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# PRICING AMERICAN PUT OPTIONS UNDER STOCHASTIC VOLATILITY USING THE MALLIAVIN DERIVATIVE

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Abstract. The aim of this paper is to develop a methodology based on Malliavin calculus, in order to price American options under stochastic volatility. This leads to compute the conditional expectation $\mathbb{E}(P_t(X_t, V_t) \mid (X_l, V_l))$ for any $0 \leq l < t$, where $V_t$ is generated by the Cox-Ingersoll-Ross (CIR) process. Some simulations and comparisons are given.

## 1. Introduction

The evaluation of American options is one of the most difficult problems in the option pricing literature. This problem should be of interest to both financial academics and traders. Comparatively to the European options, the American options are more common. The American options pricing model based on a constant volatility cannot explain the reality of the financial markets. Since the dynamics of the volatility is fundamental to elaborate strategies for hedging and for arbitrage, the pricing of options under the stochastic volatility model is required. The introduction of an additional stochastic volatility factor enormously complicates the pricing of American options. Many works have been published about option pricing with stochastic volatility models. In the last few years, the importance of applying Malliavin calculus was demonstrated ([1, 2, 6]). Malliavin calculus is a suitable tool to compute the value of the conditional expectation in order to resolve several problems in the field of financial mathematics, and in particular for the American options pricing problem.

The papers developed by Fournié et al. [4, 5] are considered as the background basis for Malliavin calculus in financial mathematics. In [2], Bally et al. have developed a representation formula for the conditional expectation using Malliavin calculus in order to evaluate the American option for a constant volatility. Abbas-Turki and Lapeyre [1] have developed a new method to price American options under stochastic volatility. The basic idea of their work was to use the Cholesky decomposition. Using Malliavin calculus, Jerbi and Kharrat [6] have shown the equivalence between the stochastic volatility model and the unidimensional model. In [7], Kharrat extended this previous work to the multidimensional case.

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In this paper, we propose a new approach to resolve the American option pricing problem. In order to compute the Malliavin weights, our work identifies two new stochastic processes, searching to determine the Malliavin derivative process associated to the stochastic process $V_t$ which is generated by the CIR process [3].

To obtain our results, we first introduce the dynamic of the model. Let $X_t$ and $V_t$ represent two stochastic processes such that $X_t$ is generated by the process

$$dX_t = rX_t\,dt + X_t\sqrt{V_t}\,dW_t^S \tag{1.1}$$

and $V_t$ follows a mean reverting and a square-root diffusion process (CIR process) given by

$$dV_t = k_V(\theta_V - V_t)\,dt + \sigma_V\sqrt{V_t}\,dW_t^V, \tag{1.2}$$

where $r$ is supposed to be constant, $W_t^S$ and $W_t^V$ are two correlated standard Brownian motions, i.e. $W_t^S = \sqrt{1-\rho^2}B_t^1 + \rho B_t^2$ and $W_t^V = B_t^2$, with $B$ a standard 2-dimensional Brownian motion and $\rho \in ]-1,1[$. The parameters $\theta_V$, $k_V$, and $\sigma_V$ are respectively the long-term mean, the rate of mean reversion, and the volatility of the stochastic process $V_t$. We assume that the volatility process $V_t$ is strictly positive a.s.

At time $l$, with $l < t$, the American put option price under stochastic volatility is given by the equation

$$P_l(X_l, V_l) = \max\left\{\max(K - X_l, 0); e^{-r(t-l)}\mathbb{E}(P_t(X_t, V_t) \mid (X_l, V_l))\right\}. \tag{1.3}$$

The outline of this paper is as follows: In section 2, we give our main results. In section 3, we apply these theoretical results to price the American option with reference to the binomial approach. We generate the results and we study the efficiency of our model.

We note that all stochastic integrals and ordinary integrals in this article are well defined on the interval $[0,T]$ and in particular on any interval included in $[0,T]$.

## 2. Main results

In this section, our objective is to compute $\mathbb{E}(P_t(X_t, V_t) \mid (X_l = \alpha, V_l = \beta))$ for any $0 \le l < t$, with $\alpha$ and $\beta$ being two positive real numbers. Firstly we compute the Malliavin weights related to the volatility, then to the underlying asset price.

