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PRIMARY DECOMPOSITION AND SECONDARY REPRESENTATION OF MODULES

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Abstract. In this paper we study the notions of primary decomposition and secondary representation of modules over a commutative ring with identity. Also we review these concepts over injective and projective modules.

1. Introduction

Throughout this paper, let R denote a commutative ring (with identity). A submodule N of an R-module M is said to be primary if $N \neq M$ and whenever $r \in R$, $m \in M \backslash N$, and $rm \in N$, there exists a positive integer n such that $r^n M \subseteq N$. An R-module M is said to be secondary if $M \neq 0$ and, for each $a \in R$, the endomorphism $\varphi_a : M \to M$ defined by $\varphi_a(m) = am$ (for $m \in M$) is either surjective or nilpotent. If M is secondary, then $\mathfrak{p} = \operatorname{Rad}(0 : M)$ is a prime ideal, and M is said to be $\mathfrak{p}$-secondary. Any non-zero quotient of a $\mathfrak{p}$-secondary module is $\mathfrak{p}$-secondary.

In this paper, we shall follow Macdonald's terminology concerning secondary representation. We refer the reader to [4, 1, 3] for more details about primary decomposition and secondary representation. For each R-module L, we denote by $m \operatorname{Ass}_R L$ the minimal elements of the set $\operatorname{Ass}_R L = \{p \in \operatorname{Spec}(R) \mid p = 0 :_R x, \text{ for some } 0 \neq x \in L\}$. For each R-module L, we denote by $\operatorname{Att}_R L$ the set of all attached prime ideals of L over the ring R. Also, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\sqrt{\mathfrak{b}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$, and set $\Gamma_{\mathfrak{b}}(M) = \cup_{n \in \mathbb{N}}(0 :_M \mathfrak{b}^n)$, the set of elements of M which are annihilated by some power of $\mathfrak{b}$. Finally we denote $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$. For any unexplained notation and terminology we refer the reader to [4, 1].

2. The results

Lemma 2.1. *Let M be an R-module such that $m \operatorname{Ass}_R M = \operatorname{Ass}_R M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $I_j = \bigcap_{\substack{i=1 \\ i \neq j}}^{n} \mathfrak{p}_i$. Then $\bigcap_{j=1}^{n} \Gamma_{I_j}(M) = 0$ is a minimal primary decomposition for the zero module.*

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Proof. It is clear that

$$\operatorname{Ass} \frac{M}{\Gamma_{I_j}(M)} = \operatorname{Ass} M \backslash V(I_j) = \{\mathfrak{p}_j\}.$$

Hence by [4, Theorem 6.6], $\Gamma_{I_j}(M)$ is a $\mathfrak{p}_j$-primary submodule of M. Now let $0 \neq x \in \bigcap_{j=1}^{n} \Gamma_{I_j}(M)$. Since $\operatorname{Ann}_R(x) \subseteq Z_R(M) = \bigcup_{i=1}^{n} \mathfrak{p}_i$, it follows that there exists $1 \leq j \leq n$, such that $\operatorname{Ann}_R(x) \subseteq \mathfrak{p}_j$. On the other hand, $xI_j^n = 0$ for some $n \in \mathbb{N}$ and so $I_j^n \subseteq \operatorname{Ann}_R(x) \subseteq p_j$. Consequently $\bigcap_{\substack{i=1 \\ i\neq j}}^{n} \mathfrak{p}_i = I_j \subseteq \mathfrak{p}_j$ and this implies that $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ for some $i \in \mathbb{N}$, which is a contradiction. Therefore $\bigcap_{j=1}^{n} \Gamma_{I_j}(M) = 0$. Now by [4, Theorem 6.8], this is a minimal primary decomposition and $\Gamma_{I_j}(M)$, for all $1 \leq j \leq n$, are uniquely determined. $\square$

Now we want to present a direct proof for the following corollary, using Lemma 2.1.

