\mathbb{C})$ has the Drazin inverse. Thus, any matrix from $M_n(\mathbb{C})$ has the core-nilpotent decomposition (5). For a matrix $A \in M_n(\mathbb{C})$, let $\operatorname{Im} A$ denote the column space of A and $\operatorname{Ker} A$ the null space of A. A matrix $A \in M_n(\mathbb{C})$ is said to be range-Hermitian (or EP) if $\operatorname{Im} A = \operatorname{Im} A^*$, or equivalently if $\operatorname{Ker} A = \operatorname{Ker} A^*$. Note that all Hermitian matrices and all non-singular matrices in $M_n(\mathbb{C})$ are range-Hermitian.

Let $\mathfrak{C}$ be the subset of all matrices in $M_n(\mathbb{C})$ whose core part is range-Hermitian. Let $A = C_A + N_A$ and $B = C_B + N_B$ be the core-nilpotent decompositions of A and B, respectively, where C_A is the core part of A, C_B is the core part of B, N_A is the nilpotent part of A, and N_B is the nilpotent part of B. Mitra et al. introduced in [10] the following relations in $M_n(\mathbb{C})$.

Definition 1. Let $A, B \in M_n(\mathbb{C})$. We write $A \leq^{\sharp,*} B$ if $C_A \leq^{\sharp} C_B$ and $N_A \leq^{*} N_B$.

Definition 2. Let $A, B \in M_n(\mathbb{C})$. We write $A \leq^{\circledast} B$ if $C_A \leq^{\sharp} C_B$ and $A \leq^{*} B$.

If $A \leq^{\sharp,*} B$, we say that A is below B with respect to *the C-N-star partial order*, and if $A \leq^{\circledast} B$, we say that A is below B with respect to the *the S-star partial order.* Mitra et al. noted in [10] that both $\leq^{\sharp,*}$ and $\leq^{\circledast}$ are partial orders, and that on $\mathfrak{C}$, the C-N-star partial order $\leq^{\sharp,*}$ implies the star partial order $\leq^*$, i.e. for $A, B \in \mathfrak{C}$, $A \leq^{\sharp,*} B$ yields $A \leq^* B$. They also posed the following open question.

Problem. What are necessary and sufficient conditions under which the S-star partial order $\leq^{\circledast}$ implies the C-N-star partial order $\leq^{\sharp,*}$?

The aim of this paper to generalize Definitions 1 and 2 to unital proper $*$-rings, study the properties of these orders, and solve Problem in a more general setting of proper $*$-rings with identity.

2. Definitions and preliminary results

Let $\mathcal{A}$ be a ring with identity. For a Drazin invertible $a \in \mathcal{A}$, we will denote by c_a the core part of a and by n_a the nilpotent part of a. Mitra et al. extended in [10] the notion of the sharp order from the set $\mathcal{G}(M_n(\mathbb{F}))$ to the set $M_n(\mathbb{F})$ of all $n \times n$ matrices over a field $\mathbb{F}$. Namely, they introduced a relation on $M_n(\mathbb{F})$ using only the core part of matrices and ignoring the nilpotent part altogether. This relation has been recently generalized in [7] from $M_n(\mathbb{F})$ to the set of all Drazin invertible elements in rings with identity.

Definition 3. Let $a, b \in \mathcal{A}$ be Drazin invertible. The element a is said to be below the element b with respect to the *the Drazin order* if $c_a \leq^\sharp c_b$. When this happens, we write $a \leq^D b$.

The relation $\leq^D$ is a pre-order, i.e. it is reflexive and transitive, and it is not a partial order. Namely, the failure of anti-symmetry is due to the fact that the Drazin order ignores the nilpotent parts.

Unless stated otherwise, from now on let $\mathcal{A}$ be a proper $*$-ring with identity 1. Note that the C-N-star partial order, defined with Definition 1, is in fact a modification of the Drazin order on $M_n(\mathbb{C})$ so that the nilpotent parts are also involved. This modification transforms the Drazin pre-order to a partial order. Let us now generalize the notions of the C-N-star and the S-star partial orders from $M_n(\mathbb{C})$ to the set of Drazin invertible elements in a proper $*$-ring with identity.

Definition 4. Let $a, b \in \mathcal{A}$ be Drazin invertible. The element a is said to be below the element b with respect to *the C-N-star partial order* if $c_a \leq^\sharp c_b$ and $n_a \leq^* n_b$. When this happens, we write $a \leq^{\sharp,*} b$.