### 2.1. Malliavin weights related to the volatility.

Since we cannot write analytically and explicitly the Malliavin derivative associated to the stochastic process $V_t$, in the following proposition we may identify two stochastic processes, namely $Y_t$ and $Z_t$, which will enable us to compute the expression of $D_s^V V_l$.

**Proposition 2.1.** *For $0 < s < l < t$, let $V_l$ be the solution of the stochastic differential equation (SDE)*

$$V_l = V_s + \int_s^l k(\theta - V_r)\,dr + \int_s^l \eta\sqrt{V_r}\,dW_r^V, \tag{2.1}$$

*$Y_l$ be the solution of the SDE*

$$Y_l = 1 - k\int_0^l Y_r\,dr + \int_0^l \frac{\eta}{2\sqrt{V_r}} Y_r\,dW_r^V, \tag{2.2}$$

*and $Z_l$ be the solution of the SDE*

$$Z_l = 1 + \int_0^l (\frac{\eta^2}{4V_r} + k) Z_r\,dr - \int_0^l \frac{\eta}{2\sqrt{V_r}} Z_r\,dW_r^V. \tag{2.3}$$

*Then, for every $l$ we have $Y_l Z_l = 1$ and $D_s^V V_l = Y_l Z_s \eta\sqrt{V_s}$.*

*Proof.* We have $Y_0 Z_0 = 1$. Then, by using Ito's formula, for any $t$ we have $d(Y_t Z_t) = 0$.

For a fixed $s$, the process $D_s^V(V_l)$ satisfies the SDE

$$D_s^V(V_l) = \eta\sqrt{V_s} - k\int_s^l D_s^V(V_r)\,dr + \frac{1}{2}\eta\int_s^l \frac{D_s^V(V_r)}{\sqrt{V_s}}\,dW_r^V.$$

Moreover, we have the following SDE

$$Y_l = Y_s - k\int_s^l Y_r\,dr + \int_s^l \frac{\eta}{2\sqrt{V_r}} Y_r\,dW_r^V.$$

Multiplying the above equation by $Z_s\eta\sqrt{V_s}$ and using the fact that $Y_l Z_l = 1$, we get

$$Y_l Z_s \eta\sqrt{V_s} = Y_s Z_s \eta\sqrt{V_s} - k\int_s^l Y_r Z_s \eta\sqrt{V_s}\,dr + \int_s^l \frac{\eta}{2\sqrt{V_r}} Y_r Z_s \eta\sqrt{V_s}\,dW_r^V.$$

Hence, we prove that $D_s^V(V_l)$ and $Y_l Z_s \eta\sqrt{V_s}$ satisfy the same SDE. Knowing the uniqueness of the solution for the SDE, we deduce that $D_s^V(V_l) = Y_l Z_s \eta\sqrt{V_s}$. ☐

**Proposition 2.2.** *Let $V_l$, $Y_l$ and $Z_l$ be defined by* (2.1), (2.2) *and* (2.3) *respectively. For any $0 \le l < t$ and for any function $\Psi \in C_b^{+\infty}(\mathbb{R})$. We have:*

$$\begin{aligned}\mathbb{E}(\Psi'(V_l)P_t(X_t,V_t)) = \mathbb{E}\Bigg(\frac{\Psi(V_l)P_t(X_t,V_t)}{t-l}\Bigg[\frac{\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}}}{Y_l\eta} \\ - \int_l^t \left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right) ds \int_l^t \frac{dW_s^V}{Y_l Z_s \eta\sqrt{V_s} + D_s^V M_l}\Bigg]\Bigg),\end{aligned} \tag{2.4}$$

*where $M_l = \int_l^t k(\theta - V_r)\,dr + \int_l^t \eta\sqrt{V_r}\,dW_r^V$ and $C_b^{+\infty}(\mathbb{R})$ represents the space of bounded and infinitely differentiable functions.*

