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COMBINATORIAL AND MODULAR SOLUTIONS OF SOME SEQUENCES WITH LINKS TO A CERTAIN CONFORMAL MAP

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Abstract. If f_n is a free parameter, we give a combinatorial closed form solution of the recursion

$$(n+1)^2 u_{n+1} - f_n u_n - n^2 u_{n-1} = 0, \quad n \geq 1,$$

and a related generating function. This is used to give a solution to the Apéry type sequence

$$r_n n^3 + r_{n-1}\left\{\alpha n^3 - \frac{3\alpha}{2}n^2 + \left\{\frac{\alpha}{2} + 2\theta\right\} n - \theta\right\} + r_{n-2}(n-1)^3 = 0, \quad n \geq 2,$$

for certain parameters α, θ.

We show from another viewpoint two independent solutions of the last recursion related to certain modular forms associated with a problem of conformal mapping: Let $f(\tau)$ be a conformal map of a zero-angle hyperbolic quadrangle to an open half plane with values 0, ρ, 1, ∞ $(0 < \rho < 1)$ at the cusps and define $t = t(\tau) := \frac{1}{\rho} f(\tau) \frac{f(\tau)-\rho}{f(\tau)-1}$. Then the function

$$E(\tau) = \frac{1}{2\pi i} \frac{f'(\tau)}{f(\tau)} \frac{1}{1 - \frac{f(\tau)}{\rho}}$$

is a solution, as a generating function in the variable t, of the above recurrence. In other words, $E(\tau) = r_0 + r_1 t + r_2 t^2 + \dots$, where $r_0 = 1$, $r_1 = -\theta$, $\alpha = 2 - \frac{4}{\rho}$.

1. Introduction

Let $P(n)$ be the third degree polynomial in n defined by

$$P(n) = \alpha n^3 + \frac{3\alpha}{2}n^2 + \left\{\frac{\alpha}{2} + 2\theta\right\} n + \theta, \tag{1}$$

with α, θ complex or real numbers.

One should notice that

$$P(n-1) = -P(-n) = \alpha n^3 - \frac{3\alpha}{2}n^2 + \left\{\frac{\alpha}{2} + 2\theta\right\} n - \theta.$$

This paper is devoted to the study of sequences $(r) = (r_0, r_1, r_2, \dots)$ defined by

$$r_n n^3 + r_{n-1} P(n-1) + r_{n-2}(n-1)^3 = 0, \quad n \geq 2. \tag{2}$$

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Case	$P(n-1)$	(a)
1	$-98n^3+147n^2-147n+49$	$(1,49,2701,171549,11951001,\dots)$
2	$-158n^3+237n^2-197n+59$	$(1,59,4801,473859,52189101,\dots)$
3	$-222n^3+333n^2-253n+71$	$(1,71,7801,1064671,163373801,\dots)$
4	$-222n^3+333n^2-221n+55$	$(1,55,5713,762775,115712941,\dots)$
5	$-222n^3+333n^2-333n+111$	$(1,111,13861,1994411,314768301,\dots)$
6	$-286n^3+429n^2-333n+95$	$(1,95,13573,2395355,474461701,\dots)$
7	$-318n^3+477n^2-189n+15$	$(1,15,1873,336095,70689441,\dots)$
8	$-322n^3+483n^2-291n+65$	$(1,65,9433,1800985,393370541,\dots)$
9	$-382n^3+573n^2-253n+31$	$(1,31,4801,1046431,265873201,\dots)$
10	$-482n^3+723n^2-603n+181$	$(1,181,45001,13558581,4557147201,\dots)$
11	$-898n^3+1347n^2-459n+5$	$(1,5,1693,846185,499129441,\dots)$
12	$-1890n^3+2835n^2-1195n+125$	$(1,125,94453,101362025,\dots)$

TABLE 1.

We will be interested in the solutions $(a),(b)$ of the above recurrence starting with $a_0=1$, $a_1=-\theta$ and $b_0=0$, $b_1=1$. Of course any solution (r) is a linear combination of (a) and (b).

Example 1. If $P(n-1)=-34n^3+51n^2-27n+5$, that is, $\alpha=-34$, $\theta=-5$, then one gets Apéry's famous sequence $a_n=\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2$. Here b_n is the more complicated expression

$$b_n=\frac{1}{6}\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2\left\{\sum_{m=1}^{n}\frac{1}{m^3}+\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}\right\};$$

see [15]. Apéry used these sequences to prove the irrationality of $\zeta(3)$.

Example 2. One can find many sequences $(a_0,a_1,a_2,\dots)$ solutions of the above recurrence having the notorious property of being integers for a long string before becoming rational numbers. A few examples are given in Table 1; in all cases α,θ are real and negative. Cases 1 and 11 (expanded) in that table are respectively

$$1,\ 49,\ 2701,\ 171549,\ 11951001,\ 885337929,\ 68479711021,\ 5468036535229,$$
$$447382621294021,\ \frac{335828273871136861}{9},\ \frac{28448771913258275929}{9},\dots$$

and

$$1,\ 5,\ 1693,\ 846185,\ 499129441,\ 322896384725,\ 221579880716125,$$
$$158412615229470425,\ 116716224422246465125,\ 88003121433329789819225,$$
$$67576191815704841837662513,\ \frac{63723961714863745985643924724485}{121},\dots.$$

This paper is, in some sense, an attempt to find the solutions $(a),(b)$ of the recurrence (2) in closed form.

Our main results are Theorems 1–4, which we briefly discuss. We exhibit the solutions of (2) from two different viewpoints. Our first point of view is combinatorial and is developed in sections 2, 3 and 5. Firstly, in Theorem 1, which we believe is interesting in its own right, we solve the easier recursion $(n+1)^2 u_{n+1} - f_n u_n - n^2 u_{n-1} = 0$, where f_n is a free parameter. This result can be seen as a variant of a certain recursion given in an interesting paper of A. Schmidt [13] and should be compared to it. To solve the recursion we need to introduce certain combinatorial numbers linked to the Stirling numbers of first and second kind. In section 3, namely Theorem 2, we show how a particular case of Theorem 1 can be used to solve in closed form the recursion (2) and is, in some sense, a combinatorial solution of it. This solves also a particular case of Heun's equation. In Theorem 4 of section 5 we present a generating function related to the combinatorial numbers appearing in Theorem 1.

Our second point of view is a modular one: F. Beukers showed the connection of Apéry's sequences, that is those of Example 1, with modular forms. Section 4 is inspired by his remarkable paper [3] and this section can be read almost independently from sections 2 and 3. Here we begin with a problem of a certain conformal mapping: describe the function $f(\tau)$ mapping a hyperbolic quadrangle, having angles all equal to zero at all four cusps, to a half plane. As shown in Theorem 3, we construct the solutions of the recursion (2) as a generating function of certain modular forms attached to $f(\tau)$ with certain parameters α, θ depending on this last function.

2. A second order recursion

The aim of this section is to prove Theorem 1, which solves, in a combinatorial way, a second order recursion. It is inspired by Asmus Schmidt's paper [13] and it could be seen as a generalization of Example 2 in [14]. We need first some definitions.

We write $s(i,k)$ for the Stirling numbers of first kind, which may be defined by the binomial

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-(i-1))}{i!} = \sum_{k=0}^{i} s(i,k)\frac{x^k}{i!}.$$

Recall that $s(j,j)=1$ if $j \geq 0$, $s(i,0)=0$ if $1 \leq i$, that is, $\binom{x}{0}=1$.

Definition. We will write for short, if $0 \leq k \leq n$,

$$d_{n,k} := \sum_{i=k}^{n} \frac{s(i,k)}{i!}\binom{n}{i}\binom{n+i}{i}. \tag{3}$$

By definition we put $d_{n,k}=0$ if $0 \leq n < k$ and $d_{n,-1}=0$ if $0 \leq n$.

Observe that $d_{n,0} = 1$ for all $n \geq 0$. One has

$$\begin{aligned}
&d_{0,0} = 1,\\
&d_{1,0} = 1, \quad d_{1,1} = 2,\\
&d_{2,0} = 1, \quad d_{2,1} = 3, \quad d_{2,2} = 3,\\
&d_{3,0} = 1, \quad d_{3,1} = 11/3, \quad d_{3,2} = 5, \quad d_{3,3} = 10/3,\\
&d_{4,0} = 1, \quad d_{4,1} = 25/6, \quad d_{4,2} = 85/12, \quad d_{4,3} = 35/6, \quad d_{4,4} = 35/12.
\end{aligned}$$

We write for short

$$\beta_{i,j,k} := \sum_{u=i}^{\infty}\sum_{\ell=j}^{\infty} \binom{u+\ell}{k}\binom{k}{u}\binom{k}{\ell}\frac{s(u,i)}{u!}\frac{s(\ell,j)}{\ell!}.$$

Observe that this is a finite sum because $\binom{k}{u} = 0$ if $k < u$. Note that $\beta_{i,j,k} = 0$ if $k < i$ or $k < j$.

Definition. We define $\alpha_{i,j,k}$ by

$$\sum_{r=0}^{k} s(k,r)\alpha_{i,j,r} = k!\beta_{i,j,k}.$$

Recall the well known fact that Stirling matrices are inverse to each other. This yields that the last equation can be inverted to give

$$\sum_{r=0}^{k} S(k,r)r!\beta_{i,j,r} = \alpha_{i,j,k}$$

where $S(i,j)$ are the Stirling numbers of second kind. Recall that these numbers may be defined by $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$, where $(x)_n = x(x-1)\cdots(x-n+1)$ (here $(x)_0 = 1$) is the falling factorial.