Definition 5. Let $a, b \in \mathcal{A}$ be Drazin invertible. The element a is said to be below the element b with respect to *the S-star partial order* if $c_a \leq^\sharp c_b$ and $a \leq^* b$. When this happens, we write $a \leq^{\circledast} b$.

Recall that the sharp order is a partial order on the set of all group invertible elements in a general ring with identity. Since the star order is also a partial order in a general proper $*$-ring, we obtain the following results.

Theorem 1. *The order relation $\leq^{\sharp,*}$, defined with Definition 4, is a partial order on the set of all Drazin invertible elements in $\mathcal{A}$.*

Theorem 2. *The order relation $\leq^{\circledast}$, defined with Definition 5, is a partial order on the set of all Drazin invertible elements in $\mathcal{A}$.*

For an element a in a ring $\mathcal{A}$, we will denote by a° the right annihilator of a, i.e. the set $a^\circ = \{x \in \mathcal{A} : ax = 0\}$. Similarly we denote the left annihilator ${}^\circ a$ of a, i.e. the set ${}^\circ a = \{x \in \mathcal{A} : xa = 0\}$. Let us now present some auxiliary results.

Lemma 2.1. *Let $\mathcal{A}$ be a proper $*$-ring and $a, b \in \mathcal{A}$. If $a \leq^* b$, then ${}^\circ b \subseteq {}^\circ a$ and $b^\circ \subseteq a^\circ$.*

Proof. For $a, b \in \mathcal{A}$, let $a \leq^* b$. Let $zb = 0$, $z \in \mathcal{A}$. From $aa^* = ba^*$, we have $0 = zba^* = zaa^*$. So, $zaa^*z^* = 0$ and therefore $za(za)^* = 0$. Since $\mathcal{A}$ is a proper $*$-ring, it follows that $za = 0$, i.e. ${}^\circ b \subseteq {}^\circ a$. The equation $a^*a = a^*b$ similarly implies $b^\circ \subseteq a^\circ$. □

Lemma 2.2. *Let $A \in M_n(\mathbb{C})$. Then A is range-Hermitian (or EP) if and only if ${}^\circ A = {}^\circ(A^*)$, which is equivalent to $A^\circ = (A^*)^\circ$.*

Proof. Let $A, B \in M_n(\mathbb{C})$. By Lemma 2.1 in [8], we have ${}^\circ A = {}^\circ B$ if and only if $\operatorname{Im} A = \operatorname{Im} B$, and $A^\circ = B^\circ$ if and only if $\operatorname{Ker} A = \operatorname{Ker} B$. It follows that ${}^\circ A = {}^\circ(A^*)$ if and only if $\operatorname{Im} A = \operatorname{Im} A^*$ if and only if $\operatorname{Ker} A = \operatorname{Ker} A^*$ if and only if $A^\circ = (A^*)^\circ$. □

We will use the following definition of EP elements in rings (see [5]). An element a of a ring $\mathcal{A}$ with involution * is said to be EP if a has the group inverse $a^\#$ and the Moore-Penrose inverse $a^\dagger$, and $a^\# = a^\dagger$.

Let $a \in \mathcal{A}$ be a $*$-regular element. Observe (see, e.g. [14]) that then ${}^\circ(a^*) = {}^\circ(a^\dagger)$ and $(a^*)^\circ = (a^\dagger)^\circ$. For $b \in \mathcal{A}$, it follows that $a^*a = a^*b$ if and only if $a^\dagger a = a^\dagger b$, and similarly $aa^* = ba^*$ if and only if $aa^\dagger = ba^\dagger$. Since for an EP element a, $a^\# = a^\dagger$, we arrive at the following result.

Lemma 2.3. *Let $a \in \mathcal{A}$ be an EP element. For $b \in \mathcal{A}$, we have $a \leq^* b$ if and only if $a \leq^\sharp b$.*

It turns out (see [11, 12]) that $a \in \mathcal{A}$ is an EP element if and only if $a\mathcal{A} = a^*\mathcal{A}$ and a has the group inverse. For $a, b \in \mathcal{A}$, where a and b are regular, the following statement holds by [14, Lemmas 2.5 and 2.6]: ${}^\circ a = {}^\circ b$ if and only if $a\mathcal{A} = b\mathcal{A}$. Note that for a group invertible $a \in \mathcal{A}$, a^* has the group inverse $(a^\#)^*$. So, since a group invertible element is also regular, we may conclude that $a \in \mathcal{A}$ is EP if and only if a has the group inverse and ${}^\circ a = {}^\circ(a^*)$.