Corollary 2.2. *Let R be a Cohen–Macaulay ring and $x_1, x_2, \dots, x_t$ be an R-sequence such that $\sqrt{I} = I = (x_1, \dots, x_t)$ and $\operatorname{Ass}_R \frac{R}{I} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. Then for all $n \geq 1$, $I^n = \mathfrak{p}_1^{(n)} \cap \dots \cap \mathfrak{p}_k^{(n)}$ is a minimal primary decomposition for I^n.*

Proof. Since $\sqrt{I} = \bigcap_{j=1}^{k} \mathfrak{p}_j = I$, it follows that for all $s_i \in (\bigcap_{\substack{j=1 \\ j\neq i}}^{k} \mathfrak{p}_j) \backslash \mathfrak{p}_i$, $s_i \mathfrak{p}_i \subseteq \bigcap_{j=1}^{k} \mathfrak{p}_j = I$. Therefore $\mathfrak{p}_i R_{\mathfrak{p}_i} = I R_{\mathfrak{p}_i}$ and consequently $\mathfrak{p}_i^n R_{\mathfrak{p}_i} = I^n R_{\mathfrak{p}_i}$ for all $n \in \mathbb{N}$. Also R/I^n is Cohen–Macaulay by [4, Ex. 17.4], and we have the exact sequence

$$0 \longrightarrow \frac{\mathfrak{p}_i^{(n)}}{I^n} \longrightarrow \frac{R}{I^n} \longrightarrow \frac{R}{\mathfrak{p}_i^{(n)}} \longrightarrow 0, \tag{2.1}$$

which implies that $\operatorname{Ass}_R \frac{\mathfrak{p}_i^{(n)}}{I^n} \subseteq \operatorname{Ass} \frac{R}{I^n}$. On the other hand, $\frac{\mathfrak{p}_i^{(n)} R_{\mathfrak{p}_i}}{I^n R_{\mathfrak{p}_i}} = \frac{\mathfrak{p}_i^n R_{\mathfrak{p}_i}}{I^n R_{\mathfrak{p}_i}} = \bar{0}$ and so $\mathfrak{p}_i \notin \operatorname{Ass} \frac{\mathfrak{p}_i^{(n)}}{I^n}$. Hence

$$\operatorname{Ass}_R \frac{\mathfrak{p}_i^{(n)}}{I^n} \subseteq \operatorname{Ass}_R \frac{R}{I^n} \backslash \{\mathfrak{p}_i\} = \{\mathfrak{p}_j\}_{j=1}^{k} \backslash \{\mathfrak{p}_i\}. \tag{2.2}$$

Also $\mathfrak{p}_j \in \operatorname{Supp} \frac{R}{I^n}$ and for all $i \neq j$, $\mathfrak{p}_j \notin \operatorname{Supp}_R \frac{R}{\mathfrak{p}_i^{(n)}}$, hence $\mathfrak{p}_j \in \operatorname{Supp} \frac{\mathfrak{p}_i^{(n)}}{I^n}$. This implies that $\mathfrak{p}_j \in m \operatorname{Ass} \frac{\mathfrak{p}_i^{(n)}}{I^n}$. Therefore

$$\{\mathfrak{p}_j\}_{j=1}^{k} \backslash \{\mathfrak{p}_i\} \subseteq \operatorname{Ass}_R \frac{\mathfrak{p}_i^{(n)}}{I^n}. \tag{2.3}$$

By relations (2.2) and (2.3), we conclude that $\operatorname{Ass}\dfrac{\mathfrak{p}_i^{(n)}}{I^n} = \{\mathfrak{p}_j\}_{j=1}^{k}\backslash\{\mathfrak{p}_i\}$. Now let $J_i := \bigcap_{\substack{j=1\\ j\neq i}}^{k} \mathfrak{p}_j$; then $\sqrt{\operatorname{Ann}_R\left(\dfrac{\mathfrak{p}_i^{(n)}}{I^n}\right)} = J_i$ and for a large $l \in \mathbb{N}$, we have $J_i^l\left(\dfrac{\mathfrak{p}_i^{(n)}}{I^n}\right) = 0$, which implies that $\dfrac{\mathfrak{p}_i^{(n)}}{I^n} \subseteq \Gamma_{J_i}\left(\dfrac{R}{I^n}\right)$. Since $\Gamma_{J_i}\left(\dfrac{R}{\mathfrak{p}_i^{(n)}}\right) = 0$, it follows from (2.1), $\dfrac{\mathfrak{p}_i^{(n)}}{I^n} = \Gamma_{J_i}\left(\dfrac{R}{I^n}\right)$, and so by Lemma 2.1 we obtain $\bar{0} = \bigcap_{i=1}^{k}\Gamma_{J_i}\left(\dfrac{R}{I^n}\right) = \dfrac{\mathfrak{p}_1^{(n)}}{I^n} \cap \ldots \cap \dfrac{\mathfrak{p}_k^{(n)}}{I^n}$ which implies that $I^n = \bigcap_{j=1}^{k}\mathfrak{p}_j^{(n)}$.