*Proof.* Applying stochastic integration by parts (Lemma A.7) to the left side of (2.4), we have

$$\begin{aligned}\mathbb{E}(\Psi'(V_l)P_t(X_t,V_t)) &= \mathbb{E}(\Psi'(V_l)P_t(X_t,V_l+M_l)) \\ &= \mathbb{E}\left(\Psi(V_l)\int_l^t \frac{u_s P_t(X_t,V_l+M_l)}{\int_l^t u_{s'} D_{s'}^V V_l ds'} \diamond dW_s^V\right),\end{aligned}$$

where $D^V_{s'}V_l = Y_l Z_{s'}\eta\sqrt{V_{s'}}$, and let $u_s = \frac{1}{Z_s\sqrt{V_s}}$; we get

$$\mathbb{E}(\Psi'(V_l)P_t(X_t,V_t)) = \mathbb{E}\left(\frac{\Psi(V_l)}{Y_l\eta(t-l)}\int_l^t \frac{P_t(X_t,V_l+M_l)}{Z_s\sqrt{V_s}} \diamond dW_s^V\right),$$

where $\frac{1}{Z_s\sqrt{V_s}}$ is adapted. Hence, by using Lemma A.6, we have

$$\begin{aligned}&\mathbb{E}(\Psi'(V_l)P_t(X_t,V_t))\\ &= \mathbb{E}\left(\frac{\Psi(V_l)}{Y_l\eta(t-l)}\left(P_t(X_t,V_l+M_l)\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}} - \int_l^t \frac{D_s^V(P_t(X_t,V_l+M_l))}{Z_s\sqrt{V_s}}ds\right)\right).\end{aligned}$$

Applying the Malliavin derivative, we get:

$$\begin{aligned}\mathbb{E}(\Psi'(V_l)P_t(X_t,V_t)) &= \mathbb{E}\left(\frac{\Psi(V_l)P_t(X_t,V_l+M_l)}{Y_l\eta(t-l)}\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}}\right)\\ &\quad - \mathbb{E}\left(\Psi(V_l)P_t'(X_t,V_l+M_l)\int_l^t\left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right)ds\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathbb{E}\left(\Psi(V_l)P_t'(X_t,V_l+M_l)\int_l^t\left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right)ds\right)\\ &\quad = \mathbb{E}\left(\mathbb{E}\left(\Psi(z')P_t'(X_t,V_l+M_l)\int_l^t\left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right)ds\right)\Big|_{z'=V_l}\right) \qquad (2.5)\\ &\quad = \mathbb{E}\left(\Psi(z')\int_l^t\left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right)ds\,\mathbb{E}(P_t'(X_t,z'+M_l))\Big|_{z'=V_l}\right).\end{aligned}$$

Applying stochastic integration by parts (Lemma A.7) to $\mathbb{E}(P_t'(X_t,V_l+M_l))$, we get

$$\mathbb{E}(P_t'(X_t,V_l+M_l)) = \mathbb{E}\left(P_t(X_t,V_l+M_l)\int_l^t \frac{u_s}{\int_l^t u_{s'}D_{s'}^V(V_l+M_l)ds'} \diamond dW_s^V\right).$$

For $u_s = \frac{1}{Y_l Z_{s'}\eta\sqrt{V_{s'}}+D^V_{s'}M_l}$ which is adapted, we have:

$$\mathbb{E}(P_t'(X_t,V_l+M_l)) = \mathbb{E}\left(\frac{P_t(X_t,V_l+M_l)}{t-l}\int_l^t \frac{1}{Y_l Z_s\eta\sqrt{V_s}+D_s^V M_l}dW_s^V\right). \qquad (2.6)$$

Combining (2.5) with (2.6) we obtain (2.4). □

Next, we give the expression of the Malliavin weights of the conditional expectation: $\mathbb{E}(P_t(X_t,V_t) \mid V_l = \beta)$.

**Theorem 2.3.** *Let $V_l$, $Y_l$, $Z_l$, and $M_l$ be as previously defined, and $\alpha, \beta$ be two positive real numbers, for any $0 \leq l < t$. We have the following Malliavin weights:*

$$\Upsilon_{V_l}(P_t(X_t,V_t)) = \frac{P_t(X_t,V_t)}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}}}{Y_l\eta} - \int_l^t \left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right) ds \int_l^t \frac{dW_s^V}{Y_l Z_s \eta\sqrt{V_s} + D_s^V M_l}\right)$$

$$\Upsilon_{V_l}(1) = \frac{1}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}}}{Y_l\eta} - \int_l^t \left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right) ds \int_l^t \frac{dW_s^V}{Y_l Z_s \eta\sqrt{V_s} + D_s^V M_l}\right),$$