Let now $a \in \mathcal{A}$ be Drazin invertible. Since the core part c_a of a is group invertible and since all matrices in $M_n(\mathbb{C})$ are Drazin invertible, we may generalize the notion of the set $\mathfrak{C}$, of all matrices whose core part is range-Hermitian (or EP), from $M_n(\mathbb{C})$ to a ring $\mathcal{A}$ with involution * (see Lemma 2.2): Let $\mathfrak{C}^{\mathcal{A}}$ be the subset of all Drazin invertible elements $a \in \mathcal{A}$ where ${}^\circ c_a = {}^\circ(c_a^*)$.

Mitra et al. observed that on $\mathfrak{C} = \mathfrak{C}^{M_n(\mathbb{C})}$, the C-N-star partial order $\leq^{\sharp,*}$ implies the star partial order $\leq^*$. In the next section, we will prove that a similar result holds also on $\mathfrak{C}^{\mathcal{A}}$ where $\mathcal{A}$ is a general proper $*$-ring with identity. First, let us present some useful tools.

The equality $1 = e_1 + e_2 + \cdots + e_n$, where 1 is the identity of $\mathcal{A}$, $e_1, e_2, \ldots, e_n$ are idempotent elements in $\mathcal{A}$, and $e_ie_j = 0$ for $i \neq j$, is called a decomposition of the identity of $\mathcal{A}$. Let $1 = e_1 + \cdots + e_n$ and $1 = f_1 + \cdots + f_n$ be two decompositions of the identity of $\mathcal{A}$. We have

$$x = 1 \cdot x \cdot 1 = (e_1 + e_2 + \cdots + e_n)x(f_1 + f_2 + \cdots + f_n) = \sum_{i,j=1}^{n} e_i x f_j.$$

Then any $x \in \mathcal{A}$ can be uniquely represented in the following matrix form:

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f}, \tag{7}$$

where $x_{ij} = e_i x f_j \in e_i \mathcal{A} f_j$. If $x = (x_{ij})_{e\times f}$ and $y = (y_{ij})_{e\times f}$, then $x + y = (x_{ij} + y_{ij})_{e\times f}$. Moreover, if $1 = g_1 + \cdots + g_n$ is a decomposition of the identity of $\mathcal{A}$ and $z = (z_{ij})_{f\times g}$, then, by the orthogonality of the idempotents involved, $xz = (\sum_{k=1}^{n} x_{ik} z_{kj})_{e\times g}$. Thus, if we have decompositions of the identity of $\mathcal{A}$, then the usual algebraic operations in $\mathcal{A}$ can be interpreted as simple operations between appropriate $n \times n$ matrices over $\mathcal{A}$. When $n = 2$ and $p, q \in \mathcal{A}$ are idempotent elements, we may write

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}_{p\times q}.$$

Here $x_{1,1} = pxq$, $x_{1,2} = px(1-q)$, $x_{2,1} = (1-p)xq$, $x_{2,2} = (1-p)x(1-q)$.

If $\mathcal{A}$ is a ring with involution *, then we may by (7) write

$$x^* = \begin{bmatrix} x_{11}^* & \cdots & x_{n1}^* \\ \vdots & \ddots & \vdots \\ x_{1n}^* & \cdots & x_{nn}^* \end{bmatrix}_{f^*\times e^*} \tag{8}$$

where this matrix representation of x^* is given relative to the decompositions of the identity $1 = f_1^* + \cdots + f_n^*$ and $1 = e_1^* + \cdots + e_n^*$.

Let $a \in \mathcal{A}$ be Drazin invertible. It turns out (for details see [7]) that we may present the core nilpotent decomposition $c_a + n_a$ of a in the following matrix form:

$$a = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p\times p} \tag{9}$$

where $p = aa^D$.