Also $\operatorname{Ass}\dfrac{\frac{R}{I^n}}{\frac{\mathfrak{p}_i^{(n)}}{I^n}} = \operatorname{Ass}\dfrac{\frac{R}{I^n}}{\Gamma_{J_i}(\frac{R}{I^n})} = \operatorname{Ass}\dfrac{R}{I^n}\backslash V(J_i) = \{\mathfrak{p}_i\}$. This shows that $\operatorname{Ass}\dfrac{R}{\mathfrak{p}_i^{(n)}} = \{\mathfrak{p}_i\}$ and so $\mathfrak{p}_i^{(n)}$ is p_i-primary. Consequently $I^n = \bigcap_{i=1}^{k}\mathfrak{p}_i^{(n)}$ is a minimal primary decomposition. □

We shall prove the following well-known theorem as a dual of Lemma 2.1 with a new proof.

Theorem 2.3 (Dual of Lemma 2.1)**.** *Let R be an Artinian ring and M be a non-zero finitely generated R-module such that* $\operatorname{Att}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. *Then M has a minimal secondary representation as $M = \Gamma_{\mathfrak{p}_1}(M) + \Gamma_{\mathfrak{p}_2}(M) + \cdots + \Gamma_{\mathfrak{p}_n}(M)$, for $1 \leq i \leq n$, with $\Gamma_{\mathfrak{p}_i}(M)$ a $\mathfrak{p}_i$-secondary submodule of M.*

Proof. Let $M = S_1 + S_2 + \cdots + S_n$ be a minimal secondary representation with S_i a $\mathfrak{p}_i$-secondary submodule of M, for $1 \leq i \leq n$. We shall stablish the Theorem by showing (a) that $\mathfrak{p}_i = \sqrt{0 :_R \Gamma_{\mathfrak{p}_i}(M)}$ for each $1 \leq i \leq n$, (b) that $\Gamma_{\mathfrak{p}_i}(M)$ is a secondary module, and (c) $M = \Gamma_{\mathfrak{p}_1}(M) + \Gamma_{\mathfrak{p}_2}(M) + \cdots + \Gamma_{\mathfrak{p}_n}(M)$.

(a) Since $\Gamma_{\mathfrak{p}_i}(M)$ is a finitely generated submodule of M, there exists $t \geq 1$ such that $\Gamma_{\mathfrak{p}_i}(M) = 0 :_M \mathfrak{p}_i^t$. Thus $\mathfrak{p}_i \subseteq \sqrt{0 :_R \Gamma_{\mathfrak{p}_i}(M)}$. It is enough to prove that $\sqrt{0 :_R \Gamma_{\mathfrak{p}_i}(M)} \neq R$. Suppose this is not true; then $\Gamma_{\mathfrak{p}_i}(M) = 0$. So $\operatorname{Supp}\left(\dfrac{R}{\mathfrak{p}_i^t}\right) \cap \operatorname{Ass}_R(M) = \emptyset$ and hence $\mathfrak{p}_i \notin \operatorname{Ass}_R(M)$. By [4, Exer. 6.8], $\operatorname{Ass}_R(S_i) = \{\mathfrak{p}_i\}$ and by the exact sequence $0 \to S_i \to S_i \oplus S_{i+1} \to S_{i+1} \to 0$ we have

$$\begin{aligned}\emptyset &\neq \operatorname{Ass}_R(S_i)\\ &\subseteq \operatorname{Ass}_R(S_i \oplus S_{i+1})\\ &\subseteq \operatorname{Ass}_R(S_{i+1}) \cup \operatorname{Ass}_R(S_i).\end{aligned}$$

Since $\operatorname{Ass}_R(S_i \oplus S_{i+1}) \subseteq \operatorname{Ass}_R(M)$, we have $\operatorname{Ass}_R(S_i \oplus S_{i+1}) = \mathfrak{p}_{i+1}$, which shows that $\mathfrak{p}_{i+1} = \mathfrak{p}_i$ and this is contradiction.

