FINITE-DIMENSIONAL HOPF ALGEBRAS OVER THE KAC–PALJUTKIN ALGEBRA $H_8$

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Abstract. Let $H_8$ be the Kac–Paljutkin algebra [Trudy Moskov. Mat. Obšč. 15 (1966), 224–261], which is the neither commutative nor cocommutative semisimple eight dimensional Hopf algebra. All simple Yetter–Drinfel’d modules over $H_8$ are given, and finite-dimensional Nichols algebras over $H_8$ are determined completely. It turns out that they are all of diagonal type. In fact, they are of Cartan types $A_1$, $A_2$, $A_2 \times A_2$, $A_1 \times \cdots \times A_1$, and $A_1 \times \cdots \times A_1 \times A_2$, respectively. By the way, we calculate Gelfand–Kirillov dimensions for some Nichols algebras. As an application, we complete the classification of the finite-dimensional Hopf algebras over $H_8$ according to the lifting method.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. The question of classification of all Hopf algebras over $\mathbb{K}$ of a given dimension up to isomorphism was posed by Kaplansky in 1975 [40]. Some progress has been made but, in general, it is a difficult question for lack of standard methods. One breakthrough is the so-called lifting method introduced by Andruskiewitsch and Schneider in 1998 [3], under the assumption that the coradical is a Hopf subalgebra.

We describe the procedure for the lifting method briefly. Let $H$ be a Hopf algebra whose coradical $H_0$ is a Hopf subalgebra. The associated graded Hopf algebra of $H$ is isomorphic to $R\#H_0$, where $R = \bigoplus_{n \in \mathbb{N}_0} R(n)$ is a braided Hopf algebra in the category $H_0^\# \mathcal{YD}$ of Yetter–Drinfel’d modules over $H_0$, $\#$ stands for the Radford biproduct or bosonization of $R$ with $H_0$. As explained in [14], to classify finite-dimensional Hopf algebras $H$ whose coradical is isomorphic to $H_0$ we have to deal with the following questions:

(a) Determine all Yetter–Drinfel’d modules $V$ over $H_0$ such that the Nichols algebra $\mathfrak{B}(V)$ has finite dimension; find an efficient set of relations for $\mathfrak{B}(V)$.
(b) If $R = \oplus_{n \in \mathbb{N}_0} R(n)$ is a finite-dimensional Hopf algebra in $H_0^H \mathcal{YD}$ with $V = R(1)$, decide if $R \simeq \mathcal{B}(V)$. Here $V = R(1)$ is a braided vector space called the infinitesimal braiding.

(c) Given $V$ as in (a), classify all $H$ such that $\text{gr} \, H \simeq \mathcal{B}(V) \# H_0^H$ (lifting).

A lifting of $V \in H_0^H \mathcal{YD}$ is a Hopf algebra $L$ such that $\text{gr} \, L = \mathcal{B}(V) \# H$, where $\text{gr} \, L$ is the graded Hopf algebra associated to the coradical filtration. In other words [16 Proposition 2.4], $L$ is a lifting of $V$ iff there is an epimorphism of Hopf algebras $\phi : T(V) := T(V) \# H \to L$ such that $\phi|_H = \text{id}_H$ and

$$\phi|_{H \oplus V \# H} : H \oplus V \# H \to L_1$$

is an isomorphism of Hopf bimodules. (1.1)

Such $\phi$ is called a lifting map. If emphasis on $H$ is needed, then we say that $L$ is a lifting of $V$ over $H$.

The lifting method was extensively used in the classification of finite-dimensional pointed Hopf algebras such as [15], [12], [25], [23], [2], [1], [9], [8] and so on. It is also effective to study finite-dimensional copointed Hopf algebras ([16], [27], [22]). We note that there are very few classification results on finite-dimensional Hopf algebras whose coradical is neither a group algebra nor the dual of a group algebra, some exceptions being [19], [26], [11]. It should be mentioned that [11] constructed Hopf algebras with the Chevalley property over a semisimple Hopf algebra $H$ that is Morita-equivalent to a group algebra $\mathbb{K}G$ (in the sense of $H^H \mathcal{YD} \simeq \mathbb{K}G \mathcal{YD}$ as braided tensor categories). It doesn’t cover our case since $H_8$ can be obtained from a group algebra by a 2-pseudo-cocycle twist but not by a 2-cocycle twist [45].

Here we would like to initiate a project for the study of Hopf algebras whose coradicals are low-dimensional neither commutative nor cocommutative semisimple Hopf algebras by running procedures of the lifting method. One important step is to study the Nichols algebras over those low-dimensional semisimple Hopf algebras. Nichols algebras were studied first by Nichols [44]. These are connected graded braided Hopf algebras [1] generated by primitive elements, and all primitive elements are of degree one. In the past decades, the study of Nichols algebras was mainly focused on categories of Yetter–Drinfeld modules over group algebras. Under the assumption that the base field has characteristic 0, the classification of finite-dimensional Nichols algebras over abelian groups was completely solved in [30, 31] by using Lie theoretic structures, and the result of the classification played an important role later in the significant work [15]. The problem of classifying finite-dimensional Nichols algebras over non-abelian groups is difficult in general for lack of systematic method; for related works please refer to [12], [24], [29], [32], [33], [36], [35], etc.