Remark. For a ring $\mathcal{A}$ with involution *, $a \in \mathcal{A}$ is EP if and only if a has the group inverse $a^\#$ and $aa^\#$ is self-adjoined (see [5, Theorem 7.3] or [11, Theorem 1.2]). Suppose ${}^\circ c_a = {}^\circ(c_a^*)$. It follows that c_a is then EP, which implies $(c_a c_a^\#)^* = c_a c_a^\#$. Recall (see the first section) that $a^D = c_a^\#$. Therefore,

$$p = aa^D = (c_a + n_a)c_a^\# = c_a c_a^\#,$$

which implies that p is a self-adjoined idempotent.

Let us conclude this section with a characterization [7, Theorem 1] of the Drazin order $\leq^D$ which we will use in the continuation.

Proposition. *Let $a, b \in \mathcal{A}$ be Drazin invertible. The following statements are then equivalent.*

(i) $a \leq^D b$.
(ii) $aa^D = ba^D = a^D b = a^D a$.
(iii) *There exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ such that*

$$a = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_3 & c_4 \\ 0 & c_5 & c_6 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & n_2 \end{bmatrix}_{e\times e}$$

where $n_a = c_3 + c_4 + c_5 + c_6$ is the nilpotent part of a, c_2 has the group inverse, and n_2 is nilpotent.

3. Main results

3.1. **The C-N-star partial order.** Recall that $\mathfrak{C}^{\mathcal{A}}$ is the set of all Drazin invertible elements $a \in \mathcal{A}$ where ${}^\circ c_a = {}^\circ(c_a^*)$, i.e. the core part c_a of a is an EP element. We shall now present a new characterization of the C-N-star partial order on $\mathfrak{C}^{\mathcal{A}}$ where $\mathcal{A}$ is a proper $*$-ring with identity.

Theorem 3. *Let $a, b \in \mathfrak{C}^{\mathcal{A}}$. Then $a \leq^{\sharp,*} b$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, where $e_1, e_2, e_3 \in \mathcal{A}$ are self-adjoined, such that*

$$a = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_1 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & n_2 \end{bmatrix}_{e\times e} \tag{10}$$

where c_1 and c_2 have the group inverse, and n_1 and n_2 are nilpotent with $n_1 \leq^* n_2$.

Proof. Suppose $a, b \in \mathfrak{C}^{\mathcal{A}}$. Let $a = c_a + n_a$ and $b = c_b + n_b$ be the core-nilpotent decompositions of a and b, respectively. By (9) and Remark we may present element a in the following matrix form:

$$a = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p\times p}$$

where $p = aa^D$ is self-adjoined.

Suppose $a \leq^{\sharp,*} b$. It follows that $a \leq^D b$ and therefore by Proposition there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ such that

$$a = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_3 & c_4 \\ 0 & c_5 & c_6 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & n_2 \end{bmatrix}_{e\times e} \tag{11}$$

where $n_a = c_3 + c_4 + c_5 + c_6$ is the nilpotent part of a, c_2 has the group inverse, and n_2 is nilpotent. Since $c_1 = a - n_a$, we may observe that $c_1 = c_a$ is the core part of a. Note that for $d \in \mathcal{G}(\mathcal{A})$, we have ${}^\circ(d^\#) = {}^\circ d$ and $(d^\#)^\circ = d^\circ$. It follows that $0 = c_2^\# c_a = c_a c_2^\# = c_2 c_a^\# = c_a^\# c_2$ and thus $(c_a + c_2)^\# = c_a^\# + c_2^\#$, i.e. $c_a + c_2$ is group invertible. So, since $(c_a + c_2)n_2 = n_2(c_a + c_2) = 0$, it follows that the core and the nilpotent parts of b are $c_a + c_2 = c_b$ and $n_2 = n_b$, respectively. Observe that by the proof of Theorem 1 ((ii) implies (iii)) in [7] we may without loss of generality assume that $e_1 = p = p^*$, $e_2 = c_2 c_2^{\#}$, and $e_3 = 1 - e_1 - e_2$. Since $b \in \mathfrak{C}^{\mathcal{A}}$, Remark implies $c_b c_b^{\#}$ is self-adjoined. So,

$$((c_a + c_2)(c_a + c_2)^{\#})^* = (c_a + c_2)(c_a + c_2)^{\#}$$

and therefore

$$(c_a c_a^{\#})^* + (c_2 c_2^{\#})^* = c_a c_a^{\#} + c_2 c_2^{\#}.$$

Recall that $a \in \mathfrak{C}^{\mathcal{A}}$. So, $(c_a c_a^{\#})^* = c_a c_a^{\#}$ which yields that $(c_2 c_2^{\#})^* = c_2 c_2^{\#}$. We may conclude that the idempotents e_1, e_2, and e_3 are all self-adjoined.