*where $H$ is the Heaviside function.*

*Proof.* According to (2.4), for any $\Psi \in C_b^\infty(\mathbb{R})$ we have

$$\begin{aligned}\mathbb{E}(\Psi(V_l)\Upsilon_{V_l}(P_t(X_t,V_t))) &= \mathbb{E}(\Psi'(V_l)P_t(X_t,V_t)) \\ &= \mathbb{E}\left(\frac{\Psi(V_l)P_t(X_t,V_t)}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}}}{Y_l\eta} - \int_l^t \left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right) ds \int_l^t \frac{dW_s^V}{Y_l Z_s \eta\sqrt{V_s} + D_s^V M_l}\right)\right).\end{aligned}$$

So, the square integrable weight $\Upsilon_{V_l}(P_t(X_t,V_t))$ is equal to

$$\Upsilon_{V_l}(P_t(X_t,V_t)) = \frac{P_t(X_t,V_t)}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}}}{Y_l\eta} - \int_l^t \left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right) ds \int_l^t \frac{dW_s^V}{Y_l Z_s \eta\sqrt{V_s} + D_s^V M_l}\right).$$

Using the same approach, we compute the square integrable weight $\Upsilon_{V_l}(1)$ which is equal to

$$\Upsilon_{V_l}(1) = \frac{1}{t-l}\left(\frac{\int_l^t \frac{dW_s^V}{Z_s\sqrt{V_s}}}{Y_l\eta} - \int_l^t \left(\eta Y_l + \frac{D_s^V M_l}{Z_s\sqrt{V_s}}\right) ds \int_l^t \frac{dW_s^V}{Y_l Z_s \eta\sqrt{V_s} + D_s^V M_l}\right).$$

□

2.2. **Computation of Malliavin weights related to the underlying asset price.** We have computed the first conditional expectation. In the following, using the Malliavin calculus, we shall write $\mathbb{E}(P_t(X_t,V_t) \mid (X_l = \alpha, V_l = \beta))$ as a ratio of ordinal expectations. In order to do this we need the following result.

**Proposition 2.4.** *Let* $X_t = X_l \exp\left(\int_l^t r\,ds + \int_l^t \sqrt{V_s}\,dW_s^S - \frac{1}{2}\int_l^t V_s\,ds\right)$ *for any* $0 \le l < t$ *and for any* $\Psi \in C_b^{+\infty}(\mathbb{R})$*. We have*

$$\mathbb{E}(\Psi'(X_l)A_{X_t}(\beta)) = \mathbb{E}\left(\Psi(X_l)\frac{A_{X_t}(\beta)}{X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right). \tag{2.7}$$

*Proof.* Let $X_t = X_l Y$, with $X_l$ and $Y$ being independent, such that $Y = \exp\left(\int_l^t r\,ds + \int_l^t \sqrt{V_s}\,dW_s^S - \frac{1}{2}\int_l^t V_s\,ds\right)$. For any $0 \le l < t$ and for any $\Psi \in C_b^{+\infty}(\mathbb{R})$, applying stochastic integration by parts to the left side of (2.7) we obtain

$$\mathbb{E}(\Psi'(X_l)A_{X_lY}(\beta)) = \mathbb{E}\left(\Psi(X_l)\int_0^l \frac{u_{s'}A_{X_lY}(\beta)}{\int_0^l u_s D_s^S X_l ds} \diamond dW_{s'}^S\right).$$

Knowing that $D_s^S(X_l) = \sqrt{V_s}X_l$, assuming that $u_s = \frac{1}{\sqrt{V_s}}$ which is adapted, and using Lemma A.6, we get:

$$\begin{aligned}
\mathbb{E}(\Psi'(X_l)A_{X_lY}(\beta)) &= \mathbb{E}\left(\frac{\Psi(X_l)}{l}\left(\frac{A_{X_lY}(\beta)}{X_l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \int_0^l D_s^S(\frac{A_{X_lY}(\beta)}{X_l})\frac{ds}{\sqrt{V_s}}\right)\right)\\
&= \mathbb{E}\left(\frac{\Psi(X_l)}{l}\left(\frac{A_{X_lY}(\beta)}{X_l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}}\right.\right.\\
&\qquad\left.\left. - \int_0^l \left(A'_{X_lY}(\beta)Y\sqrt{V_s} - \frac{A_{X_lY}(\beta)\sqrt{V_s}}{X_l}\right)\frac{ds}{\sqrt{V_s}}\right)\right)\\
&= \mathbb{E}\left(\frac{\Psi(X_l)A_{X_lY}(\beta)}{X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right)\\
&\qquad - \mathbb{E}(Y\Psi(X_l)A'_{X_lY}(\beta)).
\end{aligned}$$