In this paper, we mainly focus on the Kac–Paljutkin algebra $H_8$. The structure of our paper is as follows. In Section 2 we recall the fundamental notions related to Yetter–Drinfeld modules, Nichols algebras and Gelfand–Kirillov dimension. In Section 3 we construct all the simple left Yetter–Drinfeld modules over $H_8$ according to Radford’s method. In section 4 we get all the possible finite-dimensional Nichols algebras from Yetter–Drinfeld modules over $H_8$. It turns out that they are of Cartan types $A_1$, $A_2$, $A_2 \times A_2$, $A_1 \times \cdots \times A_1$, and $A_1 \times \cdots \times A_1 \times A_2$. Here is our first main result.
Theorem A. Let $M \in H_{H_8}^\mathcal{YD}$. Then the Nichols algebra $\mathcal{B}(M)$ is finite-dimensional if $M$ is isomorphic to one of the following Yetter–Drinfel’d modules:

1. $\Omega_1(n_1, n_2, n_3, n_4) := \bigoplus_{j=1}^{n_1} M(b_j, g_j)^{\oplus n_j}$ with $\sum_{j=1}^{n_1} n_j \geq 1$, $(b_1, g_1) = (i, x)$, $(b_2, g_2) = (-i, x)$, $(b_3, g_3) = (i, y)$ and $(b_4, g_4) = (-i, y)$, the infinitesimal braiding is of type $A_1 \times \cdots \times A_1$.

2. $\Omega_2(n_1, n_2) := M(i, x)^{\oplus n_1} \oplus M(-i, x)^{\oplus n_2} \oplus M((xy, x))$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is of type $A_1 \times \cdots \times A_1 \times A_2$.

3. $\Omega_3(n_1, n_2) := M(i, y)^{\oplus n_1} \oplus M(-i, y)^{\oplus n_2} \oplus M((y, xy))$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is of type $A_1 \times \cdots \times A_1 \times A_2$.

4. $\Omega_4(n_1, n_2) := M(i, x)^{\oplus n_1} \oplus M(i, y)^{\oplus n_2} \oplus W_1^{1,-1}$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is of type $A_1 \times \cdots \times A_1 \times A_2$.

5. $\Omega_5(n_1, n_2) := M(-i, x)^{\oplus n_1} \oplus M(-i, y)^{\oplus n_2} \oplus W_1^{1,-1}$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is of type $A_1 \times \cdots \times A_1 \times A_2$.

6. $\Omega_6 := M((xy, x)) \oplus M((y, xy))$, the infinitesimal braiding is of type $A_2 \times A_2$.

7. $\Omega_7 := W_1^{1,-1} \oplus W_1^{1,-1}$, the infinitesimal braiding is of type $A_2 \times A_2$.

Remark 1.1. We point out which of the Yetter–Drinfel’d modules have a principal realization and which not, since the liftings are known when there is a principal realization and not otherwise [3 Subsection 2.2]. Let $(h)$ and $(\delta_h)$ be dual bases of $H_8$ and $H_8^*$, and $b \in \{\pm 1, \pm i\}$. Define $\chi_b := \delta_1 + \delta_{xy} + b^2(\delta_x + \delta_y) + b(\delta_z + \delta_{xy}) + b^3(\delta_{zx} + \delta_{zy}) \in \text{Alg}(H_8, \mathbb{K})$, then $(g, \chi_b)$ is a YD-pair if $\mathbb{K}^{\chi_b} \simeq M(b, g)$ is a one-dimensional Yetter–Drinfel’d module. $\mathcal{B}((g_1, g_2))$ for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$ and $W_1^{b_1,-1}$ for $b_1 = \pm 1$ don’t have a principal realization. So only $\Omega_1(n_1, n_2, n_3, n_4)$ has a principal realization.

In section [5], according to the lifting method, we give a classification for finite-dimensional Hopf algebras over $H_8$. Here is the second main result.

Theorem B. Let $H$ be a finite-dimensional Hopf algebra over $H_8$ such that its infinitesimal braiding is in $H_8^\mathcal{YD}$. Then $H$ is isomorphic to either of:

1. $\mathcal{A}_1(n_1, n_2, n_3, n_4; I_1)$, see Definition [5.4];
2. $\mathcal{B}(\Omega_2(n_1, n_2))\#H_8$, see Proposition [5.10];
3. $\mathcal{A}_4(n_1, n_2; I_4)$, see Definition [5.18];
4. $\mathcal{A}_6(\lambda)$, see Definition [5.11];
5. $\mathcal{A}_7(\lambda)$, see Definition [5.15].