Thus the last equation is

$$\begin{pmatrix}\alpha_{i,j,0}\\ \vdots \\ \alpha_{i,j,k}\end{pmatrix} = M\begin{pmatrix}0!\beta_{i,j,0}\\ \vdots \\ k!\beta_{i,j,k}\end{pmatrix},$$

where M is the square matrix with $k+1$ rows defined by

$$\begin{pmatrix} S(0,0) & 0 & 0 & \cdots & 0 \\ S(1,0) & S(1,1) & 0 & \cdots & 0 \\ S(2,0) & S(2,1) & S(2,2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S(k,0) & S(k,1) & S(k,2) & \cdots & S(k,k) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 7 & 6 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & S(k,1) & S(k,2) & S(k,3) & S(k,4) & \cdots & 1 \end{pmatrix}.$$

Definition. We define the real numbers δ_k by the equation

$$\sum_{k=0}^{n} d_{n,k}\delta_k = 0,$$

for $n \geq 1$, and by definition $\delta_0 = 1$.

From (3) one sees that $d_{n,n} = \frac{1}{n!}\binom{2n}{n}$, thus δ_k is well defined. One computes

$$\delta_0 = 1, \quad \delta_1 = -1/2, \quad \delta_2 = 1/6, \quad \delta_3 = 0, \quad \delta_4 = -1/30, \quad \delta_5 = 0,$$
$$\delta_6 = 1/42, \quad \delta_7 = 0, \quad \delta_8 = -1/30, \quad \delta_9 = 0, \quad \delta_{10} = 5/66,$$
$$\delta_{11} = 0, \quad \delta_{12} = -691/2730, \quad \delta_{13} = 0, \quad \delta_{14} = 7/6.$$

Our objective is to prove the following result.

Theorem 1. *Let $(x_0, x_1, \ldots, x_j, \ldots)$ be any sequence of complex numbers. Let*

$$f_n := (2n+1)\Big(1 + 2\sum_{j=0}^{n} x_j d_{n,j}\Big),$$

and consider sequences $(u) = (u_0, u_1, \ldots)$ satisfying the recursion formula

$$(n+1)^2 u_{n+1} - f_n u_n - n^2 u_{n-1} = 0.$$

Then the recursion has two independent solutions $(p), (q)$ as follows:

The element p_n is represented as

$$p_n = \sum_{k=0}^{n} c_k d_{n,k},$$

where

$$c_0 = 1,$$
$$c_{k+1} = \sum_{i=0}^{k}\sum_{j=0}^{k} \alpha_{i,j,k} x_j c_i.$$

The element q_n is represented as

$$q_n = \sum_{k=0}^{n} e_k d_{n,k},$$

where

$$e_0 = 0,$$
$$e_{k+1} = \sum_{i=0}^{k}\sum_{j=0}^{k} \alpha_{i,j,k} x_j e_i + \delta_k.$$

Our proof will follow from some lemmas. Firstly we have the following result of A. Schmidt as given in [14, Example 2, p. 366].

Lemma 1. *Set $g_n = g_n(x) := \sum_{i=0}^{n} \binom{x}{i}\binom{n}{i}\binom{n+i}{i} = \sum_{k=0}^{n} d_{n,k}\, x^k$. If $n \geq 0$ then*

$$(n+1)^2 g_{n+1} - (2n+1)(1+2x)g_n - n^2 g_{n-1} = 0.$$

We will need the following lemma.

Lemma 2. *For $0 \leq k \leq n+1$,*

$$(n+1)^2 d_{n+1,k} - (2n+1)d_{n,k} - n^2 d_{n-1,k} = (4n+2)d_{n,k-1}.$$

Proof. The identity of the last lemma can be written as

$$(n+1)^2 g_{n+1}(x) - (2n+1)g_n(x) - n^2 g_{n-1}(x) = (4n+2)x g_n(x).$$

Taking out the coefficient of x^k in this recurrence one gets the desired identity. $\square$

Lemma 3. *The following identity holds:*

$$\binom{n+\ell}{\ell}\binom{n}{\ell}\binom{n+u}{u}\binom{n}{u} = \sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}\binom{u+\ell}{k}\binom{k}{u}\binom{k}{\ell}.$$

Proof. This is basically Lemma 1 of [13] which uses the *Pfaff-Saalschütz* identity. See page 196 of that paper. $\square$

Lemma 4. *Let $\alpha_{i,j,k}$ be the real numbers defined at the beginning of this section. Then $\alpha_{i,j,k} = 0$ if $k < i$ or $k < j$ and $\alpha_{i,j,k} = \alpha_{j,i,k}$. Also*

$$d_{n,i} d_{n,j} = \sum_{k=0}^{n} \alpha_{i,j,k} d_{n,k}.$$

Proof. Recall that

$$\beta_{i,j,k} = \sum_{u=i}^{\infty}\sum_{\ell=j}^{\infty}\binom{u+\ell}{k}\binom{k}{u}\binom{k}{\ell}\frac{s(u,i)}{u!}\frac{s(\ell,j)}{\ell!}.$$

Thus $\beta_{i,j,k}=\beta_{j,i,k}$ and this implies $\alpha_{i,j,k}=\alpha_{j,i,k}$. Also $\beta_{i,j,k}=0$ if $i>k$ or $j>k$ which implies $\alpha_{i,j,k}=0$ if $i>k$ or $j>k$.

Next we prove the stated identity. Firstly observe that if $k\le n$ one may write $\beta_{i,j,k}=\sum_{u=i}^{n}\sum_{\ell=j}^{n}\binom{u+\ell}{k}\binom{k}{u}\binom{k}{\ell}\frac{s(u,i)}{u!}\frac{s(\ell,j)}{\ell!}$.

Now multiply the identity of Lemma 3 by $\frac{s(u,i)}{u!}\frac{s(\ell,j)}{\ell!}$ and add from $u=i$ up to n and $\ell=j$ up to n. The left-hand side gives

$$\sum_{u=i}^{n}\sum_{\ell=j}^{n}\frac{s(u,i)}{u!}\frac{s(\ell,j)}{\ell!}\binom{n+\ell}{\ell}\binom{n}{\ell}\binom{n+u}{u}\binom{n}{u} = d_{n,i}d_{n,j}$$

using the definition of $d_{n,k}$, while the right-hand side is equal to

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}\sum_{u=i}^{n}\sum_{\ell=j}^{n}&\binom{u+\ell}{k}\binom{k}{u}\binom{k}{\ell}\frac{s(u,i)}{u!}\frac{s(\ell,j)}{\ell!}\\ &=\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}\beta_{i,j,k},\end{aligned}$$

due to the definition of $\beta_{i,j,k}$ and because $k\le n$. That is, we have proved that

$$d_{n,i}d_{n,j}=\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}\beta_{i,j,k}.$$

By definition of $\alpha_{i,j,k}$ one has that $\sum_{r=0}^{k}\frac{s(k,r)}{k!}\alpha_{i,j,r}=\beta_{i,j,k}$. Therefore

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}\beta_{i,j,k}&=\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}\sum_{r=0}^{k}\frac{s(k,r)}{k!}\alpha_{i,j,r}\\ &=\sum_{r=0}^{n}\alpha_{i,j,r}\sum_{u=r}^{n}\frac{s(u,r)}{u!}\binom{n+u}{u}\binom{n}{u}=\sum_{r=0}^{n}\alpha_{i,j,r}d_{n,r},\end{aligned}$$

which proves the lemma. □

Finally we prove Theorem 1.

Proof of Theorem 1. Set

$$\tilde{r}_n := (n+1)^2p_{n+1}-f_np_n-n^2p_{n-1}.$$

Our aim is to prove that $\tilde{r}_n=0$ for all $n\ge 1$.

Writing the definition of p_n without any explicit c_k one has that $\tilde{r}_n$ is equal to

$$(n+1)^2\sum_{k=0}^{n+1}c_kd_{n+1,k}-(2n+1)\Big(1+2\sum_{j=0}^{n}x_jd_{n,j}\Big)\sum_{k=0}^{n}c_kd_{n,k}-n^2\sum_{k=0}^{n-1}c_kd_{n-1,k}.$$

We collect the terms with c_k alone. Remembering that $d_{n,n+1} = d_{n-1,n+1} = d_{n-1,n} = 0$, this can be rearranged to give that $\tilde{r}_n$ is equal to

$$\sum_{k=0}^{n+1} c_k \left\{(n+1)^2 d_{n+1,k} - (2n+1)d_{n,k} - n^2 d_{n-1,k}\right\} - (4n+2)\Big(\sum_{j=0}^{n} x_j d_{n,j}\Big)\sum_{k=0}^{n} c_k d_{n,k}$$
$$= (4n+2)\Big\{\sum_{k=0}^{n+1} c_k d_{n,k-1} - \Big(\sum_{j=0}^{n} x_j d_{n,j}\Big)\sum_{k=0}^{n} c_k d_{n,k}\Big\}$$
$$= (4n+2)\Big\{\sum_{k=0}^{n} c_{k+1} d_{n,k} - \Big(\sum_{j=0}^{n} x_j d_{n,j}\Big)\sum_{k=0}^{n} c_k d_{n,k}\Big\},$$

where we have used Lemma 2 and the fact that $d_{n,-1} = 0$. Putting the definition of c_{k+1} in the first sum one gets that $\tilde{r}_n$ is equal to

$$(4n+2)\Big\{\sum_{k=0}^{n} d_{n,k} \sum_{i=0}^{k}\sum_{j=0}^{k} \alpha_{i,j,k} x_j c_i - \Big(\sum_{j=0}^{n} x_j d_{n,j}\Big)\sum_{k=0}^{n} c_k d_{n,k}\Big\}.$$