Let us now show that in (11), $c_3 = c_4 = c_5 = 0$. From $a \leq^{\sharp,*} b$, we have $n_a \leq^* n_b$. Since $n_a = c_3 + c_4 + c_5 + c_6$ and $n_b = n_2$, it follows by Lemma 2.1 that

$$^\circ n_2 \subseteq {}^\circ(c_3 + c_4 + c_5 + c_6) \quad \text{and} \quad n_2^\circ \subseteq (c_3 + c_4 + c_5 + c_6)^\circ. \tag{12}$$

We have $c_3 + c_4 + c_5 + c_6 = e_2 a e_2 + e_2 a e_3 + e_3 a e_2 + e_3 a e_3$ and $n_2 = e_3 b e_3$. Since $e_2 e_3 = 0 = e_3 e_2$, we obtain $e_2 n_2 = 0 = n_2 e_2$, i.e. $e_2 \in {}^\circ n_2 \cap n_2^\circ$. So, by (12),

$$e_2 \in {}^\circ(c_3 + c_4 + c_5 + c_6) \cap (c_3 + c_4 + c_5 + c_6)^\circ$$

and therefore

$$0 = e_2(c_3 + c_4 + c_5 + c_6) = e_2 a e_2 + e_2 a e_3 = c_3 + c_4$$

and

$$0 = (c_3 + c_4 + c_5 + c_6)e_2 = e_2 a e_2 + e_3 a e_2 = c_3 + c_5.$$

So, $c_4 = c_5 = -c_3$. Note that $c_4 \in e_2 \mathcal{A} e_3$ and $c_5 \in e_3 \mathcal{A} e_2$. Since $e_2 \mathcal{A} e_3 \cap e_3 \mathcal{A} e_2 = \{0\}$, we may conclude that $0 = c_3 = c_4 = c_5$. If we denote $c_6 = n_1$, we obtain the matrix form (10) of a and b.

Conversely, let $a, b \in \mathfrak{C}^{\mathcal{A}}$ be of the matrix form (10). Since c_1 is group invertible and n_1 is nilpotent with $c_1 n_1 = n_1 c_1 = 0$, the uniqueness of the core-nilpotent decomposition implies $c_a = c_1$ and $n_a = n_1$. Similarly, $c_b = c_1 + c_2$ and $n_b = n_2$. By Proposition it follows that $a \leq^D b$, i.e. $c_a \leq^\sharp c_b$. Therefore, since by assumption $n_a \leq^* n_b$, we may conclude that $a \leq^{\sharp,*} b$. ☐

The nilpotent part of a Drazin invertible element $a \in \mathcal{A}$ is by (6), $n_a = a - a^2 a^D = a - a a^D a$. Thus, directly by Definitions 3 and 4, we obtain another characterization of the C-N-star partial order on a proper $*$-ring $\mathcal{A}$ with identity.

Theorem 4. *Let $a, b \in \mathcal{A}$ be Drazin invertible. Then $a \leq^{\sharp,*} b$ if and only if $a \leq^D b$ and $a - a a^D a \leq^* b - b b^D b$.*

With the next result we will show that on $\mathfrak{C}^{\mathcal{A}}$, the C-N-star partial order $\leq^{\sharp,*}$ implies the star partial order $\leq^*$.

Theorem 5. *Let $a, b \in \mathfrak{C}^{\mathcal{A}}$. If $a \leq^{\sharp,*} b$, then $a \leq^* b$.*

Proof. Suppose $a, b \in \mathfrak{C}^{\mathcal{A}}$ and $a \leq^{\sharp,*} b$, i.e. $c_a \leq^\sharp c_b$ and $n_a \leq^* n_b$. The star partial order (1) and the sharp partial order (3) are by Lemma 2.3 equivalent on the set of EP elements in $\mathcal{A}$. Since the core part c_a of a is an EP element, we may conclude that $c_a \leq^* c_b$. It follows that $c_a^* c_a = c_a^* c_b$ and $c_a c_a^* = c_b c_a^*$. Also, $n_a^* n_a = n_a^* n_b$ and $n_a n_a^* = n_b n_a^*$. Since $a = c_a + n_a$ and $b = c_b + n_b$, we obtain

$$a^* a = c_a^* c_a + c_a^* n_a + n_a^* c_a + n_a^* n_a \quad \text{and} \quad a^* b = c_a^* c_b + c_a^* n_b + n_a^* c_b + n_a^* n_b. \tag{13}$$