$\mathcal{A}_1(n_1, n_2, n_3, n_4; I_1)$ comprises two 16-dimensional nonisomorphic nonpointed self-dual Hopf algebras with coradical $H_8$ described in [19] as special cases. Except for the case (2), the remaining four families of Hopf algebras contain non-trivial lifting relations.
2. Preliminaries

2.1. Conventions. Let $H$ be a Hopf algebra over $\mathbb{K}$, with antipode $S$. We will use Sweedler’s notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for the comultiplication ([43]). Let $H^H \mathcal{YD}$ be the category of left Yetter–Drinfel’d modules over $H$. A left Yetter–Drinfel’d module $M$ over $H$ is a left $H$-module $(M, \cdot)$ and a left $H$-comodule $(M, \rho)$ satisfying

$$
\rho(h \cdot m) = h_{(1)} m_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}, \quad \forall m \in M, h \in H,
$$

where $\rho(m) = m_{(-1)} \otimes m_{(0)}$. The category $H^H \mathcal{YD}$ is a braided monoidal category. The braiding $c \in \text{End}_H(M \otimes M)$ of $M$ is defined by $c(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, and the inverse braiding is defined by $c^{-1}(v \otimes w) = w_{(0)} \otimes (S^{-1}(w_{(-1)}) \cdot v)$.

Definition 2.1 ([13] Definition 2.1]). Let $H$ be a Hopf algebra and $V \in H^H \mathcal{YD}$. A braided $\mathbb{N}$-graded Hopf algebra $R = \bigoplus_{n \geq 0} R(n) \in H^H \mathcal{YD}$ is called the Nichols algebra of $V$ if

(i) $\mathbb{K} \simeq R(0), V \simeq R(1) \in H^H \mathcal{YD}$.

(ii) $R(1) = \mathcal{P}(R) = \{ r \in R \mid \Delta_R(r) = r \otimes 1 + 1 \otimes r \}$.

(iii) $R$ is generated as an algebra by $R(1)$.

In this case, $R$ is denoted by $\mathfrak{B}(V) = \bigoplus_{n \geq 0} \mathfrak{B}^n(V)$.

Remark 2.2. The Nichols algebra $\mathfrak{B}(V)$ is completely determined by the braiding. More precisely, as proved in [49] and noted in [13],

$$
\mathfrak{B}(V) = K \oplus V \oplus \bigoplus_{n=2}^{\infty} V^{\otimes n} / \ker \mathfrak{S}_n = T(V) / \ker \mathfrak{S},
$$

where $\mathfrak{S}_{n,1} \in \text{End}_H(V^{\otimes n+1})$, $\mathfrak{S}_n \in \text{End}_H(V^{\otimes n})$,

$$
\mathfrak{S}_{n,1} := \text{id} + c_n + c_n c_{n-1} + \cdots + c_n c_{n-1} \cdots c_1 = \text{id} + c_n \mathfrak{S}_{n-1,1},
$$

$\mathfrak{S}_1 := \text{id}$, $\mathfrak{S}_2 := \text{id} + c$, $\mathfrak{S}_n := (\mathfrak{S}_{n-1} \otimes \text{id}) \mathfrak{S}_{n-1,1}$.

Lemma 2.3 ([28] Theorem 2.2], [6] Remark 1.4]). Let $M_1, M_2 \in H^H \mathcal{YD}$ be both finite-dimensional and assume $c_{M_1, M_2} c_{M_2, M_1} = \text{id}_{M_2 \otimes M_1}$. Then $\mathfrak{B}(M_1 \oplus M_2) \simeq \mathfrak{B}(M_1) \otimes \mathfrak{B}(M_2)$ as graded vector spaces and $\text{GKdim} \mathfrak{B}(M_1 \oplus M_2) = \text{GKdim} \mathfrak{B}(M_1) + \text{GKdim} \mathfrak{B}(M_2)$.

Proposition 2.4 ([40] Radford biproduct]). Let $H$ be a Hopf algebra and $A \in H^H \mathcal{YD}$ be a braided Hopf algebra. Then $A \# H$ is a Hopf algebra with

$$
\Delta(a \# h) = \sum [a_{(1)} \# (a_{(2)})_{(-1)} h_{(1)}] \otimes [(a_{(2)})_{(0)} \# h_{(2)}],
$$

$$
S(a \# h) = \sum (1 \# S_H(h) S_H(a_{(-1)})) (S_A(a_{(0)}) \# 1),
$$

$$
(a \# h)(a' \# h') = \sum a h_{(1)} \cdot a' \# h_{(2)} h', \quad a, a' \in A, h, h' \in H.
$$

The map $\iota : H \to A \# H$ given by $\iota(h) = 1 \# h$ for all $h \in H$ is an injective Hopf algebra map, and the map $\pi : A \# H \to H$ given by $\pi(a \# h) = \varepsilon_A(a) h$ for all $a \in A$, $h \in H$ is a surjective Hopf algebra map such that $\pi \circ \iota = \text{id}_H$. Moreover, it holds that $A = (A \# H)^{co \pi}$.
Conversely, let $B$ be a Hopf algebra with bijective antipode and $\pi : B \rightarrow H$ a Hopf algebra epimorphism admitting a Hopf algebra section $\iota : H \rightarrow B$ such that $\pi \circ \iota = \text{id}_H$. Then $A = B^{\text{co}\pi}$ is a braided Hopf algebra in $H_H \mathcal{YD}$ and $B \simeq A \# H$ as Hopf algebras.