By Lemma 4, the inner double sum in the first term could be summed up to n (in both summands i, j) instead of k because $\alpha_{i,j,k} = 0$ if $i, j > k$. Changing the order of summation and using the identity of Lemma 4 yields

$$(4n+2)\Big\{\sum_{i=0}^{n}\sum_{j=0}^{n} x_j c_i \sum_{k=0}^{n} a_{i,j,k} d_{n,k} - \Big(\sum_{j=0}^{n} x_j d_{n,j}\Big)\sum_{k=0}^{n} c_k d_{n,k}\Big\}$$
$$= (4n+2)\Big\{\sum_{i=0}^{n}\sum_{j=0}^{n} x_j c_i d_{n,i} d_{n,j} - \Big(\sum_{j=0}^{n} x_j d_{n,j}\Big)\sum_{k=0}^{n} c_k d_{n,k}\Big\} = 0.$$

If one puts (q) then one obtains, with exactly the same proof, the additional term

$$(4n+2)\Big\{\sum_{k=0}^{n} d_{n,k}\delta_k\Big\},$$

which is zero if $1 \leq n$ by definition of δ_k. □

We record the first values of $\alpha_{i,j,k}$; recall that $\alpha_{i,j,k} = \alpha_{j,i,k}$. One has

$$\alpha_{0,0,0} = 1,$$
$$\alpha_{0,0,1} = 0, \quad \alpha_{0,1,1} = 1, \quad \alpha_{1,1,1} = 2,$$
$$\alpha_{0,0,2} = 0, \quad \alpha_{0,1,2} = 0, \quad \alpha_{1,1,2} = 1, \quad \alpha_{0,2,2} = 1, \quad \alpha_{2,1,2} = \alpha_{2,2,2} = 3.$$

3. Connection with Apéry type sequences and Heun's equation

Our aim is to prove the following theorem which solves, in a certain closed form, the recursion (2).

Theorem 2. *Let θ be a complex number, α real and $\alpha < -2$. Set*

$$B_0 := \frac{i(\alpha-6)}{2\sqrt{2-\alpha}}, \quad K_0 := \frac{i(\theta-1)}{\sqrt{2-\alpha}}$$

and assume that $(x_0, x_1, x_2, \ldots)$ *is a complex sequence such that*

$$(-B_0 n^2 - B_0 n - K_0) = (2n+1)\Big(1 + 2\sum_{j=0}^{n} x_j d_{n,j}\Big),$$

for all $n \geq 1$. *Also define* $A_1 := \frac{2-\alpha}{4}$ *and* $B_1 := \frac{1-\theta}{2}$.

Then there exists $(u) = (u_0, u_1, u_2, \ldots)$ *which is a linear combination of* $(p), (q)$, *the solutions given in Theorem 1, such that if one writes* $U(x) = u_0 + u_1 x + u_2 x^2 + \ldots$, *then*

i) $V(x) := U(-\frac{\sqrt{2-\alpha}}{2}x)$ *is a holomorphic solution around* $x = 0$ *of*

$$x(x-1)(A_1 x - 1)V'' + (3A_1 x^2 - 2(A_1+1)x + 1)V' + (A_1 x - B_1)V = 0. \tag{4}$$

ii) The coefficients of

$$R(t) := (1-x)V^2(x) = r_0 + r_1 t + r_2 t^2 + \cdots,$$

where

$$t = \frac{x(A_1 x - 1)}{x - 1},$$

that is

$$x = \frac{-\sqrt{1 + \alpha t + t^2} + 1 + t}{2A_1},$$

satisfy the recursion (2). *Also,*

$$r_0 = u_0^2, \quad r_1 = -u_0(u_0 + u_1\sqrt{2-\alpha}).$$

Note: The above equation (4) is a particular case of Heun's equation and is connected to the problem of mapping the half plane onto a hyperbolic quadrangle.

We first prove some lemmas.

Lemma 5. *Let* B_0, K_0 *be complex numbers and let* $(x_0, x_1, x_2, \ldots)$ *be a complex sequence such that*

$$(-B_0 n^2 - B_0 n - K_0) = (2n+1)\Big(1 + 2\sum_{j=0}^{n} x_j d_{n,j}\Big),$$

for all $n \geq 1$. *Let* $(u) = (u_0, u_1, u_2, \ldots)$ *be a solution of the recursion*

$$(n+1)^2 u_{n+1} - (-B_0 n^2 - B_0 n - K_0)u_n - n^2 u_{n-1} = 0, \quad n \geq 1,$$
$$K_0 u_0 + u_1 = 0.$$

Then (u) *is a linear combination of* (p) *and* (q) *of Theorem 1.*

Proof. This lemma is immediate observing that the hypothesis gives $f_n = (-B_0 n^2 - B_0 n - K_0)$ in Theorem 1. □

Lemma 6. *Let* B_0, K_0 *be complex numbers. Set*

$$B_1 := \frac{K_0(-B_0 \pm \sqrt{4 + B_0^2})}{2}, \quad A_1 := -\frac{(B_0 \mp \sqrt{4 + B_0^2})^2}{4}.$$

The sequence of complex numbers $(u) = (u_0, u_1, u_2, \dots)$ *is a solution of the recursion*

$$(n+1)^2 u_{n+1} - (-B_0 n^2 - B_0 n - K_0) u_n - n^2 u_{n-1} = 0, \ n \geq 1$$
$$K_0 u_0 + u_1 = 0,$$

if and only if $V(x) := U\left(i\left\{\frac{-B_0 \pm \sqrt{4+B_0^2}}{2}\right\} x\right)$ *(here* $U(x) := u_0 + u_1 x + u_2 x^2 + \cdots$*) is a holomorphic solution at* $x = 0$ *of*

$$x(x-1)(A_1 x - 1)V'' + (3A_1 x^2 - 2(A_1+1)x + 1)V' + (A_1 x - B_1)V = 0.$$

Proof. The sequence of complex numbers $(u) = (u_0, u_1, u_2, \dots)$ satisfies the conditions of Lemma 6 if and only if $U(x) = u_0 + u_1 x + u_2 x^2 + \cdots$ is a holomorphic function at $x = 0$ which satisfies

$$(-x^3 + B_0 x^2 + x)U'' + (-3x^2 + 2B_0 x + 1)U' + (-x + K_0)U = 0.$$

Now $V(x) = U\left(i\left\{\frac{-B_0 \pm \sqrt{4+B_0^2}}{2}\right\} x\right)$ gives the required result after some algebraic manipulation. $\square$

Definition. We define the linear operators L, L_1 as follows:

$$\begin{aligned} L &:= \left\{t^4 + \alpha t^3 + t^2\right\} \frac{d^3}{dt^3} + \left\{6t^3 + \frac{9}{2}\alpha t^2 + 3t\right\} \frac{d^2}{dt^2} \\ &\quad + \left\{7t^2 + (3\alpha + 2\theta)t + 1\right\} \frac{d}{dt} + (t + \theta), \qquad (5) \\ L_1 &:= \left\{t^4 + \alpha t^3 + t^2\right\} \frac{d^2}{dt^2} + \left\{2t^3 + \frac{3}{2}\alpha t^2 + t\right\} \frac{d}{dt} + \frac{\left\{t^2 + 2\theta t\right\}}{4}. \end{aligned}$$

For the next two lemmas we will write for short $L_1 = P_1(t)\frac{d^2}{dt^2} + P_2(t)\frac{d}{dt} + \frac{P_3(t)}{4}$.

Lemma 7. *Set* $A_1 := \frac{2-\alpha}{4}$ *and* $B_1 := \frac{1-\theta}{2}$. *Let* $V(x)$ *be a solution of*

$$x(x-1)(A_1 x - 1)V'' + (3A_1 x^2 - 2(A_1+1)x + 1)V' + (A_1 x - B_1)V = 0.$$

Set $W(t) := \sqrt{1-x}\,V(x)$*; here* $t = \frac{x(A_1 x - 1)}{x-1}$*, that is,* $x = \frac{-\sqrt{1+\alpha t + t^2} + 1 + t}{2A_1}$*. Then*

$$L_1 W(t) = 0.$$

Proof. If all the functions involved are smooth enough, one has the following general formula. Set $W(t) := g(x)V(x)$ where $t := f(x)$. Then $W(t)$ satisfies (here $\dot{}$ denotes the derivative with respect to t)

$$P_1(t)\ddot{W}(t) + P_2(t)\dot{W}(t) + \frac{P_3(t)}{4} W(t) = H(t)$$

if and only if $V(x)$ satisfies (here $'$ is the derivative with respect to x)

$$\begin{aligned}&V''(x)\frac{P_1(f(x))g(x)}{f'(x)^2}\\ &\quad+V'(x)\Big\{\frac{P_1(f(x))2g'(x)}{f'(x)^2}-\frac{P_1(f(x))f''(x)g(x)}{f'(x)^3}+\frac{P_2(f(x))g(x)}{f'(x)}\Big\}\\ &\quad+V(x)\Big\{\frac{P_1(f(x))g''(x)}{f'(x)^2}-\frac{P_1(f(x))f''(x)g'(x)}{f'(x)^3}+\frac{P_2(f(x))g'(x)}{f'(x)}+\frac{P_3(f(x))g(x)}{4}\Big\}\\ &=H(f(x)).\end{aligned}$$

Hint: Just put the derivatives of $W(t=f(x)):=g(x)V(x)$ with respect to x into one equation to get the other. We note that this is a general formula valid for smooth functions P_i.