(b) Since $\Gamma_{\mathfrak{p}_i}(M)$ is Noetherian and $\mathfrak{p}_i^t.\Gamma_{\mathfrak{p}_i}(M) = 0$, $\Gamma_{\mathfrak{p}_i}(M)$ is Artinian. Therefore, this module has finite length. On the other hand,

$$\begin{aligned} \emptyset \neq \operatorname{Ass}_R \Gamma_{\mathfrak{p}_i}(M) \\ &= \{\mathfrak{p}_i\} \cap \operatorname{Ass}_R(M) \\ &= \{\mathfrak{p}_i\}. \end{aligned}$$

Therefore $\Gamma_{\mathfrak{p}_i}(M)$ is coprimary, and so $\Gamma_{\mathfrak{p}_i}(M)$ is a secondary module by [4, Exer. 6.9].

(c) Let $m \in M$, so $m = s_1 + s_2 + \cdots + s_n$ for some $s_i \in S_i$ (for $1 \leq i \leq n$). For $1 \leq i \leq n$, $\mathfrak{p}_i^{t_i}.S_i = 0$ for some $t_i \geq 1$. Hence $\mathfrak{p}_i^{t_i}.s_i = 0$ and so $s_i \in \Gamma_{\mathfrak{p}_i}(M)$ for $1 \leq i \leq n$. □

3. Injective and projective modules

Lemma 3.1. *Let E be an injective R-module and let N be a $\mathfrak{p}$-primary submodule of E. Then $(0 :_E (N :_R E))$, if non-zero, is $\mathfrak{p}$-secondary.*

Proof. Let $a \in R$. If $a \in \mathfrak{p}$, then $a^n \in (N :_R E)$ for some positive integer n, so that a^n annihilates $(0 :_E (N :_R E))$. On the other hand, if $a \notin \mathfrak{p}$, then we can show that $a(0 :_E (N :_R E)) = (0 :_E (N :_R E))$.

Let $x \in (0 :_E (N :_R E))$. There is a well-known homomorphism $\varphi : \dfrac{R}{(N :_R E)} \longrightarrow E$ for which $\varphi(b + (N :_R E)) = bx$ for all $b \in R$. Multiplication by a on $\dfrac{R}{(N :_R E)}$ is a monomorphism. So the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \dfrac{R}{(N :_R E)} & \xrightarrow{\ a\ } & \dfrac{R}{(N :_R E)} \\ & & {\scriptstyle\varphi}\big\downarrow & & \\ & & E & & \end{array}$$

can be completed with a homomorphism $\psi : \dfrac{R}{(N :_R E)} \longrightarrow E$ which makes the extended diagram commute. Thus

$$\begin{aligned} x &= \varphi(1 + (N :_R E) \\ &= \psi(a + (N :_R E)) \\ &= \psi(a(1 + (N :_R E)) \\ &= a\psi(1 + (N :_R E)). \end{aligned}$$

Hence $x \in a(0 :_E (N :_R E))$, because

$$(N :_R E)\psi(1 + (N :_R E)) = \psi(N :_R E) = 0_E;$$

this implies that $\psi(1 + (N :_R E)) \in (0 :_E (N :_R E))$ and the proof is complete. □

Lemma 3.2. *Let $I_1, I_2, \dots, I_n$ be ideals of R and E an injective R-module. Then*

$$\sum_{i=1}^{n}(0 :_E I_i) = (0 :_E \bigcap_{i=1}^{n} I_i).$$

Proof. See [5, Lemma 2.2]. □

We recall that an injective R-module E is said to be an injective cogenerator of R if, for every R-module M and every non-zero $m \in M$, there is a homomorphism $\varphi : M \to E$ such that $\varphi(m) \neq 0$.

Theorem 3.3. *Let E be an injective Noetherian R-module. Then E has a secondary representation, and $\operatorname{Att}(E) \subseteq \operatorname{Ass}(E)$. More precisely, let $0 = N_1 \cap N_2 \cap \cdots \cap N_n$ be a minimal primary decomposition for the zero submodule of E, with N_i a $\mathfrak{p}_i$-primary submodule of E, for $i = 1, 2, \dots, n$. Then*

$$E = (0 :_E (N_1 :_R E)) + (0 :_E (N_2 :_R E)) + \cdots + (0 :_E (N_n :_R E)) \qquad (*)$$

and for $i = 1, 2, \dots, n$, $(0 :_E (N_i :_R E))$ is either zero or $\mathfrak{p}_i$-secondary.