2.2. GK-dimension. Let $A$ be a finitely generated algebra over a field $\mathbb{K}$, and assume $a_1, \ldots, a_m$ generate $A$. Set $V$ to be the span of $a_1, \ldots, a_m$, and denote $V^n$ the span of all monomials in the $a_i$’s of length $n$. As $a_i$’s generate $A$ one has $A = \bigcup_{k=0}^{\infty} A_k$, where $A_k = \mathbb{K} + V + V^2 + \cdots + V^k$. The function $d_V(n) = \dim A_n$ is the growth function of $A$. The Gelfand–Kirillov dimension of a $\mathbb{K}$-algebra $A$ is $\text{GKdim } A = \lim_{n \rightarrow \infty} \log n \dim \sum_{0 \leq j \leq n} V^j$. Clearly, if $V$ is a GK-deterministic subspace of $A$, then any finite-dimensional subspace of $A$ containing $V$ is GK-deterministic. Let $A$ and $B$ be two algebras. Then

$$\text{GKdim}(A \otimes B) \leq \text{GKdim } A + \text{GKdim } B,$$

but the equality does not hold in general. However, it does hold when $A$ or $B$ has a GK-deterministic subspace, see [41, Proposition 3.11]. The Gelfand–Kirillov dimension is a useful tool in ring theory and Hopf algebraic theories. We shall not discuss in detail its importance but we refer the reader to [41] as a basic reference and [51, 50, 18, 6] for additional information related with Hopf algebras.

3. Simple Yetter–Drinfel’d modules of $H_8$

Recall that the neither commutative nor cocommutative semisimple 8-dimensional Hopf algebra $H_8$ in [42] is constructed as an extension of $\mathbb{K}[C_2 \times C_2]$ by $\mathbb{K}[C_2]$. A basis for $H_8$ is given by $\{1, x, y, xy = yx, z, xz, yz, xyz\}$ with the relations

$x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad xy = yx, \quad xz = yz, \quad zy = xz.$

The coalgebra structure and the antipode are defined by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \varepsilon(x) = \varepsilon(y) = 1, \quad S(x) = x, \quad S(y) = y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes y + y \otimes 1 - y \otimes x)(z \otimes z), \quad \varepsilon(z) = 1, \quad S(z) = z.$$

The automorphism group of $H_8$ is the Klein four-group [18]. These automorphisms are given in Table 1 they are going to be used in Corollary 5.3.

Denote a set of central orthogonal idempotents of $H_8$ as

$$e_1 = \frac{1}{8}(1 + x)(1 + y)(1 + z), \quad e_2 = \frac{1}{8}(1 + x)(1 + y)(1 - z),$$

$$e_3 = \frac{1}{8}(1 - x)(1 - y)(1 + iz), \quad e_4 = \frac{1}{8}(1 - x)(1 - y)(1 - iz),$$

Table 1. Hopf algebra automorphisms of $H_8$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>1</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_1 = \text{id}$</td>
<td>1</td>
<td>$x$</td>
<td>$y$</td>
<td>$z$</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>1</td>
<td>$x$</td>
<td>$y$</td>
<td>$xyz$</td>
</tr>
<tr>
<td>$\tau_3$</td>
<td>1</td>
<td>$y$</td>
<td>$x$</td>
<td>$\frac{1}{2}(z + xz + yz - xyz)$</td>
</tr>
<tr>
<td>$\tau_4$</td>
<td>1</td>
<td>$y$</td>
<td>$x$</td>
<td>$\frac{1}{2}(-z + xz + yz + xyz)$</td>
</tr>
</tbody>
</table>

$e_5 = \frac{1 - xy}{2}$, $e_j e_k = \delta_{jk}$, $j, k = 1, \ldots, 5$; $i = \sqrt{-1}$; and denote idempotents $e_5' = \frac{1}{4}(1 - xy)(1 + z)$, $e_5'' = \frac{1}{4}(1 - xy)(1 - z)$. Then

$$H_8 = H_8 e_1 \oplus H_8 e_2 \oplus H_8 e_3 \oplus H_8 e_4 \oplus H_8 e_5$$

$$= H_8 e_1 \oplus H_8 e_2 \oplus H_8 e_3 \oplus H_8 e_4 \oplus (H_8 e_5' + H_8 e_5''),$$

where $H_8 e_5' \simeq H_8 e_5''$ as left $H_8$-modules, via $e_5' \mapsto xe_5''$, $xe_5' \mapsto e_5''$.