By using the independence between $X_l$ and $Y$, we have:

$$\mathbb{E}(Y\Psi(X_l)A'_{X_lY}(\beta)) = \mathbb{E}(\Psi(x)\mathbb{E}(YA'_{xY}(\beta))|_{X_l=x}). \tag{2.8}$$

Using stochastic integration by parts, and for $\frac{1}{\sqrt{V_s}}$ which is adapted we have

$$\begin{aligned}
\mathbb{E}(YA'_{X_lY}(\beta)) &= \mathbb{E}\left(A_{X_lY}(\beta)\int_l^t \frac{u_{s'}Y}{\int_l^t u_s D_s^S(X_lY)ds} \diamond dW_{s'}^S\right)\\
&= \mathbb{E}\left(\frac{A_{X_lY}(\beta)}{X_l(t-l)}\int_l^t \frac{dW_s^S}{\sqrt{V_s}}\right).
\end{aligned} \tag{2.9}$$

Combining (2.9) with (2.8) we obtain (2.7). □

The next theorem gives the analytic value of the Malliavin weights of the conditional expectation $\mathbb{E}(P_t(X_t, V_t) \mid (X_l = \alpha, V_l = \beta))$.

**Theorem 2.5.** *Let* $X_t = X_l \exp\left(\int_l^t r\,ds + \int_l^t \sqrt{V_s} dW_s^S - \frac{1}{2}\int_l^t V_s\,ds\right)$, *with* $0 \leq l < t$. *We have:*

$$\Upsilon_{X_l}(A_{X_t}(\beta)) = \frac{A_{X_t}(\beta)}{X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)$$

*and*

$$\Upsilon_{X_l}(1) = \frac{1}{X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right),$$

*where* $H$ *is the Heaviside function.*

*Proof.* According to (2.7), for any $\Psi \in C_b^\infty(\mathbb{R})$ we have

$$\begin{aligned}\mathbb{E}(\Psi(X_l)\Upsilon_{X_l}(A_{X_t}(\beta))) &= \mathbb{E}(\Psi'(X_l)A_{X_t}(\beta)) \\ &= \mathbb{E}\left(\Psi(X_l)\frac{A_{X_t}(\beta)}{X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right).\end{aligned}$$

Then, the square integrable weight $\Upsilon_{X_l}(A_{X_t}(\beta))$ is defined as follows:

$$\Upsilon_{X_l}(A_{X_t}(\beta)) = \frac{A_{X_t}(\beta)}{X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right).$$

Applying the same methodology, the expression of the square integrable weight $\Upsilon_{X_l}(1)$ is

$$\Upsilon_{X_l}(1) = \frac{1}{X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right). \qquad \square$$

## 3. Implementations and simulations

At time $l$, with $l < t$, the American put option price under stochastic volatility is given by the equation

$$P_l(X_l) = \max(\max(K - X_l, 0); e^{-r(t-l)}\mathbb{E}(P_t(X_t, V_t) \mid (X_l, V_l))).$$

Using Theorem 2.3 and Theorem 2.5, we can compute the conditional expectation

$$\mathbb{E}(P_t(X_t, V_t) \mid (X_l, V_l)).$$

At each time step ranging between the current time and maturity, we compute an iterative calculation, comparing this conditional expectation with the intrinsic value. This calculus is performed step by step backwards from maturity up to the current time. With our formula, the American put option under stochastic volatility can be performed with the Monte Carlo simulation. In the following, to verify the efficiency (accuracy) of our approach, we compare our algorithm (Mall) with the binomial model under stochastic volatility (Bin). To this end, we take an interest rate $r = 0.05$, a rate of mean reversion $k_V = 3$, a long term mean $\theta_V = 0.04$, a volatility of the stochastic process $V_t$ $\sigma_V = 0.1$, and a strike price $K = 100$.