Moreover, if j is an integer such that $1 \leq j \leq n$, and $J = \{1, \dots, j-1, j+1, \dots, n\}$, then $E = \sum_{i \in J}(0 :_E (N_i :_R E))$ if and only if $\bigcap_{i \in J}(N_i :_R E)$ annihilates E. Consequently, if E is an injective cogenerator of R, then (*) is a minimal secondary representation for E, and $\operatorname{Att}(E) = \operatorname{Ass}(E)$.

Proof. By Lemma 3.1, $(0 :_E (N_i :_R E))$ is either zero or $\mathfrak{p}_i$-secondary. Generally, $(0 :_R E) = (\bigcap_{i=1}^{n} N_i :_R E) = \bigcap_{i=1}^{n}(N_i :_R E)$. Since $(0 :_E \bigcap_{i=1}^{n}(N_i :_R E)) = (0 :_E (0 :_R E))$, we have $e(0 :_R E) = 0$ for each $e \in E$. Thus $E \subseteq (0 :_E (0 :_R E))$, and it follows that $E = (0 :_E \bigcap_{i=1}^{n}(N_i :_R E))$. Now, we have $E = \sum_{i=1}^{n}(0 :_E (N_i :_R E))$ by Lemma 3.2. Hence $\operatorname{Att}(E) \subseteq \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$, which shows that $\operatorname{Att}(E) \subseteq \operatorname{Ass}(E)$.

For each set such as J, we can write:

$$\sum_{i \in J}(0 :_E (N_i :_R E)) = (0 :_E \bigcap_{i \in J}(N_i :_R E)).$$

Now, if $\bigcap_{i \in J}(N_i :_R E)$ annihilates E, then $E \subseteq \sum_{i \in J}(0 :_E (N_i :_R E))$ and so clearly $E = \sum_{i \in J}(0 :_E (N_i :_R E))$ if and only if $\bigcap_{i \in J}(N_i :_R E)$ annihilates E.

Now, let E be an injective cogenerator of R. It is enough to show that, for each $j = 1, \dots, n$, the ideal $\bigcap_{i \in J}(N_i :_R E))$ does not annihilate E. To prove this, let I be a non-zero arbitrary ideal of R, and $y \in I$. Since E is an injective cogenerator of R, there exists a homomorphism $\varphi : R \longrightarrow E$ such that $\varphi(y) \neq 0$. Hence $0 \neq \varphi(y) = y\varphi(1)$. Then $\varphi(1)$ is an element of E which is not annihilated by y, and so not annihilated by I. Thus E is not annihilated by $\bigcap_{i \in J}(N_i :_R E))$ and so $E = \sum_{i=1}^{n}(0 :_E (N_i :_R E))$, which shows that $\operatorname{Att}(E) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\} = \operatorname{Ass}(E)$. □

Remark 3.4. If E is an injective module over the Noetherian ring R, then there is a family $\{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$ of prime ideals of R for which $E \cong \bigoplus_{\alpha \in \Lambda} E\left(\frac{R}{\mathfrak{p}_\alpha}\right)$, and if $\{\mathfrak{q}_\beta\}_{\beta \in \Phi}$ is a second family of prime ideals of R for which $E \cong \bigoplus_{\beta \in \Phi} E\left(\frac{R}{\mathfrak{q}_\beta}\right)$, then there is a bijection $\gamma : \Lambda \to \Phi$ such that $\mathfrak{p}_\alpha = \mathfrak{q}_{\gamma(\alpha)}$ for all $\alpha \in \Lambda$. The set $\{\mathfrak{p}_\alpha | \alpha \in \Lambda\}$ is thus uniquely determined by E; we shall denote this set by $\operatorname{Occ}(E)$, and refer to its members as the *prime ideals which occur in the direct decomposition of E.*

Theorem 3.5. *Let E be an injective Noetherian R-module. Then*

$$\operatorname{Att}(E) = \{\mathfrak{p}' \in \operatorname{Ass}(E) \mid \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \operatorname{Occ}(E)\}.$$