**Definition 3.1.** Denote $V_1(b) := \mathbb{K}\{v \mid x \cdot v = b^2 v, y \cdot v = b^2 v, z \cdot v = bv, b \in \{\pm 1, \pm i\}\}$, where $v$ is a vector. Let $V_2 \simeq H_8 e_5'$ as left $H_8$-modules; the actions of the generators are given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**Proposition 3.2.** All simple left modules of $H_8$ are classified by $V_1(b), V_2, b \in \{\pm 1, \pm i\}$.

**Remark 3.3.** The result was also obtained in [20] under a different basis (thanks to referee for reminding us about this fact).

In the remaining part of the article, $V_1(b)$ and $V_2$ always mean simple left $H_8$-modules.

**Lemma 3.4 ([47, Proposition 2]).** Let $H$ be a bialgebra over a field $\mathbb{K}$ and suppose $S$ is the antipode of $H$.

1. If $L \in H \mathcal{M}$, then $L \otimes H \in H \mathcal{YD}^H$; the module and comodule actions are given by

$$h \cdot (\ell \otimes a) = h_{(2)} \cdot \ell \otimes h_{(3)} a S^{-1}(h_{(1)}), \quad \rho(\ell \otimes h) = (\ell \otimes h_{(1)}) \otimes h_{(2)}, \quad \forall h, a \in H, \ell \in L.$$

Let $M \in H \mathcal{YD}^H$.

2. Suppose that $L \in H \mathcal{M}$ and $p : M \rightarrow L$ is a map of left $H$-modules. Then the linear map $f : M \rightarrow L \otimes H$ defined by $f(m) = p(m_{(0)}) \otimes m_{(1)}$ for all $m \in M$ is a map of Yetter–Drinfel’d $H$-modules, where $L \otimes H$ has the structure described in part (1). Furthermore ker $f$ is the largest Yetter–Drinfel’d $H$-submodule, indeed the largest subcomodule, contained in ker $p$.

3. $M$ is isomorphic to a Yetter–Drinfel’d submodule of some $L \otimes H$ described above.
Similarly, according to Radford’s method, any simple left Yetter–Drinfel’d module over \( H_8 \) could be constructed by the submodule of the tensor product of a left module \( V \) of \( H_8 \) and \( H_8 \) itself, where the module and comodule structures are given by

\[
h \cdot (\ell \boxtimes g) = (h_{(2)} \cdot \ell) \boxtimes h_{(1)} g S(h_{(3)}),
\]

\[
\rho(\ell \boxtimes h) = h_{(1)} \otimes (\ell \boxtimes h_{(2)}), \quad \forall h, g \in H_8, \ell \in V.
\] (3.1)

Here we use \( \boxtimes \) instead of \( \otimes \) to avoid confusion by using too many symbols of the tensor product. We are going to construct all simple left Yetter–Drinfel’d modules over \( H_8 \) in this way. Keeping in mind that \( H_8 \) is semisimple, it is possible to do so. In fact, it is much easier than making use of the fact that \( H_8 \coprod H_8 \cong \mathcal{D}(H_8) \). The following is a list of useful formulae for looking for simple objects of \( H_8 \coprod H_8 \).

**Lemma 3.5.**

\[
(id \otimes S) \Delta^{(2)}(z) = \frac{1}{4} [(1 + y)z \otimes z \otimes z(1 + x) + (1 - y)z \otimes zx \otimes z(1 + x) + (1 + y)z \otimes yz \otimes z(1 - x) + (1 - y)z \otimes xy \otimes z(1 - x)]
\]

\[
= \frac{1}{4} [z \otimes (1 + x)(1 + y) + xz \otimes (1 + x)(1 - y) + yz \otimes (1 - x)(1 + y) + xy \otimes (1 - x)(1 - y)],
\] (3.2)

\[
z_{(2)} \otimes z_{(1)} S(z_{(3)}) = \frac{1}{4} [z \otimes (1 + x)(1 + y) + xz \otimes (1 + x)(1 - y) + yz \otimes (1 - x)(1 + y) + xy \otimes (1 - x)(1 - y)],
\] (3.3)

\[
z_{(2)} \otimes z_{(1)} y S(z_{(3)}) = \frac{1}{4} [z \otimes (1 + x)(1 + y) + xz \otimes (1 + x)(1 - y) + yz \otimes (1 - x)(1 + y) + xy \otimes (1 - x)(1 - y)],
\] (3.4)

\[
z_{(2)} \otimes z_{(1)} xy S(z_{(3)}) = \frac{1}{4} [z \otimes (1 + x)(1 + y) + xz \otimes (1 + x)(1 - y) + yz \otimes (1 - x)(1 + y) + xy \otimes (1 - x)(1 - y)],
\] (3.5)

\[
z_{(2)} \otimes z_{(1)} z S(z_{(3)}) = \frac{1}{2} [z \otimes (1 + y)z + xz \otimes x(y - 1)z],
\] (3.6)

\[
z_{(2)} \otimes z_{(1)} xz S(z_{(3)}) = \frac{1}{2} [z \otimes (1 + y)z + xz \otimes x(1 - y)z],
\] (3.7)

\[
z_{(2)} \otimes z_{(1)} yz S(z_{(3)}) = \frac{1}{2} [z \otimes x(1 + y)z + xy \otimes (y - 1)z],
\] (3.8)

\[
z_{(2)} \otimes z_{(1)} xyz S(z_{(3)}) = \frac{1}{2} [z \otimes x(1 + y)z + xy \otimes (1 - y)z].
\] (3.9)

**Definition 3.6.** Define \( M(b, g) := \mathbb{K}\{v \boxtimes g \mid v \in V_1(b)\} \), where \( b \in \{\pm 1, \pm i\} \) and \( g \in \{1, x, y, xy\} \).