| $V_0$ | $\rho$ | $S_0$ | Mall(1000) | Mall(5000) | Bin(400) | Mall(1000)-Bin(400) | Mall(5000)-Bin(400) |
|---|---|---|---|---|---|---|---|
| 0.2 | $-0.1$ | 90 | 10 | 10 | 10.0000 | 0 | 0 |
| 0.2 | $-0.1$ | 100 | 2.2008 | 2.1248 | 2.1254 | $4.6810^{-2}$ | $0.0610^{-2}$ |
| 0.2 | $-0.1$ | 110 | 0.1546 | 0.1128 | 0.1091 | $4.5510^{-2}$ | $0.3710^{-2}$ |
| 0.2 | $-0.7$ | 90 | 10 | 10 | 9.9997 | $0.0310^{-2}$ | $0.0310^{-2}$ |
| 0.2 | $-0.7$ | 100 | 2.1388 | 2.1252 | 2.1267 | $1.2110^{-2}$ | $0.1510^{-2}$ |
| 0.2 | $-0.7$ | 110 | 0.1337 | 0.1275 | 0.1274 | $0.6310^{-2}$ | $0.0110^{-2}$ |
| 0.4 | $-0.1$ | 90 | 10.7626 | 10.7087 | 10.7100 | $5.2610^{-2}$ | $0.1310^{-2}$ |
| 0.4 | $-0.1$ | 100 | 4.2812 | 4.2154 | 4.2158 | $6.5410^{-2}$ | $0.0410^{-2}$ |
| 0.4 | $-0.1$ | 110 | 1.1852 | 1.1676 | 1.1667 | $1.8510^{-2}$ | $0.0910^{-2}$ |
| 0.4 | $-0.7$ | 90 | 10.6915 | 10.6844 | 10.6804 | $1.1110^{-2}$ | $0.410^{-2}$ |
| 0.4 | $-0.7$ | 100 | 4.2244 | 4.2173 | 4.2140 | $1.0410^{-2}$ | $0.3310^{-2}$ |
| 0.4 | $-0.7$ | 110 | 1.2057 | 1.1968 | 1.1939 | $1.1810^{-2}$ | $0.2910^{-2}$ |

Table 1. Comparisons of American put options, Malliavin stochastic solution, and the binomial one when the maturity is $\frac{1}{12}$.

| $V_0$ | $\rho$ | $S_0$ | Mall(1000) | Mall(5000) | Bin(400) | Mall(1000)-Bin(400) | Mall(5000)-Bin(400) |
|---|---|---|---|---|---|---|---|
| 0.2 | $-0.1$ | 90 | 10.1809 | 10.1733 | 10.1706 | $1.0310^{-2}$ | $0.2710^{-2}$ |
| 0.2 | $-0.1$ | 100 | 3.4917 | 3.4832 | 3.4747 | $1.7010^{-2}$ | $0.8510^{-2}$ |
| 0.2 | $-0.1$ | 110 | 0.7904 | 0.7683 | 0.7736 | $1.6810^{-2}$ | $0.5310^{-2}$ |
| 0.2 | $-0.7$ | 90 | 10.1431 | 10.1244 | 10.1206 | $2.2510^{-2}$ | $0.3810^{-2}$ |
| 0.2 | $-0.7$ | 100 | 3.5020 | 3.4873 | 3.4807 | $2.1310^{-2}$ | $0.6610^{-2}$ |
| 0.2 | $-0.7$ | 110 | 0.8569 | 0.8441 | 0.8416 | $1.5310^{-2}$ | $0.2510^{-2}$ |
| 0.4 | $-0.1$ | 90 | 12.1901 | 12.1853 | 12.1819 | $0.8210^{-2}$ | $0.3410^{-2}$ |
| 0.4 | $-0.1$ | 100 | 6.5106 | 6.4952 | 6.4964 | $1.4210^{-2}$ | $0.1210^{-2}$ |
| 0.4 | $-0.1$ | 110 | 3.1285 | 3.0919 | 3.0914 | $3.7110^{-2}$ | $0.0510^{-2}$ |
| 0.4 | $-0.7$ | 90 | 12.1537 | 12.1141 | 12.1122 | $4.1510^{-2}$ | $0.1910^{-2}$ |
| 0.4 | $-0.7$ | 100 | 6.5 105 | 6.4924 | 6.4899 | $2.0610^{-2}$ | $0.2510^{-2}$ |
| 0.4 | $-0.7$ | 110 | 3.1551 | 3.1443 | 3.1456 | $0.9410^{-2}$ | $0.1310^{-2}$ |