Proof. Let $0 = N_1 \cap N_2 \cap \cdots \cap N_n$ be a minimal primary decomposition for the zero submodule of E, with N_i a $\mathfrak{p}_i$-primary submodule of E (for $i = 1, 2, \ldots, n$). Then

$$E = (0 :_E (N_1 :_R E)) + (0 :_E (N_2 :_R E)) + \cdots + (0 :_E (N_n :_R E)), \qquad (**)$$

and (for $i = 1, 2, \ldots, n$) $(0 :_E (N_i :_R E))$ is either zero or $\mathfrak{p}_i$-secondary. Also, a minimal secondary representation for E can be written from (**). For proving the Theorem, we shall show: (a) that $(0 :_E (N_i :_R E)) = 0$ for each i for which $\mathfrak{p}_i$ is not contained in any $\mathfrak{p}$ in $\operatorname{Occ}(E)$, and (b) that if j is an integer (with $1 \leq j \leq n$) for which $\mathfrak{p}_j$ is contained in some $\mathfrak{p}$ belonging to $\operatorname{Occ}(E)$, then $\sum_{i\in J}(0 :_E (N_i :_R E)) \neq E$, where $J = \{1, \ldots, j-1, j+1, \ldots, n\}$ so that $(0 :_E (N_j :_R E))$ cannot be omitted from (**).

(a) Let i be an integer (with $1 \leq i \leq n$) such that $\mathfrak{p}_i \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Occ}(E)$. If $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ is a family of prime ideals of R for which $E = \bigoplus_{\alpha\in\Lambda} E(\frac{R}{\mathfrak{p}_\alpha})$, then

$$\begin{aligned}(0 :_E (N_i :_R E)) &\cong (0 :_{\bigoplus_{\alpha\in\Lambda} E(\frac{R}{\mathfrak{p}_\alpha})} (N_i :_R E)) \\ &\cong \bigoplus_{\alpha\in\Lambda} (0 :_{E(\frac{R}{\mathfrak{p}_\alpha})} (N_i :_R E)).\end{aligned}$$

Now, in order to show that $(0 :_E (N_i :_R E)) = 0$, it is enough to show that $(0 :_{E(\frac{R}{\mathfrak{p}})} (N_i :_R E)) = 0$ for all $\mathfrak{p} \in \operatorname{Occ}(E)$. For such a $\mathfrak{p}$, we have $(N_i :_R E) \nsubseteq \mathfrak{p}$, since $\mathfrak{p}_i \nsubseteq \mathfrak{p}$. Thus, there is $r \in (N_i :_R E)\setminus\mathfrak{p}$ such that multiplication by r on $E(\frac{R}{\mathfrak{p}})$ provides an automorphism of $E(\frac{R}{\mathfrak{p}})$, and consequently $(0 :_{E(\frac{R}{\mathfrak{p}})} (N_i :_R E)) = 0$.

(b) Assume that j is an integer (with $1 \leq j \leq n$) for which $\mathfrak{p}_j \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Occ}(E)$. Suppose that $E = \sum_{i\in J}(0 :_E (N_i :_R E))$ and J defined as before. Then, by Theorem 3.3, E is annihilated by $a_j = \bigcap_{i\in J}(N_i :_R E)$ and so $E(\frac{R}{\mathfrak{p}})$ is annihilated by a_j. Hence $a_j \subseteq \bigcap_{k=1}^{\infty} \mathfrak{p}^{(k)}$. Now $a_j \nsubseteq (N_j :_R E)$; let $r \in a_j \setminus (N_j :_R E)$. Then $r \in \bigcap_{k=1}^{\infty} \mathfrak{p}^{(k)}$. We know that if $f : R \longrightarrow R_{\mathfrak{p}}$ is a natural ring homomorphism, then $f \otimes id_E : E \longrightarrow E_{\mathfrak{p}}$. Since $\ker f \otimes E \subseteq \ker(f \otimes id_E)$, for each $e \in E$ we have $r \otimes e = re \in \ker(f \otimes id_E)$. Thus, there exists $s \in R \setminus \mathfrak{p}$ such that $s(re) = 0 \in N_j$. But $s \notin \mathfrak{p}_j$ and $re \notin N_j$, this is in contradiction with the fact that N_j is $\mathfrak{p}_j$-primary. This completes the proof. □

Lemma 3.6. *Let M be an R-module. Then qM is a $\mathfrak{p}$-secondary submodule of M for every $\mathfrak{p}$-secondary ideal $\mathfrak{q}$ of R.*