Now take as P_i the polynomials defined by the linear operator L_1, $g(x)=\sqrt{1-x}$, $t=f(x)=\frac{x(A_1x-1)}{x-1}$ and $H(t)=0$. A tedious routine check gives the result. □

Lemma 8. *Let $W(t)$ be a function such that*

$$L_1W(t)=H(t).$$

Then

$$L\left\{W(t)^2\right\}=6W'(t)H(t)+2W(t)H'(t).$$

In particular, if $L_1W(t)=0$ then $L\left\{W(t)^2\right\}=0$.

Proof. We write for short $P_i=P_i(t)$, $W=W(t)$ and $P_4(t):=-t-\theta$. Then

$$\begin{aligned}&\frac{d}{dt}\left\{\left(P_1\frac{d^2}{dt^2}+P_2\frac{d}{dt}+P_3\right)W^2\right\}+P_4W^2\\ &\qquad=\left\{P_1\frac{d^3}{dt^3}+(P_1'+P_2)\frac{d^2}{dt^2}+(P_2'+P_3)\frac{d}{dt}+(P_3'+P_4)\right\}W^2=L\{W^2\},\end{aligned}$$

where the last equality follows checking that $P_1'+P_2=6t^3+\frac{9}{2}\alpha t^2+3t$ and so on.

Also,

$$\begin{aligned}&\frac{d}{dt}\left\{\left(P_1\frac{d^2}{dt^2}+P_2\frac{d}{dt}+P_3\right)W^2\right\}+P_4W^2\\ &\qquad=\frac{d}{dt}\left\{P_1(2W'^2+2WW'')+P_22WW'+P_3W^2\right\}+P_4W^2\\ &\qquad=\frac{d}{dt}\left\{P_12W'^2+\frac{P_3}{2}W^2+2WH\right\}+P_4W^2,\end{aligned}$$

where we have used in the last equality the hypothesis $L_1W(t)=H(t)$, that is, $P_1W''+P_2W'=-P_3W/4+H$, and written $H=H(t)$ for short. The last formula is equal to

$$P_14W'W''+2P_1'W'^2+P_3WW'+\frac{P_3'}{2}W^2+2W'H+2WH'+P_4W^2$$

which, noticing that $\frac{P_3'}{2}+P_4=0$, equals

$$4W'\left\{P_1W''+\frac{P_1'}{2}W'+\frac{P_3}{4}W\right\}+2W'H+2WH'$$
$$=4W'\left\{L_1W\right\}+2W'H+2WH'=6W'H+2WH',$$

and the lemma follows. ☐

Lemma 9. *Assume $P(n)$ is the polynomial defined by* (1). *Then a holomorphic function around zero $R(t)=r_0+r_1t+r_2t^2+\cdots$ satisfies*

$$LR(t)=r_1+\theta r_0$$

if and only if the coefficients r_i satisfy the recurrence

$$r_nn^3+r_{n-1}P(n-1)+r_{n-2}(n-1)^3=0,$$

for $n\geq 2$ with inital conditions r_0,r_1.

Proof. After grouping the coefficients of t^n in the operator L one obtains the above recursion. ☐

Finally we give the proof of Theorem 2.

Proof of Theorem 2. Assuming that $\alpha<-2$ and putting $B_0:=\frac{i(\alpha-6)}{2\sqrt{2-\alpha}}$, $K_0:=\frac{i(\theta-1)}{\sqrt{2-\alpha}}$ then one has $+\sqrt{4+B_0^2}=-i\frac{(2+\alpha)}{2\sqrt{2-\alpha}}$ and $-B_0+\sqrt{4+B_0^2}=i\sqrt{2-\alpha}$. In the notation of Lemma 6 this gives $A_1=\frac{2-\alpha}{4}$ $B_1=\frac{1-\theta}{2}$.

By Lemmas 5 and 6 one has that

$$V(x)=(1-x)U\Big(-\frac{\sqrt{2-\alpha}}{2}x\Big)$$

satisfies part (i) of the theorem.

Part (ii) of the theorem follows from Lemmas 7, 8 and 9. ☐

4. Connection with modular forms and conformal mapping

In this section we start anew and we connect our sequences with a certain conformal mapping $f(\tau)$ described below and certain modular forms $E(\tau),E(\tau)F_0(\tau)$ related to $f(\tau)$. We show in Theorem 3 that, choosing constants α,θ in (1) depending on $f(\tau)$, the coefficients of these modular forms (viewed in an appropriate variable) are the sought sequences $(a),(b)$ solutions of (2) described in the introduction.

The function $f(\tau)$ is described as follows. Let Q_0 be the open region in the upper open complex plane H, described by the variable $\tau=\tau_1+i\tau_2$ with both $\tau_1,\tau_2\in\mathbb{R}$ (i.e., the τ-plane) surrounded by the lines $i\tau_2$ and $1/2+i\tau_2$ with $0\leq\tau_2$ and the (half) circles $\mathcal{C}_2,\mathcal{C}_1$ whose centers are real, with radii r_2 and $r_1=1/4-r_2$ respectively, $0<r_2<1/4$. See figure 1. This region Q_0 is a hyperbolic quadrangle whose interior angles are all zero and whose vertices are 0, $2r_2$, $1/2$, $i\infty$.

By the Riemann mapping theorem there exists a conformal mapping $f(\tau)$ of this region Q_0 onto the upper open half plane which can be extended to the boundary

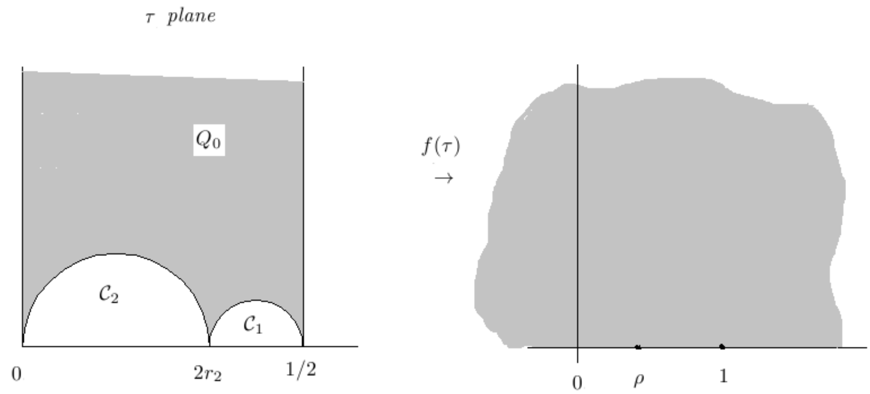

FIGURE 1. Q_0 is mapped conformally onto the upper open half plane by the function $f(\tau)$.

of the region. Moreover, by applying a bilineal map from the upper half plane into itself, one may normalize this mapping sending $i\infty \to 0$, $0 \to \rho$, $2r_2 \to 1$, $1/2 \to \infty$ with $0 < \rho < 1$. As in the construction of the modular invariant one may apply the Schwarz reflection principle an infinite number of times to the sides to get a function which is an extension of $f(\tau)$ which we call in the same way. This function is the Hauptmodul of the discrete group generated by the bilinear transformations (not necessarily related to the modular group):

$$T_1\tau = \tau + 1, \quad T_2\tau = \frac{\tau}{\frac{\tau}{r_2} + 1}, \quad T_3\tau = \frac{\tau\frac{1+4r_2}{1-4r_2} + \frac{4r_2}{1-4r_2}}{\tau\frac{4}{1-4r_2} + \frac{1+4r_2}{1-4r_2}}.$$

Lemma 10. *Under the above construction one has that the function $f(\tau) : H \to \mathbb{C}$ is a holomorphic function, mapping Q_0 conformally onto the upper plane, where*

$$f(T_i\tau) = f(\tau) \tag{6}$$

for $i = 1, 2, 3$ and $(0 < \rho < 1)$

$$f(i\infty) = 0, \quad f(0) = \rho, \quad f(\pm 2r_2) = 1, \quad f(\pm 1/2) = \infty.$$

Also, $f(\tau)$ takes real values on the lines $i\tau_2$ and $1/2 + i\tau_2$ $(0 \leq \tau_2)$ and the half circles $\mathcal{C}_2$, $\mathcal{C}_1$. Moreover, it has the mirror symmetry

$$f(-\bar{\tau}) = \overline{f(\tau)},$$

and $f(\tau) \neq 0, \rho, 1$ in the open upper half plane H. See figure 1.

The last two statements follow from the construction of $f(\tau)$.