Table 2. Comparisons of American put options, Malliavin stochastic solution, and the binomial one when the maturity is $\frac{1}{4}$.

| $V_0$ | $\rho$ | $S_0$ | Mall(1000) | Mall(5000) | Bin(400) | Mall(1000)-Bin(400) | Mall(5000)-Bin(400) |
|---|---|---|---|---|---|---|---|
| 0.2 | $-0.1$ | 90 | 10.6659 | 10.6491 | 10.6478 | $1.8110^{-2}$ | $0.1310^{-2}$ |
| 0.2 | $-0.1$ | 100 | 4.6703 | 4.6451 | 4.6473 | $2.3\ 10^{-2}$ | $0.2210^{-2}$ |
| 0.2 | $-0.1$ | 110 | 1.7124 | 1.6871 | 1.6832 | $2.92\ 10^{-2}$ | $0.3910^{-2}$ |
| 0.2 | $-0.7$ | 90 | 10.5979 | 10.5596 | 10.5637 | $3.4210^{-2}$ | $0.4110^{-2}$ |
| 0.2 | $-0.7$ | 100 | 4.6943 | 4.6649 | 4.6636 | $3.0710^{-2}$ | $0.1310^{-2}$ |
| 0.2 | $-0.7$ | 110 | 1.8291 | 1.787 | 1.7874 | $4.1710^{-2}$ | $0.0410^{-2}$ |
| 0.4 | $-0.1$ | 90 | 13.3551 | 13.3117 | 13.3142 | $4.0910^{-2}$ | $0.2510^{-2}$ |
| 0.4 | $-0.1$ | 100 | 8.0315 | 8.0078 | 8.0083 | $2.3210^{-2}$ | $0.0510^{-2}$ |
| 0.4 | $-0.1$ | 110 | 4.5811 | 4.5436 | 4.5454 | $3.7510^{-2}$ | $0.1810^{-2}$ |
| 0.4 | $-0.7$ | 90 | 13.2641 | 13.2205 | 13.2172 | $4.6910^{-2}$ | $0.3310^{-2}$ |
| 0.4 | $-0.7$ | 100 | 8.0259 | 8.0011 | 7.9998 | $2.6110^{-2}$ | $0.1310^{-2}$ |
| 0.4 | $-0.7$ | 110 | 4.6536 | 4.6197 | 4.6201 | $3.3510^{-2}$ | $0.0410^{-2}$ |

Table 3. Comparisons of American put options, Malliavin stochastic solution, and the binomial one when the maturity is $\frac{1}{2}$.

In Tables 1–3, under different moneyness scenarios, and for different values of the maturity, we give the American put price computed by our model.

In order to ensure the convergence of our algorithm we use the Monte Carlo simulation under different values of path ($N = 1000$ and $N = 5000$). For all the different scenarios, comparing to the binomial model (400 time steps), we clearly notice that the error decreases in a considerable extent (Mall(5000)-Bin(400) more than Mall(1000)-Bin(400)). Hence, we can say that for $N = 5000$ our algorithm converges. With this number of samples, the obtained results present a good compromise in terms of accuracy and convergence speed. (The relative pricing errors of our model comparing the Bin(400) in different scenarios are less than one percent.)

## 4. Conclusion

We have performed a methodology based on Malliavin calculus to compute the conditional expectation related to the problem of the pricing of the American put option when the volatility is stochastic (generated by the CIR process). We have identified two new processes allowing us to compute and write the general analytical solution of the conditional expectation $\mathbb{E}(P_t(X_t, V_t) \mid (X_l, V_l))$ for $l \leq t$, which appears as a suitable ratio of ordinal expectations. Our results extend the work by Bally et al. (2005) from the constant volatility to the stochastic case. From now on, and with this new formula, the above conditional expectation becomes much easier to estimate using the Monte Carlo simulation. The theoretical results are applied to the American option pricing with reference to the binomial approach (400 steps). We simulate the results and we show that our model is efficient and more accurate when the number of samples is 5000.

## Appendix A.

In what follows, we give some definitions and lemmas which constitute the basis of our work.