**Lemma 3.7.** There are eight pairwise non-isomorphic one dimensional Yetter–Drinfel’d modules over \( H_8 \) as \( M(b, g) \) with \((b, g) \in \{(\pm 1, 1), (\pm 1, xy), (\pm i, x), (\pm i, y)\}\). The actions and coactions are given by

\[
x \cdot (v \boxtimes g) = b^2(v \boxtimes g), \quad y \cdot (v \boxtimes g) = b^2(v \boxtimes g), \quad z \cdot (v \boxtimes g) = b(v \boxtimes g),
\]
\[ \rho(v \otimes g) = g \otimes (v \otimes g), \quad v \otimes g \in M(b, g), \quad v \in V_1(b). \]

**Proof.** Let \( v \in V_1(b) \). Then

\[
\begin{align*}
z \cdot (v \otimes 1) &= \frac{3b}{4} v \otimes [1 + x + b^2(1 - x)][1 + y + b^2(1 - y)], \quad (3.10) \\
z \cdot (v \otimes xy) &= \frac{3b}{4} v \otimes [1 + x + b^2(x - 1)][1 + y + b^2(y - 1)], \quad (3.11) \\
z \cdot (v \otimes x) &= \frac{3b}{4} v \otimes [1 + x + b^2(1 - x)][1 + y + b^2(1 - y)], \quad (3.12) \\
z \cdot (v \otimes y) &= \frac{3b}{4} v \otimes [1 + x + b^2(x - 1)][1 + y + b^2(1 - y)]. \quad (3.13)
\end{align*}
\]

so

\[
\begin{align*}
z \cdot (v \otimes 1) &= bv \otimes 1, \quad z \cdot (v \otimes xy) = bv \otimes xy, \quad \text{when } b = \pm 1; \\
z \cdot (v \otimes x) &= bv \otimes x, \quad z \cdot (v \otimes y) = bv \otimes y, \quad \text{when } b = \pm 1.
\end{align*}
\]

Now it is easy to see that \( M(b, g) \) defined above is a one-dimensional Yetter–Drinfel’d module by Radford’s method and the eight one-dimensional Yetter–Drinfel’d modules are pairwise non-isomorphic by observations on their actions and coactions. \( \square \)

**Definition 3.8.** Let \((g_1, g_2) \in \{(1, y), (x, 1), (xy, x), (y, xy)\}\) and denote three vector spaces as

\[
\begin{align*}
M\langle (1, xy) \rangle &:= \mathbb{K}\{v \otimes 1, v \otimes xy \mid v \in V_1(i)\}, \\
M\langle (x, y) \rangle &:= \mathbb{K}\{v \otimes x, v \otimes y \mid v \in V_1(1)\}, \\
M\langle (g_1, g_2) \rangle &:= \mathbb{K}\{(v_1 + v_2) \otimes g_1, (v_1 - v_2) \otimes g_2 \mid v_1, v_2 \in V_2\}.
\end{align*}
\]

**Lemma 3.9.** There are six pairwise non-isomorphic two-dimensional simple Yetter–Drinfel’d modules over \( H_8 \) as below, where the action and coaction are given by formulae (3.1).

1. \( M\langle (1, xy) \rangle \), the actions of generators on \((v \otimes 1, v \otimes xy)\) are given by

\[
\begin{align*}
x &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\end{align*}
\]

2. \( M\langle (x, y) \rangle \), the actions of generators on \((v \otimes x, v \otimes y)\) are given by

\[
\begin{align*}
x &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

3. \( M\langle (g_1, g_2) \rangle \), where \((g_1, g_2) \in \{(1, y), (x, 1), (xy, x), (y, xy)\}\); the actions of generators on the row vector \((v_1 + v_2) \otimes g_1, (v_1 - v_2) \otimes g_2\) are given by

\[
\begin{align*}
x &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]
Proof. Since the coactions are easy to see, we can focus on their structures as left $H_8$-modules. Parts (1) and (2) of the lemma can be checked by formulae (3.10) to (3.13). Let $v_1, v_2 \in V_2$. Then

\[ z \cdot (v_1 \otimes 1) = \frac{1}{2} [v_1 \otimes (x + y) + v_2 \otimes (x - y)], \]

\[ z \cdot (v_2 \otimes 1) = \frac{1}{2} [v_1 \otimes (-x + y) + v_2 \otimes (-x - y)], \]

\[ z \cdot (v_1 \otimes xy) = \frac{1}{2} [v_1 \otimes (x + y) + v_2 \otimes (y - x)], \]