Writing $q = e^{2\pi i\tau}$ one may write $f(\tau)$ as a Taylor series in q with radius of convergence 1, because $f(T_1\tau) = f(\tau + 1) = f(\tau)$. Such series will be of the form $e_0 q + O(q^2)$ with $e_0 > 0$, because $f(\tau)$ is univalent at $\tau = i\infty$ (that is at $q = 0$) and $f(\tau)$ is real and increasing if τ moves from $i\infty$ to $i0$ (on the line $i\tau_2$) or if τ

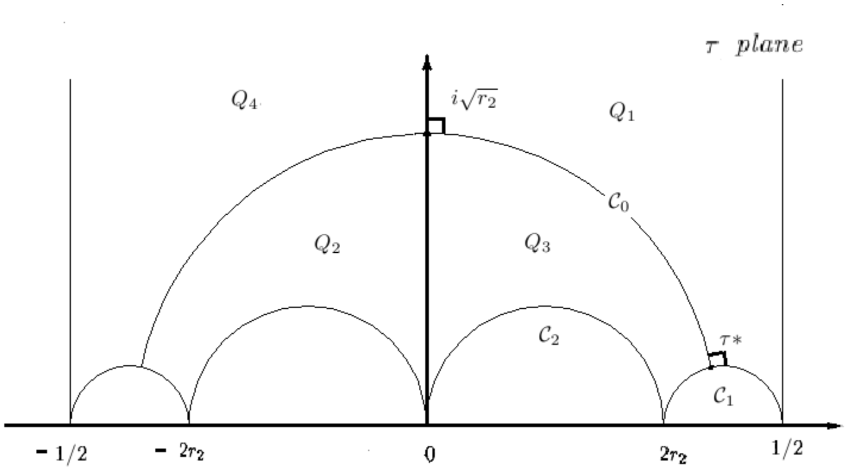

FIGURE 2. The regions Q_1, Q_2, Q_3, Q_4.

moves from $1/2$ to $1/2 + i\infty$ (on the line $1/2 + i\tau_2$). All this gives that $f(\tau)$ is real and increasing at $q = 0$ if q is real and then forces that all the coefficients of its Taylor series must be real. Moreover one can see that there exists $e_n \in \mathbb{R}$, $e_0 > 0$ such that (see [6])

$$f(\tau) = e_0\, q \prod_{n=1}^{\infty} (1 - q^n)^{e_n} = e_0 q - e_0 e_1 q^2 + e_0 \frac{(-e_1 + e_1^2 - 2e_2)}{2} q^3 + e_0 \frac{(3e_1^2 - e_1^3 + e_1(-2 + 6e_2) - 6e_3)}{6} q^4 + \cdots ,$$

around $q = 0$.

In this section our aim is to show how $f(\tau)$, e_i ρ and the radius r_2 are related to the solutions $(a), (b)$ of the recursion (2).

If we choose the circle $\mathcal{C}_0$ as the circle centered at zero of radius $\sqrt{r}_2$, then $\mathcal{C}_0$ is orthogonal to the circle $\mathcal{C}_1$ (defined at the begining of this section), see figure 2. We define the open regions Q_i, $i = 1, 2, 3, 4$ in the same figure. For example, Q_1 is the exterior of the circles $\mathcal{C}_0$, $\mathcal{C}_1$, surrounded by the lines $i\tau_2$ and $1/2 + i\tau_2$; it is a hyperbolic quadrangle with angles $0, \frac{\pi}{2}, \frac{\pi}{2}, 0$.

Lemma 11. *If $\tau \in H$ then*

$$f\left(-\frac{r_2}{\tau}\right) = \frac{f(\tau) - \rho}{f(\tau) - 1}.$$

Proof. This follows from the formulae (6). Indeed,

$$f\Big(-\frac{r_2}{T_1 \tau}\Big) = f\Big(-\frac{r_2}{\tau + 1}\Big) = f\Big(\frac{-r_2/\tau}{1 + 1/\tau}\Big) = f\Big(T_4(-\frac{r_2}{\tau})\Big)$$

and $T_4 := \frac{\tau}{-\frac{\tau}{r_2}+1}$ is the inverse of T_2. Therefore by (6) one has $f(T_4(-\frac{r_2}{\tau})) = f(-\frac{r_2}{\tau})$, that is, $f(-\frac{r_2}{T_1\tau}) = f(-\frac{r_2}{\tau})$. In the same way one proves that, for $i = 1, 2, 3$,

$$f\Big(-\frac{r_2}{T_i\tau}\Big) = f\Big(-\frac{r_2}{\tau}\Big).$$

Observe that the function $-r_2/\tau$ interchanges conformally Q_1 with Q_2 and Q_3 with Q_4. As $f(\tau)$ is a Hauptmodul for the group generated by T_i then $f(-\frac{r_2}{\tau})$ must be a Hauptmodul also. The lemma follows by matching the values at the cusps. □

Some explicit known examples are the following:

i) If $r_2 = 1/8$ then $\rho = 1/2$ and $f(\tau) = \frac{1}{2}\left\{1 - \sqrt{1 - 16q\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n+1}}\right)^8}\right\}$.

ii) If $r_2 = 1/6$ then $\rho = 1/9$, and $f(\tau) = q\prod_{n=1}^{\infty}(1-q^{6n-5})^4(1-q^{6n-1})^4(1-q^{6n-4})^{-4}(1-q^{6n-2})^{-4}$.

iii) If $r_2 = 1/5$ then $\rho = \frac{1}{2} - \frac{11}{10\sqrt{5}}$ and $f(\tau) = \frac{f_0(\tau)}{f_0(\tau)+\frac{11}{2}+\frac{5\sqrt{5}}{2}}$, where $f_0(\tau) := q\prod_{n=1}^{\infty}(1-q^n)^{5\left(\frac{n}{5}\right)}$. Here $\left(\frac{n}{5}\right)$ is the Legendre symbol.

From now on we write $'$ to denote the derivative with respect to τ.

Definition. Define

$$\begin{aligned}
t(\tau) &:= \frac{1}{\rho}f(\tau)f\Big(-\frac{r_2}{\tau}\Big) = \frac{1}{\rho}f(\tau)\frac{f(\tau)-\rho}{f(\tau)-1},\\
t_0(\tau) &:= f(\tau) - f\Big(-\frac{r_2}{\tau}\Big) = f(\tau) - \frac{f(\tau)-\rho}{f(\tau)-1},\\
E(\tau) &:= -\frac{1}{t(\tau)\left\{\rho - 2 + \rho t(\tau)\right\}}\frac{t_0'(\tau)}{2\pi i},\\
F(\tau) &:= -\frac{1}{\rho}E(\tau)^2 t_0(\tau)t(\tau).
\end{aligned}$$

From this definition it is seen that

$$\begin{aligned}
t(\tau) &= t\left(-\frac{r_2}{\tau}\right),\\
t_0(\tau) &= -t_0\left(-\frac{r_2}{\tau}\right).
\end{aligned} \tag{7}$$

We denote by $\tau*$ the point of intersection of the circles $\mathcal{C}_0$, $\mathcal{C}_1$. Recall that the point $i\sqrt{r_2}$ belongs to $\mathcal{C}_0$. See figure 2.

Lemma 12. *The function $t(\tau)$ maps Q_1 univalently onto the upper half plane. One has the mirror symmetry $\overline{t(\tau)} = t(-\overline{\tau})$ and $t(i\infty) = 0$, $t\Big(\frac{1}{2}\Big) = \infty$,*

$$t(i\sqrt{r_2}) = \frac{2-\rho-2\sqrt{1-\rho}}{\rho} =: \rho_{\min}, \quad t(\tau*) = \frac{2-\rho+2\sqrt{1-\rho}}{\rho} =: \rho_{\max},$$

where $0 < \rho_{\min} < 1 < \rho_{\max}$. See figure 3. Also,

$$f(i\sqrt{r_2}) = 1 - \sqrt{1-\rho}.$$

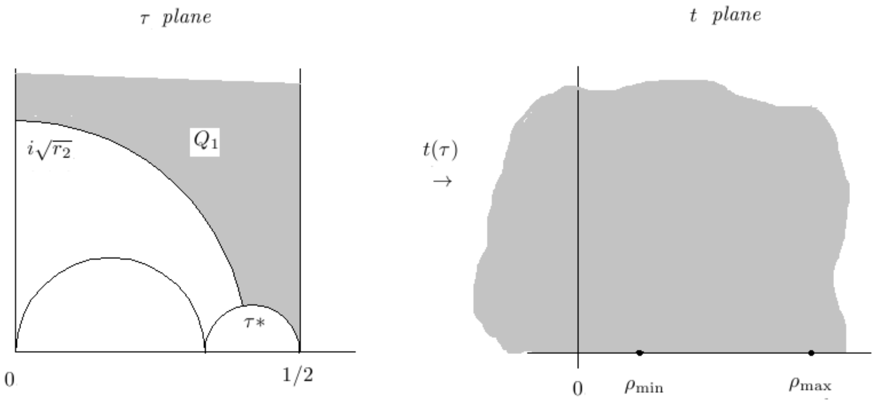

FIGURE 3. Q_1 is mapped conformally onto the upper open half plane by the function $t(\tau)$.

Proof. The mirror symmetry of $t(\tau)$ and the values $t(i\infty) = 0$, $t(1/2) = \infty$ follow trivially from the properties of $f(\tau)$.

We prove the mapping property of $t(\tau)$: as $f(\tau)$ takes real values on the lines $i\tau_2$, $i\tau_2 + 1/2$ and the circle $\mathcal{C}_1$ then $t(\tau)$ takes real values there. Also any point τ on the circle $\mathcal{C}_0$ goes to $-\bar{\tau}$ by the transformation $-\frac{r_2}{\tau}$ which gives $\rho t(\tau) = f(\tau)f(-\bar{\tau}) = |f(\tau)|^2 \in \mathbb{R}$ using the mirror symmetry of f. Therefore $t(\tau)$ takes real values on the boundary of Q_1.