Observe that for any $d \in \mathcal{A}$, ${}^\circ d = {}^\circ(d^*)$ if and only if $d^\circ = (d^*)^\circ$. So, since $a \in \mathfrak{C}^{\mathcal{A}}$ and therefore ${}^\circ c_a = {}^\circ(c_a^*)$, we have $c_a^* n_a = 0 = n_a^* c_a$. By Theorem 3, elements a and b have the matrix representation (10), where $c_1 = c_a$ is the core part of a, $n_1 = n_a$ is the nilpotent part of a, $c_1 + c_2 = c_b$ is the core part of b, and $n_2 = n_b$ is the nilpotent part of b. Clearly, by (10) we have $c_1 n_2 = 0$, i.e. $c_a n_b = 0$, which yields $0 = c_a^* n_b$. Similarly, $(c_1 + c_2)n_1 = 0$, i.e. $c_b n_a = 0$. Since $b \in \mathfrak{C}^{\mathcal{A}}$ and therefore ${}^\circ c_b = {}^\circ(c_b^*)$, we obtain $0 = c_b^* n_a$ and thus $0 = n_a^* c_b$. By (13), we may conclude that

$$a^* a = a^* b.$$

We may similarly prove that $aa^* = ba^*$. Therefore, $a \leq^* b$. $\square$

3.2. **The S-star partial order.** With Theorem 5 we showed that on $\mathfrak{C}^{\mathcal{A}}$, where $\mathcal{A}$ is a proper $*$-ring with identity, the C-N-star partial order $\leq^{\sharp,*}$ implies the star partial order $\leq^*$. It follows (compare Definitions 4 and 5) that the C-N-star partial order implies also the S-star partial order $\leq^{\circledast}$. With the next theorem, we will present some new characterizations of the C-N-star partial order and thus find some conditions under which the S-star partial order implies the C-N-star partial order.

Theorem 6. *Let $a, b \in \mathfrak{C}^{\mathcal{A}}$, let $k = \max\{i(a), i(b)\}$, and suppose $a \leq^{\circledast} b$. The following statements are then equivalent.*

(i) $a \leq^{\sharp,*} b$
(ii) $b^k abb^D = b^k a$, $bb^D ab^k = ab^k$, *and* $bb^D a = aa^D a$
(iii) ${}^\circ(b^k) \subseteq {}^\circ(ab^k)$ *and* $b^k a = a^{k+1}$
(iv) ${}^\circ((b^k)^*) \subseteq {}^\circ((b^k a)^*)$ *and* $ab^k = a^{k+1}$
(v) $ab^k = b^k a = a^{k+1}$

Proof. Some steps of the proof will be similar to the corresponding steps in the proof of Theorem 8 in [7]. For the sake of completeness, we will not skip these details. Let $a \leq^{\sharp,*} b$. By Theorem 3, there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, where $e_1, e_2, e_3 \in \mathcal{A}$ are self-adjoined, such that

$$a = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_1 \end{bmatrix}_{e \times e}, \qquad b = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & n_2 \end{bmatrix}_{e \times e} \tag{14}$$

where c_1 and c_2 have the group inverse, and n_1 and n_2 are nilpotent with $n_1 \leq^* n_2$. Note that $c_a = c_1$, $n_a = n_1$, $c_b = c_1 + c_2$, and $n_b = n_2$. Since $k = \max\{i(a), i(b)\}$, we have $n_1^k = 0 = n_2^k$ and therefore

$$a^{k+1} = \begin{bmatrix} c_1^{k+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times e} \quad \text{and} \quad b^k = \begin{bmatrix} c_1^k & 0 & 0 \\ 0 & c_2^k & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times e}. \tag{15}$$

(i)⇒(ii): Observe that $a^D = c_1^\# = \begin{bmatrix} c_1^\# & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$ and $b^D = (c_1+c_2)^\# = c_1^\# + c_2^\# = \begin{bmatrix} c_1^\# & 0 & 0 \\ 0 & c_2^\# & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$. We obtain

$$b^k abb^D = \begin{bmatrix} c_1^k c_1 c_1 c_1^\# & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} c_1^{k+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = b^k a,$$

and similarly $bb^D ab^k = c_1^{k+1} = ab^k$, and $bb^D a = c_1 = aa^D a$.