**Definition A.1** (See [8])**.** Given $n \in \mathbb{N}$, the family of simple $n$-th order functionals is defined by $\mathcal{S}_n := \{\varphi(\Delta_n) \,|\, \varphi \in C^\infty_{\mathrm{pol}}(\mathbb{R}^{2^n};\mathbb{R})\}$ and we define $\mathcal{S} := \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ with $\Delta_n := (\Delta^1_n \cdots \Delta^{2n}_n)$ and $\Delta^k_n = W_{t^k_n} - W_{t^{k-1}_n}$ for $k = 0,\ldots,2^n$, where $C^\infty_{\mathrm{pol}}$ is the family of smooth functions which, together with their derivatives of any order, have at most a polynomial growth.

**Definition A.2** (See [8])**.** For every $X = \varphi(\Delta_n) \in \mathcal{S}$, the stochastic derivative of $X$ at time $t$ is defined by

$$D_t X := \frac{\partial \varphi}{\partial x_n^{k_n(t)}}(\Delta_n).$$

**Definition A.3** (See [8])**.** The space $\mathbb{D}^{1,2}$ of the Malliavin-differentiable random variables is the closure of $\mathcal{S}$ with respect to the norm $\|.\|_{1,2}$. In other words, $X \in \mathbb{D}^{1,2}$ if, and only if, there exists a sequence $(X_n)$ in $S$ such that

(1) $X = \lim_{n\to\infty} X_n$ in $L^2(\Omega)$;
(2) the limit $\lim_{n\to\infty} DX_n$ exists in $L^2([0,T]\times\Omega)$,

where $\|X\|_{1,2} = \|X\|_{L^2(\Omega)} + \|DX\|_{L^2([0,T]\times\Omega)}$.

In what follows, we introduce the adjoint operator of the Malliavin derivative.

**Definition A.4** (See [8])**.** For a fixed $n \in \mathbb{N}$, the family $\mathcal{P}_n$ of the $n$-th order simple processes consists of the processes $U$ of the form $U_t = \sum_{k=1}^{2^n} \varphi_k(\Delta_n)\mathbf{1}_{I^k_n}(t)$, where $\varphi_k \in C^\infty_{\mathrm{pol}}(\mathbb{R}^{2^n};\mathbb{R})$ for $k = 1,\ldots,2^n$, $I^k_n = ]t^{k-1}_n, t^k_n]$, and $t^k_n := \frac{k}{2^n}$.

The above formula can be rewritten more simply as $U_t = \varphi_{k_n(t)}(\Delta_n)$. We notice that $\mathcal{P}_n \subseteq \mathcal{P}_{n+1},\ n \in \mathbb{N}$, and we define $\mathcal{P} := \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ as the family of simple functionals. It is obvious that $D : \mathcal{S} \to \mathcal{P}$.

**Definition A.5** (See [8] and [9])**.** Given a simple process $U \in \mathcal{P}$ as defined in Definition A.4, we set

$$D * U = \sum_{k=1}^{2^n} (\varphi_k(\Delta_n)\Delta^k_n - \partial_{x^k_n}\varphi_k(\Delta_n)\frac{1}{2^n}).$$

$D * U$ is called the Skorohod integral of $U$; we will write $D * U = \int_0^T U_t \diamond dW_t$.

**Lemma A.6** (See [8])**.** *Let $X \in \mathbb{D}^{1,2}$ and let $U$ be a second-order Skorohod integrable process. Then,*

$$\int_0^T XU_t \diamond dW_t = X\int_0^T U_t \diamond dW_t - \int_0^T (D_tX)U_t\,dt$$

*and, when $U_t$ is adapted, the above equation becomes*

$$\int_0^T XU_t \diamond dW_t = X\int_0^T U_t\,dW_t - \int_0^T (D_tX)U_t\,dt.$$

**Lemma A.7** (Stochastic integration by parts; see [9]). *Let $F \in C_b^1$ and $X \in \mathbb{D}^{1,2}$. Then, the integration by parts*

$$E(F'(X)Y) = E\left(F(X)\int_0^T \frac{u_t Y}{\int_0^T u_s D_s X ds} \diamond dW_t\right) \tag{A.1}$$

*holds for every random variable $Y$ and for every stochastic process $u$ for which* (A.1) *is well-defined, where $C_b^1$ is the space of functions in $C^1$ bounded together with their derivatives.*

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