This last fact and the definition $t(\tau) = \frac{1}{\rho} f(\tau) \frac{f(\tau)-\rho}{f(\tau)-1}$ which gives that $t(\tau)$ is a $2:1$ map, yield that $t(\tau)$ maps Q_1 univalently onto the upper half plane. (Hint: If τ moves anticlockwise on the boundary of Q_1 then $t(\tau)$ must move on the real line from $-\infty$ to $+\infty$, without "bouncing back" for, otherwise, a real point would have three preimages at least; thus the derivative of $t(\tau)$ on the line $i\tau_2$ must be purely complex. The image $t(Q_1)$ is open, it must contain a point from the upper half plane and using the mirror symmetry, $\overline{t(Q_1)} = t(Q_4)$. Therefore $t(Q_1)$ can not touch the real line for, otherwise, again a real point would have three preimages at least. This yields that $t(Q_1)$ must be the upper half plane.)

Finally observe that the point $i\sqrt{r_2}$, which belongs to $\mathcal{C}_0$, goes to itself by the transformation $-\frac{r_2}{\tau}$. Also $f(-\frac{r_2}{\tau*}) = f(-\overline{\tau*}) = \overline{f(\tau*)} = f(\tau*)$, where the last equality follows because $f(\tau)$ takes real values on $\mathcal{C}_1$. Then, by Lemma 11, $f(i\sqrt{r_2})$ and $f(\tau*)$ are the roots of the equation $x = \frac{x-\rho}{x-1}$, i.e., $f(i\sqrt{r_2}) = 1 - \sqrt{1-\rho}$ and $f(\tau*) = 1 + \sqrt{1-\rho}$ (observe that $f(i\sqrt{r_2})$ should be the smallest root). The lemma follows from these values and the fact that $0 < \rho < 1$. □

From the definition of $t(\tau)$ one calculates that in a neighbourhood of $q = 0$

$$t = e_0 q + e_0\Big(\frac{(\rho-1)e_0}{\rho} - e_1\Big)q^2 + \dots \tag{8}$$

and therefore the local inverse in a neighbourhood of $t = 0$ is

$$q = \frac{t}{e_0} + \left(\frac{1}{\rho e_0} - \frac{1}{e_0} + \frac{e_1}{e_0^2} \right) t^2 + \dots \tag{9}$$

Thus one has $t_0(\tau) = -\rho + e_0(2-\rho)q + e_0 \{(1-\rho)e_0 + (\rho - 2)e_1\} q^2 + \dots$ and

$$E(\tau) = 1 + \left(\frac{e_0}{\rho} - e_1 \right) q + \dots,$$

Putting (9) into this last equation we get that in a neighbourhood of $t = 0$

$$E(\tau) = 1 + \left(\frac{1}{\rho} - \frac{e_1}{e_0} \right) t + \dots. \tag{10}$$

Definition. If $F(\tau) = \sum_{n=1}^{\infty} \tilde{a}_n q^n$ we define $F_0(\tau) := \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{n^3} q^n$ and $\zeta_F := \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{n^3}$.

As with $E(\tau)$ we may look at the expression of $E(\tau)F_0(\tau)$ as a function of t. A calculation gives that in a neighbourhood of $t = 0$

$$E(\tau)F_0(\tau) = t + \left(\frac{15}{8\rho} - \frac{3}{4} - \frac{3e_1}{8e_0} \right) t^2 + \dots. \tag{11}$$

We finally connect our construction with the sequences at the beginning of the paper.

Theorem 3. *Set $\alpha = 2 - \frac{4}{\rho}$ and $\theta = \frac{e_1}{e_0} - \frac{1}{\rho}$. Then the following holds:*

i) Let $(a) = (1, a_1, a_2, \dots)$ be the sequence of numbers that are the coefficients of $E(\tau) = 1 + ta_1 + t^2 a_2 + \dots$, that is (10). Then (a) satisfy the recurrence (2) and $a_0 = 1$, $a_1 = -\theta$.

ii) Let $(b) = (0, 1, b_2, b_3, \dots)$ be the sequence of numbers that are the coefficients of $E(\tau)F_0(\tau) = t + t^2 b_2 + \dots$, that is (11). Then (b) satisfy the recurrence (2) and $b_0 = 0$, $b_1 = 1$.

Moreover, $L\,E(\tau) = 0$ and $L\,E(\tau)F_0(\tau) = 1$, where L is the operator defined by (5).

Proof. i) Our aim is to show that the function $\sqrt{E(\tau)}$, viewed as a function of the variable t, satisfies

$$L_1 \sqrt{E(\tau)} = 0, \tag{12}$$

where L_1, L are the operators defined in section 3, see formula (5). If this is so, then using Lemma 8 and Lemma 9 one gets that the coefficients of $E(\tau) = 1 + a_1 t + a_2 t^2 \dots$ satisfy the recurrence (2). To ease the proof we write $'$ (resp. $\dot{}$) for the derivative with respect to τ (resp. t) and $\sqrt{E(\tau)} = \sqrt{E}$. Thus for a generic function f we have, for example, $\dot{f} = \frac{f'}{t'}$. Note: In the space of modular forms (under the group that we have) the function $\sqrt{E(\tau)}$ is a 1-form and by a theorem of P. Stiller it satisfies a differential equation of second order in the variable t where $t(\tau)$ is the Hauptmodul for that group, namely equation (12). We give here a direct and self contained proof of this fact adapted from the third proof of Proposition 21 of [17]; the reader may recognize the coefficients A, B below as certain Rankin-Cohen brackets whose definitions we do not need. For a more general point of view the reader may consult [17].

Firstly observe that one trivially has

$$\frac{d}{dt^2}\sqrt{E} + A\frac{d}{dt}\sqrt{E} + B\sqrt{E}$$
$$= \frac{1}{t'}\Big(\frac{\sqrt{E}'}{t'}\Big)' + \frac{\sqrt{E}t'' - 2\sqrt{E}'t'}{\sqrt{E}t'^2}\frac{\sqrt{E}'}{t'} - \frac{\sqrt{E}\sqrt{E}'' - 2\sqrt{E}'^2}{t'^2\sqrt{E}^2}\sqrt{E} = 0.$$

We will see that, up to a factor, this is equation (12). We calculate explicitly the factors A, B as functions of t. Recall that from the definitions

$$2\pi i E(\tau) = g(t(\tau))t_0'(\tau),$$

where

$$g(t) := -\frac{1}{t(\rho - 2 + \rho t)}.$$

The relationship between $t(\tau)$ and $t_0(\tau)$ can be read from the definitions and is given in a neighbourhood of $t = 0$, that is $i\tau = i\infty$, by

$$t_0(\tau) = -\sqrt{\rho\{-4t(\tau) + \rho(1 + t(\tau))^2\}}. \tag{13}$$

We write for short

$$h(t) := -\sqrt{\rho\{-4t + \rho(1 + t)^2\}}.$$

Thus (dropping the variables) one has $2\pi i E' = \dot{g}t't_0' + gt_0''$, $t_0' = \dot{h}t'$ and $t_0'' = \ddot{h}t'^2 + t''\dot{h}$. Therefore

$$\frac{E'}{E} = t'\Big(\frac{\dot{g}}{g} + \frac{\ddot{h}}{\dot{h}}\Big) + \frac{t''}{t'}.$$

Putting this into the definition of $A = \frac{t''}{t'^2} - \frac{E'}{Et'}$ one gets

$$A = -\frac{\dot{g}}{g} - \frac{\ddot{h}}{\dot{h}} = \frac{2t^3 + \frac{3\alpha}{2}t^2 + t}{t^4 + \alpha t^3 + t^2},$$

if $\alpha := 2 - \frac{4}{\rho}$.

The coefficient B is calculated as follows. One has by definition

$$B = -\frac{\sqrt{E}\sqrt{E}'' - 2\sqrt{E}'^2}{t'^2\sqrt{E}^2} = \left(\frac{1}{4}\left\{\frac{E'}{E}\right\}^2 - \frac{1}{2}\left\{\frac{E'}{E}\right\}'\right)\frac{1}{t'^2}$$

and also

$$\frac{E'}{E} = -t'A + \frac{t''}{t'} \tag{14}$$

$$\Big(\frac{E'}{E}\Big)' = -t'^2\dot{A} - At'' + \Big(\frac{t''}{t'}\Big)'.$$

Therefore $B = \frac{A^2}{4} + \frac{\dot{A}}{2} - \frac{S(t,\tau)}{2t'^2}$, where $S(t,\tau) := (\frac{t''}{t'})' - \frac{1}{2}(\frac{t''}{t'})^2$ is the Schwarzian. Here we recall two basic facts about the Schwarzian: $-S(t,\tau)/t'^2 = S(\tau,t)$ (see [8, Exercise 9, p. 377] or use the composition formula for the Schwarzian). Therefore

$$B = \frac{A^2}{4} + \frac{\dot{A}}{2} + \frac{S(\tau,t)}{2}, \tag{15}$$

and $S(\tau,t)$ can be calculated explicitly as a function of t as in [5, pp. 131–135] (or see [8, Theorem 10.2.1]) because by Lemma 12, the function $\tau(t)$, the inverse of $t(\tau)$ defined on that lemma, maps conformally the upper half plane (in the variable t) onto a hyperbolic quadrangle (in the variable τ). Moreover, by the same lemma, it sends the points $0, \rho_{\min}, \rho_{\max}, \infty$ to the points $i\infty, i\sqrt{r_2}, \tau*, 1/2$ respectively, and at these last points the quadrangle has angles $0, \pi/2, \pi/2, 0$. Therefore (see [5] or [8]),

$$S(\tau,t) = \frac{1}{2t^2} + \frac{3/4}{2(\rho_{\min}-t)^2} + \frac{3/4}{2(\rho_{\max}-t)^2} + \frac{\beta_1}{t} + \frac{\beta_2}{\rho_{\min}-t} + \frac{\beta_3}{\rho_{\max}-t}$$

for some constants (accessory parameters) β_i. Also at a neighbourhood of infinity one has

$$S(\tau,t) = \frac{1}{2t^2} + O(1/t^3).$$

So $S(\tau,t)t \to 0, S(\tau,t)t^2 \to 1/2$ if $t \to \infty$. These conditions imply that β_2, β_3 can be given in terms of β_1 alone.