(i)⇒(iii): Clearly, by (14) and (15), $b^k a = c_1^{k+1} = a^{k+1}$. Let $z \in {}^\circ(b^k)$, i.e. $zb^k = 0$. Since $b^k = c_1^k + c_2^k$, we obtain $zc_1^k + zc_2^k = 0$ and thus, $zc_1^{k+1} + zc_2^k c_1 = 0$. Note that $c_1c_2 = c_2c_1 = 0$. So, $zc_1^{k+1} = 0$. Observe that $ab^k = c_1^{k+1}$. Therefore, $z \in {}^\circ(ab^k)$, i.e. ${}^\circ(b^k) \subseteq {}^\circ(ab^k)$.

(i)⇒(iv): Again, clearly, $ab^k = a^{k+1} = c_1^{k+1} = b^k a$. Let $z \in {}^\circ((b^k)^*)$. Thus $z(c_1^k)^* + z(c_2^k)^* = 0$ and therefore $z(c_1^{k+1})^* + z(c_2^k)^* c_1^* = 0$. Since $c_1c_2 = 0$, we have $c_2^* c_1^* = 0$ and hence $z \in {}^\circ((c_1^{k+1})^*) = {}^\circ((b^k a)^*)$.

(i)⇒(v): It follows directly by (14) and (15).

(ii)⇒(i): Let $a \leq^{\circledast} b$, i.e. $c_a \leq^\sharp c_b$ and $a \leq^* b$, and let $b^k abb^D = b^k a$, $bb^D ab^k = ab^k$, and $bb^D a = aa^D a$. We will show that then $n_a \leq^* n_b$. By (9),

$$b = \begin{bmatrix} c_b & 0 \\ 0 & n_b \end{bmatrix}_{q\times q}$$

where $q = bb^D = c_b c_b^\#$. Note (see Remark) that $q = q^*$. Also, since n_b is the nilpotent part of b, we have $n_b^k = 0$ and therefore

$$b^k = \begin{bmatrix} c_b^k & 0 \\ 0 & 0 \end{bmatrix}_{q\times q}.$$

Let $a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{q\times q}$. From $b^k abb^D = b^k a$, we obtain $0 = b^k a(1 - bb^D) = b^k a(1-q)$ and thus

$$0 = \begin{bmatrix} c_b^k & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{q\times q} \begin{bmatrix} 0 & 0 \\ 0 & 1-q \end{bmatrix}_{q\times q} = \begin{bmatrix} 0 & c_b^k a_2(1-q) \\ 0 & 0 \end{bmatrix}_{q\times q}.$$

So, $c_b^k a_2(1-q) = 0$. This yields $c_b^k a_2 = 0$ since $a_2 \in q\mathcal{A}(1-q)$. It follows that

$$0 = (c_b^\#)^k c_b^k a_2 = q^k a_2 = a_2.$$

Similarly, from $bb^D ab^k = ab^k$ we obtain $0 = (1-q)ab^k$ and therefore $0 = (1-q)a_3q^k$. So, $a_3 = 0$ since $a_3 \in (1-q)\mathcal{A}q$. Thus,

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \end{bmatrix}_{q\times q}.$$

Since $a_1 \in q\mathcal{A}q$ and $a_4 \in (1-q)\mathcal{A}(1-q)$, we have

$$a_4 = (1-q)a_1 + (1-q)a_4 = (1-q)(a_1+a_4) = (1-q)a = a - bb^D a.$$

The assumption $bb^Da = aa^Da$ yields $a_4 = a - a^2a^D$, which is by (6) exactly the nilpotent part of a. So, $a_1 = a - a_4 = a^2a^D$ is the core part of a and thus a may be presented in the following matrix form:

$$a = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{q\times q}.$$

Recall that $a \leq^* b$, i.e. $a^*a = a^*b$ and $aa^* = ba^*$. Since $q = q^*$, the first equation yields

$$\begin{bmatrix} c_a^* & 0 \\ 0 & n_a^* \end{bmatrix}_{q\times q} \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{q\times q} = \begin{bmatrix} c_a^* & 0 \\ 0 & n_a^* \end{bmatrix}_{q\times q} \begin{bmatrix} c_b & 0 \\ 0 & n_b \end{bmatrix}_{q\times q}.$$

Thus, $n_a^*n_a = n_a^*n_b$. Similarly, the second equation implies $n_an_a^* = n_bn_a^*$. So, $n_a \leq^* n_b$ and therefore $a \leq^{\sharp,*} b$.