\[ z \cdot (v_2 \otimes xy) = \frac{1}{2} [v_1 \otimes (x - y) + v_2 \otimes (-x - y)], \]

\[ z \cdot (v_1 \otimes y) = \frac{1}{2} [v_1 \otimes (1 + xy) + v_2 \otimes (1 - xy)], \]

\[ z \cdot (v_2 \otimes y) = \frac{1}{2} [v_1 \otimes (-1 + xy) + v_2 \otimes (-1 - xy)], \]

\[ z \cdot (v_1 \otimes x) = \frac{1}{2} [v_1 \otimes (1 + xy) + v_2 \otimes (-1 + xy)], \]

\[ z \cdot (v_2 \otimes x) = \frac{1}{2} [v_1 \otimes (1 - xy) + v_2 \otimes (-1 - xy)]. \]

So we have

\[ z \cdot [(v_1 + v_2) \otimes 1] = (v_1 - v_2) \otimes y, \quad z \cdot [(v_1 - v_2) \otimes y] = (v_1 + v_2) \otimes 1, \]

\[ z \cdot [(v_1 + v_2) \otimes x] = (v_1 - v_2) \otimes 1, \quad z \cdot [(v_1 - v_2) \otimes 1] = (v_1 + v_2) \otimes x, \]

\[ z \cdot [(v_1 + v_2) \otimes xy] = (v_1 - v_2) \otimes x, \quad z \cdot [(v_1 - v_2) \otimes x] = (v_1 + v_2) \otimes xy, \]

\[ z \cdot [(v_1 + v_2) \otimes y] = (v_1 - v_2) \otimes xy, \quad z \cdot [(v_1 - v_2) \otimes xy] = (v_1 + v_2) \otimes y. \]

Part (3) is immediate to check. The six two-dimensional Yetter–Drinfel’d modules are pairwise non-isomorphic since they are pairwise non-isomorphic as comodules.

□

Lemma 3.10. Let $b_1, b_2 \in \{\pm 1\}$ and $v \in V_1(b_2)$, and denote

\[ w_{1, b_2} := v \otimes (1 + ib_1y)z, \quad w_{2, b_2} := v \otimes x(1 - ib_1y)z. \]

Then $W_{b_1, b_2} = \mathbb{K}w_{1, b_2} \oplus \mathbb{K}w_{2, b_2}$ is a family of four pairwise non-isomorphic two-dimensional simple Yetter–Drinfel’d modules over $H_8$ with the actions of generators on the row vector $(w_{1, b_1, b_2}, W_{2, b_1, b_2})$ and coactions given by

\[
\begin{pmatrix}
0 & -ib_1 \\
ib_1 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & -ib_1 \\
ib_1 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
(1 + ib_1)\frac{z}{2} & (1 - ib_1)\frac{1}{2} \\
-(1 - ib_1)\frac{z}{2} & (1 + ib_1)\frac{1}{2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\rho (w_{1, b_2}) & = \frac{(1 + y)z}{2} \otimes w_{1, b_2} + \frac{(1 - y)z}{2} \otimes w_{2, b_2}, \\
\rho (w_{2, b_2}) & = \frac{x(1 + y)z}{2} \otimes w_{2, b_2} + \frac{x(1 - y)z}{2} \otimes w_{1, b_2}.
\end{pmatrix}
\]
Proof. It is straightforward by the definition of Yetter–Drinfel’d module. When $b_2 \neq b'_2$, $W^{b_1,b_2} \neq W^{b'_1,b'_2}$ since we will see that their braidings are different in Proposition 4.1. As explained in the following remark, $W^{b_1,b_2}$ has another basis $\{p_1, p_2\}$ with $p_1 \in V_1(b_2)$ and $p_2 \in V_1(-b_1b_2)$. So $W^{b_1,b_2} \neq W^{b'_1,b'_2}$ if $b_1 \neq b'_1$. □

Remark 3.11. (1) Let $M = \mathbb{K}\{v \otimes z, v \otimes xz, v \otimes yz, v \otimes xyz \mid v \in V_1(b)\}$, $b \in \{\pm 1\}$. $z$ acts on elements of $M$ as

\[
z \cdot (v \otimes z) = \frac{bv}{2} \otimes (1 - x + y + xy)z,
\]

\[
z \cdot (v \otimes xz) = \frac{bv}{2} \otimes (1 + x + y - xy)z,
\]

\[
z \cdot (v \otimes yz) = \frac{bv}{2} \otimes (-1 + x + y + xy)z,
\]

\[
z \cdot (v \otimes xyz) = \frac{bv}{2} \otimes (1 + x - y + xy)z.
\]

Then $M \simeq W^{1,b} \oplus W^{-1,b}$ as Yetter–Drinfel’d modules over $H_8$.

(2) Let $f_{jk} \coloneqq \frac{1}{4}[1 + (-1)^j x][1 + (-1)^k y]$, $j, k = 0, 1$. Denote $p_1 = w_1^{b_1,b_2} + ib_1w_2^{b_1,b_2}$, $p_2 = w_1^{b_1,b_2} - ib_1w_2^{b_1,b_2}$.