Proof. Let $\mathfrak{q}$ be a $\mathfrak{p}$-secondary ideal of R and $r \in R$. We consider two cases and prove the lemma. Case 1: $r \in \mathfrak{p}$. Then $r \in \sqrt{0 :_R \mathfrak{q}}$ and so $r^n\mathfrak{q} = 0$ for some $n \in \mathbb{N}$. Therefore $r^n\mathfrak{q}M = 0$ and the result follows. Case 2: $r \notin \mathfrak{p}$. Then $r^n\mathfrak{q} \neq 0$ for each $n \in \mathbb{N}$, and so $r\mathfrak{q} = \mathfrak{q}$. Hence $r\mathfrak{q}M = \mathfrak{q}M$. It is clear that $\sqrt{0 :_R \mathfrak{q}M} = \mathfrak{p}$. □

Theorem 3.7. *Let M be a representable projective R-module and let $M = S_1 + S_2 + \cdots + S_n$, with S_i a $\mathfrak{p}_i$-secondary submodule (for $i = 1, 2, \ldots, n$), be a minimal secondary representation of M. Then*

$$0_R = (0 :_R S_1) \cap (0 :_R S_2) \cap \cdots \cap (0 :_R S_n)$$

is a primary decomposition for the zero ideal of R.

Proof. We can assume that R is local. Hence M is a free module and so $\mathrm{Ann}_R M = 0$. Therefore the result follows. □

Lemma 3.8. *Let M be a projective R-module. Then either $\mathfrak{q}M = M$ or $\mathfrak{q}M$ is a $\mathfrak{p}$-primary submodule of M for every $\mathfrak{p}$-primary ideal $\mathfrak{q}$ of R.*

Proof. See [2, Theorem 2.2]. □

Theorem 3.9. *Let M be a representable projective module over an integral domain R and let $M = S_1 + S_2 + \cdots + S_n$, with S_i a $\mathfrak{p}_i$-secondary submodule (for $i = 1, 2, \ldots, n$), be a minimal secondary representation of M. Also, let $(0 :_R S_i)M \neq M$ for $i = 1, 2, \ldots, n$. Then*

$$0_M = (0 :_R S_1)M \cap (0 :_R S_2)M \cap \cdots \cap (0 :_R S_n)M$$

is a primary decomposition for the zero submodule of M.

Proof. Clearly, $(0 :_R S_i)$, for $i = 1, 2, \ldots, n$, is a $\mathfrak{p}_i$-primary ideal of R. Since M is projective, $(0 :_R S_i)M$ will be a $\mathfrak{p}_i$-primary submodule of M, by Lemma 3.8. Let $x \in (0 :_R S_1)M \cap (0 :_R S_2)M \cap \cdots \cap (0 :_R S_n)M$. So x will be expressed as a finite sum $x = \sum_{t=1}^{k} r_t m_t$ for $r_t \in (0 :_R S_1)$ and $m_t \in M$. In order to show that $x = 0$, it is enough to consider $x = r_1 m_1$ for $r_1 \in (0 :_R S_1)$ and $m_1 \in M$.

If $r_1 \in (0 :_R S_i)$ for $i = 2, 3, \ldots, n$, then by Theorem 3.7, $r_1 = 0$ and so $x = 0$. Now, let $r_1 \notin (0 :_R S_j)$ for some $1 \leq j \leq n$. So, there are two cases, $r_1 \notin \mathfrak{p}_j$ or $r_1 \in \mathfrak{p}_j$. If $r_1 \notin \mathfrak{p}_j$, then $m_1 \in (0 :_R S_j)M$, since $r_1 m_1 \in (0 :_R S_j)M$. Therefore $m_1 = tm$ for some $t \in (0 :_R S_j)$ and $m \in M$. Then $x = r_1 m_1 = r_1 tm = 0$. But, if $r_1 \in \mathfrak{p}_j$, we can assume that $r_1 \in \mathfrak{p}_i$ for each $i = 2, 3, \ldots, n$. Hence there is $k \geq 1$ such that $r_1^k \in (0 :_R S_i)$ for $i = 1, 2, \ldots, n$. So $r_1^k = 0$ by Theorem 3.7 and consequently $r_1 = 0$. □

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