(iii)⇒(i) Suppose ${}^\circ(b^k) \subseteq {}^\circ(ab^k)$ and $b^ka = a^{k+1}$. Since $a \leq^{\circledast} b$, we have $c_a \leq^\sharp c_b$ and $a \leq^* b$. So, $a \leq^D b$ and thus by Proposition, there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ such that

$$a = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_3 & c_4 \\ 0 & c_5 & c_6 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & n_2 \end{bmatrix}_{e\times e}$$

where $n_a = c_3+c_4+c_5+c_6$ is the nilpotent part of a and $n_2 = n_b$ is the nilpotent part of b. Here we may without loss of generality assume (see the proof of Theorem 1 ((ii) implies (iii)) in [7]) that $e_1 = c_ac_a^\#$ and $e_2 = c_2c_2^\#$. Since $a, b \in \mathfrak{C}^{\mathcal{A}}$, we may (see Remark) as in the proof of Theorem 3 conclude that e_1, e_2, and $e_3 = 1 - e_1 - e_2$ are self-adjoined idempotents. Let us prove that $c_3 = c_4 = c_5 = 0$. Since $n_b^k = 0 = n_a^k$, we obtain

$$b^ka = \begin{bmatrix} c_1^{k+1} & 0 & 0 \\ 0 & c_2^kc_3 & c_2^kc_4 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \quad \text{and} \quad a^{k+1} = \begin{bmatrix} c_1^{k+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

The equation $b^ka = a^{k+1}$ yields $c_2^kc_3 = 0 = c_2^kc_4$. It follows that $(c_2^\#)^kc_2^kc_3 = 0 = (c_2^\#)^kc_2^kc_4$ and thus $e_2^kc_3 = 0 = e_2^kc_4$. Since $c_3 \in e_2\mathcal{A}e_2$ and $c_4 \in e_2\mathcal{A}e_3$, we may conclude that $c_3 = c_4 = 0$. From

$$e_3b^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_{e\times e} \begin{bmatrix} c_1^k & 0 & 0 \\ 0 & c_2^k & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

we obtain $e_3 \in {}^\circ(b^k) \subseteq {}^\circ(ab^k)$. So,

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_{e\times e} \begin{bmatrix} c_1^{k+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_5c_2^k & 0 \end{bmatrix}_{e\times e}$$

and hence $0 = e_3 c_5 c_2^k$. Note that $c_5 \in e_3 \mathcal{A} e_2$. Thus, $c_5 c_2^k = 0$ and therefore $0 = c_5 c_2^k (c_2^{\#})^k = c_5 e_2 = c_5$. It follows that

$$a = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_6 \end{bmatrix}_{e \times e}$$

where $c_6 = n_a$ is the nilpotent part of a and $c_1 = c_a$ is the core part of a. Finally, let us show that $n_a \leq^* n_b$. Since $a \leq^* b$, we have $a^* a = a^* b$ and $aa^* = ba^*$, and thus by (8),

$$\begin{bmatrix} c_a^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_a^* \end{bmatrix}_{e \times e} \begin{bmatrix} c_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_a \end{bmatrix}_{e \times e} = \begin{bmatrix} c_a^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_a^* \end{bmatrix}_{e \times e} \begin{bmatrix} c_a & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & n_b \end{bmatrix}_{e \times e}.$$

It follows that $n_a^* n_a = n_a^* n_b$. Similarly, we obtain $n_a n_a^* = n_b n_b^*$. So, $n_a \leq^* n_b$ and therefore $a \leq^{\sharp,*} b$.

(iv)$\Rightarrow$(i) We will omit the proof since (by using the matrix formulation (8)) the proof may be very similar to the proof that (iii) implies (i).

(v)$\Rightarrow$(iii) Let $ab^k = b^k a = a^{k+1}$ and suppose $z \in {}^\circ(b^k)$. Then $0 = zb^k a = zab^k$ and therefore ${}^\circ(b^k) \subseteq {}^\circ(ab^k)$. $\square$

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