But coefficient β_1 is given by

$$\beta_1 = \frac{e_1}{e_0} + \frac{1}{\rho} - 1,$$

which can be calculated with the formula $S(\tau,t) = -S(t,\tau)/t'^2$ using the expression of $t(\tau)$ given by (8) and knowing that $S(\tau,t) = \frac{1}{2t^2} + \frac{\beta_1}{t} + O(1)$ around $t=0$, see [8].

This yields

$$B = \frac{\frac{t^2}{4} + \frac{t}{2}(\frac{e_1}{e_0} - \frac{1}{\rho})}{t^4 + \alpha t^3 + t^2}.$$

Part (i) of the theorem is proved.

ii) From the definition of $F_0(\tau), F(\tau)$ one trivially has

$$F_0''' = (2\pi i)^3 F = -\frac{(2\pi i)^3}{\rho} E^2 t_0 t. \tag{16}$$

Also, $\dot{F}_0 = \frac{F_0'}{t'}$, $\ddot{F}_0 = (\frac{F_0'}{t'})'/t' = F_0''\frac{1}{t'^2} - F_0'\frac{t''}{t'^3}$ and $\dddot{F}_0 = F_0'''\frac{1}{t'^3} - F_0''\frac{3t''}{t'^4} - F_0'\frac{1}{t'}(\frac{t''}{t'^3})'$. Therefore with a suitable combination we can make the terms F_0' and F_0'' disappear, that is,

$$\dddot{F}_0 + \ddot{F}_0\frac{3t''}{t'^2} + \dot{F}_0\frac{t'''}{t'^3} = \frac{F_0'''}{t'^3}. \tag{17}$$

We will prove that EF_0, viewed as a function of the variable t, satisfies

$$LEF_0 = 1 \tag{18}$$

and this will prove our theorem because this equation is equivalent to the desired recursion by Lemma 9. We do this basically by showing that (17) and (18) are, up to a factor, equal.

As we already proved that $LE=0$ one has that

$$LEF_0 = \dddot{F}_0 E P_1 + \ddot{F}_0(3P_1\dot{E} + P_2 E) + \dot{F}_0(3P_1\ddot{E} + 2P_2\dot{E} + EP_3) \tag{19}$$

(here $P_i = P_i(t)$ are the polynomials defined by $L = P_1\frac{d^3}{dt^3} + P_2\frac{d^2}{dt^2} + P_3\frac{d}{dt} + P_4$.) We calculate the coefficients of the last equation: using $\frac{\dot{E}}{E} = -A + \frac{t''}{t'^2}$ (which is (14)) and the fact that

$$P_2 - 3P_1 A = 0 \tag{20}$$

one gets

$$3P_1\dot{E} + P_2 E = E\Big(P_2 - 3P_1 A + 3P_1\frac{t''}{t'^2}\Big) = EP_1\frac{3t''}{t'^2}. \tag{21}$$

In the same way $\frac{\ddot{E}}{E} = (\frac{t''}{t'^2} - A)^2 - \dot{A} + \frac{t'''}{t'^3} - \frac{2t''^2}{t'^4}$ and one gets

$$\begin{aligned} &3P_1\ddot{E} + 2P_2\dot{E} + EP_3 \\ &= E\left\{(P_3 - 2AP_2 + 3A^2P_1 - 3P_1\dot{A}) + \frac{2t''}{t'^2}(P_2 - 3P_1 A) + 3P_1\Big(\frac{t'''}{t'^3} - \frac{t''^2}{t'^4}\Big)\right\}. \end{aligned}$$

Using (20) one may simplify the first and second terms of the last inner sum obtaining

$$\begin{aligned} &E\left\{(P_3 - AP_2 - 3P_1\dot{A}) + 3P_1\Big(\frac{t'''}{t'^3} - \frac{t''^2}{t'^4}\Big)\right\} \\ &\qquad = E\left\{(P_3 - AP_2 - 3P_1\dot{A}) + 2P_1\frac{S(t,\tau)}{t'^2} + P_1\frac{t'''}{t'^3}\right\}. \end{aligned}$$

But in part (i), formula (15), we have calculated the Schwarzian $S(t,\tau)/t'^2 = -S(\tau,t) = \frac{A^2}{2} + \dot{A} - 2B$. Inserting this into the last equation one sees that everything in the inner sum cancels out except the last term. This yields

$$3P_1\ddot{E} + 2P_2\dot{E} + EP_3 = EP_1\frac{t'''}{t'^3}. \tag{22}$$

Thus using (22) and (21) in (19) one gets $LEF_0 = EP_1\left\{\dddot{F}_0 + \ddot{F}_0\frac{3t''}{t'^2} + \dot{F}_0\frac{t'''}{t'^3}\right\}$. Thus (17) and (16) yield

$$LEF_0 = -\frac{(2\pi i)^3}{t'^3\rho}P_1E^3t_0t = 1,$$

where the last equality follows using the definition of E and the derivative of (13). This ends our proof. □

The following two lemmas complement the last theorem.

Lemma 13. *The following hold:*

i) $E(\tau) = \frac{1}{2\pi i}\frac{f'(\tau)}{f(\tau)}\frac{1}{1-\frac{f(\tau)}{\rho}}$.

ii) $E(-\frac{r_2}{\tau}) = -\frac{\tau^2}{r_2}E(\tau)$ *and* $E(\tau)$ *has, as a function of* q*, radius of convergence 1.*

iii) $F(-\frac{r_2}{\tau}) = -\frac{\tau^4}{r_2^2}F(\tau)$ *and* $F(\tau)$ *has, as a function of* q*, radius of convergence 1.*

iv) $\tilde{a}_n = O(n^2)$ *and* $\zeta_F := \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{n^3}$ *is a convergent series. Also*

$$E(\tau)\left(F_0(\tau) - \zeta_F\right) = E\left(-\frac{r_2}{\tau}\right)\left(F_0(-\frac{r_2}{\tau}) - \zeta_F\right).$$

Proof. i) The definition of $t_0(\tau)$ gives

$$t_0' = \frac{f^2 - 2f + 2 - \rho}{(f-1)^2} f'.$$

Using this last formula, the definition of $t(\tau)$ and the definition of $E(\tau)$ one gets the desired formula.

ii-iii) By Lemma 10 the function $f(\tau) \neq 0, 1, \rho$ in the open upper half plane. From the expression (i) one has that $E(\tau)$ has radius of convergence 1 as a function of q. The same happens with both $t(\tau)$ and $t_0(\tau)$ and therefore with $F(\tau)$.

The transformation formulae follow from (7) and its derivative with respect to τ using the definition of $E(\tau)$.

iv) If $F(\tau) = \sum_{n=1}^{\infty} \tilde{a}_n q^n$ then $\frac{\tilde{a}_n}{n^2} = O(1)$, the proof being similar to that of Theorem 6.17 on [2, p. 134]. Also the function $\int_0^{\infty} F(i\tau)\tau^{s-1}d\tau$ has an analytic continuation to all the s-complex plane, the later integral being equal to $\frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{\tilde{a}_n}{n^s}$. Using a theorem of Ingham or Newman (see [9], [10]) one gets that $\zeta_F = \sum_{n=1}^{\infty}\frac{\tilde{a}_n}{n^3}$ is convergent.

Apply Proposition 1.2 of [3] (with $N = 1/r_2$, $\epsilon = -1$, $k = 4$ there) to the transformation formula we have already proved $F(-\frac{r_2}{\tau}) = -\frac{\tau^4}{r_2^2}F(\tau)$, giving

$$(F_0(\tau) - \zeta_F) = -\frac{\tau^2}{r_2}\left(F_0(-\frac{r_2}{\tau}) - \zeta_F\right).$$

Multiplying this by formula (ii) gives the result. Note: in Proposition 1.2 of [3] it is stated that $N = 1/r_2$ should be a natural number but this is unnecessary. □

The reason for introducing the constant ζ_F is the following result.

Lemma 14. *The functions* $E(\tau)$ *and*

$$E(\tau)\left(F_0(\tau) - \zeta_F\right) = \sum_0^{\infty}\left\{b_n - \zeta_F a_n\right\} t^n$$

have, as functions of the parameter t *(recall formulas* (10) *and* (11)*), radius of convergence* $\rho_{\min}$ *and greater than or equal to* $\rho_{\max}$*, respectively.*

Proof. Looking at the function $E(\tau)(F_0(\tau) - \zeta_F)$ as a multivalued function of the parameter $t = t(\tau)$ we get that this funcion has radius of convergence either $\rho_{\min}$, $\rho_{\max}$ or ∞. By the transformation formula (iv) of Lemma 14, looking carefully at what happens around $t = \rho_{\min}$, one gets that the function has no singularity there. For doing this one should recall, as already observed, that the transformation

$-r_2/\tau$ interchanges conformally Q_1 with Q_2 and Q_3 with Q_4, and maps the point $i/\sqrt{r_2}$ (in the τ plane) to $\rho_{\min}$ (in the t plane). So its radius of convergence is either greater than or equal to $\rho_{\max}$. Note: one may use here Poincaré's theorem (see for example [1, p. 141]) to prove that the radius of convergence is exactly $\rho_{\max}$.

Doing the same with the function $E(\tau)$ one has that now, due to the transformation formula (ii) of Lemma 14, this function has radius of convergence $\rho_{\min}$. $\square$

5. Generating functions

In this section we give some results concerning the combinatorial numbers appearing in Theorem 1.

Our first result links a double Legendre type series with an elliptic type integral. More precisely:

Lemma 15. *If x, y, t are in a neighbourhood of zero then*

$$\sum_{n=0}^{\infty} t^n \left\{ \sum_{i=0}^{n} \binom{n}{i} \binom{n+i}{i} x^i \right\} \left\{ \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} y^j \right\} = \frac{1}{\pi\sqrt{B}} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-zA_+)(1-zA_-)}}, \quad (23)$$

where

$$A_{\mp} := A_{\mp}(x,y,t) = \frac{4t\sqrt{y(1+y)}}{1+2x \mp 2\sqrt{x(1+x)} + t\left\{-1-2y+2\sqrt{y(1+y)}\right\}}$$

and

$$B := B(x,y,t) = 1 + t2(1+2x)\left\{-1-2y+2\sqrt{y(1+y)}\right\} + t^2\left\{1+8y^2-4\sqrt{y(1+y)}-8y\left\{-1+\sqrt{y(1+y)}\right\}\right\}.$$

Proof. One has the well known generating function for the Legendre type polynomials ([12, pp. 66, 78])

$$\sum_{n=0}^{\infty} t^n \left\{ \sum_{i=0}^{n} \binom{n}{i} \binom{n+i}{i} x^i \right\} = \frac{1}{\sqrt{1-2t(1+2x)+t^2}}. \quad (24)$$

Using (24) and Cauchy's formula one may write

$$\begin{aligned}
&\sum_{n=0}^{\infty} t^{2n} \left\{ \sum_{i=0}^{n} \binom{n}{i}\binom{n+i}{i} x^i \right\} \left\{ \sum_{j=0}^{n} \binom{n}{j}\binom{n+j}{j} y^j \right\} \\
&\qquad = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z\sqrt{(1-2tz(1+2x)+t^2z^2)(1-2t/z(1+2y)+t^2/z^2)}} \\
&\qquad = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{t\sqrt{(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)}} \\
&\qquad = \frac{1}{2\pi} \int_{|z|=1} \frac{dz}{t\sqrt{(z-\alpha_1)(\alpha_2-z)(z-\alpha_3)(z-\alpha_4)}},
\end{aligned}$$

where $\alpha_{1,2} = t\left\{1+2y \mp 2\sqrt{y(1+y)}\right\}$ and $\alpha_{3,4} = \frac{1}{t}\left\{1+2x \mp 2\sqrt{x(1+x)}\right\}$ for (say) x, y, t real, positive and small enough. Observe that in such case $0 < \alpha_1 < \alpha_2 < 1 < \alpha_3 < \alpha_4$. Therefore the curve $|z| = 1$, which encloses α_1, α_2, may be deformed to two circles of radius ϵ around α_1, α_2 and two segments: one from $\alpha_1+\epsilon$ to $\alpha_2-\epsilon$ and another from $\alpha_2-\epsilon$ to $\alpha_1+\epsilon$. Making ϵ tend to zero yields that the last formula is equal to

$$\frac{1}{\pi} \int_{\alpha_1}^{\alpha_2} \frac{dz}{t\sqrt{(z-\alpha_1)(\alpha_2-z)(z-\alpha_3)(z-\alpha_4)}},$$

which after making the change of variables $Z = \frac{z-\alpha_1}{\alpha_2-\alpha_1}$ yields

$$\frac{1}{\pi\sqrt{t^2(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)}} \int_0^1 \frac{dZ}{\sqrt{Z(1-Z)(1-Z\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1})(1-Z\frac{\alpha_2-\alpha_1}{\alpha_4-\alpha_1})}},$$

which proves the lemma because $B(x, y, t^2) = t^2(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)$ and $A_\mp(x, y, t^2)$ are $\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}$ and $\frac{\alpha_2-\alpha_1}{\alpha_4-\alpha_1}$, respectively. ☐

The next lemma gives a generating function related to the numbers $\beta_{i,j,k}$ defined in section 2. The sums shown are intended to be from zero to infinity, for example $\sum_{n,k}$ means $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}$ and so on.

Theorem 4. *Assume x, y, t are in a neighbourhood of zero and $A_\mp, B$ are defined as in the last lemma. Then if $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{2\pi i\tilde{\tau}}$,*

$$\begin{aligned}
&\sum_{n,i,j} t^n x^i y^j \sum_k \beta_{i,j,k} \binom{n}{k}\binom{n+k}{k} \\
&= \int_0^1\int_0^1\int_0^1 \frac{(1+q^{-1})^y(1+\tilde{q}^{-1})^x}{\pi\sqrt{B(q,\tilde{q},t)}\sqrt{z(1-z)(1-zA_+(q,\tilde{q},t))(1-zA_-(q,\tilde{q},t))}}\, dz d\tau d\tilde{\tau}.
\end{aligned}$$

Remark: The above theorem is related to the Legendre transform. Given a generic sequence β_k one may generate another sequence Δ_n called the Legendre transform as

$$\Delta_n = \sum_{k=0}^{n} \beta_k \binom{n}{k}\binom{n+k}{k}.$$

One has the inversion formula ([14])

$$\beta_k = \sum_{j=0}^{k} (-1)^{k-j} \frac{2j+1}{k+j+1} \frac{\binom{k}{j}}{\binom{k+j}{j}} \Delta_j.$$

Proof. Observe that

$$\sum_j \binom{x}{j} w^j = (1+w)^x. \tag{25}$$

Recalling that $\binom{x}{i} = \sum_{k=0}^{i} s(i,k)\frac{x^k}{i!} = \sum_k s(i,k)\frac{x^k}{i!}$ and the definition of $\beta_{i,j,k}$ one has

$$\begin{aligned}
&\sum_n t^n \sum_{i,j} x^i y^j \sum_k \beta_{i,j,k} \binom{n}{k}\binom{n+k}{k} \\
&= \sum_n t^n \sum_{i,j} x^i y^j \sum_k \sum_{u,\ell} \binom{u+\ell}{k}\binom{k}{u}\binom{k}{\ell} \frac{s(u,i)}{u!}\frac{s(\ell,j)}{\ell!}\binom{n}{k}\binom{n+k}{k} \\
&= \sum_n t^n \sum_{k,\ell,u} \binom{x}{u}\binom{y}{\ell}\binom{k}{u}\binom{k}{\ell}\binom{u+\ell}{k}\binom{n}{k}\binom{n+k}{k} \\
&\overset{\text{Lemma 3}}{=} \sum_n t^n \left\{\sum_\ell \binom{y}{\ell}\binom{n}{\ell}\binom{n+\ell}{\ell}\right\}\left\{\sum_u \binom{x}{u}\binom{n}{u}\binom{n+u}{u}\right\} \\
&= \int_0^1 \int_0^1 (1+q^{-1})^y (1+\tilde{q}^{-1})^x \\
&\qquad \times \sum_n t^n \left\{\sum_{\ell=0}^{n} q^\ell \binom{n}{\ell}\binom{n+\ell}{\ell}\right\}\left\{\sum_{u=0}^{n} \tilde{q}^u \binom{n}{u}\binom{n+u}{u}\right\} d\tau d\tilde{\tau},
\end{aligned}$$

where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{2\pi i\tilde{\tau}}$, and the last equality follows from (25) and the trivial fact that $\int_0^1 q^i d\tau = 0$ (resp. $= 1$) if $i \neq 0$ (resp. $i = 0$). Using (23) in the inner sum of the last formula gives the result. □

Our final observation is one concerning the solution $(p) = (p_1, p_2, \dots)$ in Theorem 1, where the definitions of $d_{n,k}, c_k$ are given. First, observe that from (24) and (25) one has, if $q = e^{2\pi i\tau}$,

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n \left\{\sum_{i=0}^{n} \binom{x}{i}\binom{n}{i}\binom{n+i}{i}\right\} &= \int_0^1 \frac{(1+q)^x}{\sqrt{1-2t(1+2q^{-1})+t^2}} d\tau \\
&= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} x^k d_{n,k},
\end{aligned}$$

where we have used Lemma 1. Therefore using this last formula, if c_k is any sequence and $\tilde{q} = e^{2\pi i\tilde{\tau}}$, on has formally that

$$\int_0^1 \int_0^1 \left\{ \sum_k c_k \tilde{q}^{-k} y^k \right\} \frac{(1+q)^{\tilde{q}}}{\sqrt{1-2t(1+2q^{-1})+t^2}}\, d\tau d\tilde{\tau} = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} c_k d_{n,k} y^k \right\}.$$

In particular, if c_k is the sequence of numbers defined as in Theorem 1 and putting $y = 1$ in the last equation one gets

$$\int_0^1 \int_0^1 \left\{ \sum_k c_k \tilde{q}^{-k} \right\} \frac{(1+q)^{\tilde{q}}}{\sqrt{1-2t(1+2q^{-1})+t^2}}\, d\tau d\tilde{\tau} = \sum_{n=0}^{\infty} t^n p_n,$$

which is a formal generating function of the solutions of $(n+1)^2 u_{n+1} - f_n u_n - n^2 u_{n-1} = 0$, the recurrence given in Theorem 1.

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