Then $W^{b_1,b_2} = \mathbb{K}p_1 \oplus \mathbb{K}p_2$ with the actions of generators on the row vector $(p_1, p_2)$ and coactions given by

\[
x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} b_2 & 0 \\ 0 & -ib_1b_2 \end{pmatrix},
\]

\[
\rho(p_1) = [f_{00} - ib_1f_{11}] z \otimes p_1 + [f_{10} + ib_1f_{01}] z \otimes p_2,
\]

\[
\rho(p_2) = [f_{00} + ib_1f_{11}] z \otimes p_1 + [f_{10} - ib_1f_{01}] z \otimes p_1.
\]

According to [42] Remark 2.14, $H_8$ is presented by generators $x, y, w$, where the expressions containing $z$ are replaced by

\[
w = \left( f_{00} + \sqrt{i}f_{10} + \frac{1}{\sqrt{i}}f_{01} + if_{11} \right) z,
\]

\[
w^2 = 1,
\]

\[
wx = yw, \quad S(w) = \left( \frac{1 + i}{2} x + \frac{1 - i}{2} y \right) w,
\]

\[
\Delta(w) = \left( \frac{1}{2} (1 + xy) \otimes 1 + \frac{1 + i}{4} (1 - xy) \otimes x + \frac{1 - i}{4} (1 - xy) \otimes y \right) (w \otimes w).
\]

Let $a + 1 = \pm \sqrt{2}$. We define

\[
w_1^{(1)} \coloneqq (v_1 + iav_2) \otimes \frac{i}{2} \left[ (x + y) + \sqrt{i}(x - y) \right] w
\]

\[
\quad + (av_1 - iv_2) \otimes \frac{i}{2} \left[ (x + y) - \sqrt{i}(x - y) \right] w,
\]

\[
w_2^{(1)} \coloneqq (v_1 + iav_2) \otimes \frac{i}{2} \left[ (1 + xy) + \sqrt{i}(1 - xy) \right] w
\]

\[
\quad - (av_1 - iv_2) \otimes \frac{i}{2} \left[ (1 + xy) - \sqrt{i}(1 - xy) \right] w,
\]
$w_1^{(2)} := (v_1 - iav_2) \otimes \frac{i}{2} [x + y] w + (av_1 + iv_2) \otimes \frac{1}{2} [(x + y) - \sqrt{i}(x - y)] w,\\
\quad w_2^{(2)} := (v_1 - iav_2) \otimes \frac{i}{2} [(1 + xy) + \sqrt{i}(1 - xy)] w - (av_1 + iv_2) \otimes \frac{1}{2} [(1 + xy) - \sqrt{i}(1 - xy)] w.$

Lemma 3.12. Let $a + 1 = \pm \sqrt{2}$. There are four pairwise non-isomorphic simple Yetter–Drinfel’d modules $W_1^a$ and $W_2^a$ over $H_8$ as follows:

1. Let $W_1^a = \mathbb{K}w_1^{(1)} \oplus \mathbb{K}w_2^{(1)}$. Then $W_1^a$ is a two-dimensional simple Yetter–Drinfel’d module over $H_8$ with actions given by

\[
\begin{align*}
x \cdot w_1^{(1)} &= -w_1^{(1)}, \\
y \cdot w_1^{(1)} &= w_1^{(1)}, \\
z \cdot w_1^{(1)} &= \frac{1}{2} (1 - i)(a + 1) w_1^{(1)}, \\
w \cdot w_1^{(1)} &= \frac{1}{\sqrt{2}} (1 - i)(a + 1) w_1^{(1)}, \\
x \cdot w_2^{(1)} &= w_2^{(1)}, \\
y \cdot w_2^{(1)} &= -w_2^{(1)}, \\
z \cdot w_2^{(1)} &= \frac{1}{2} (1 + i)(a + 1) w_2^{(1)}, \\
w \cdot w_2^{(1)} &= \frac{1}{\sqrt{2}} (1 + i)(a + 1) w_2^{(1)},
\end{align*}
\]

and coactions given by

\[
\rho \left( w_1^{(1)} \right) = \frac{1}{2} (x + y) w \otimes w_1^{(1)} + \frac{\sqrt{2}}{2} (x - y) w \otimes w_2^{(1)},
\]

\[
\rho \left( w_2^{(1)} \right) = \frac{1}{2} (1 + xy) w \otimes w_2^{(1)} + \frac{\sqrt{2}}{2} (1 - xy) w \otimes w_2^{(1)}.
\]

2. Let $W_2^a = \mathbb{K}w_1^{(2)} \oplus \mathbb{K}w_2^{(2)}$. Then $W_2^a$ is a two-dimensional simple Yetter–Drinfel’d module over $H_8$ with actions given by

\[
\begin{align*}
x \cdot w_1^{(2)} &= w_1^{(2)}, \\
y \cdot w_1^{(2)} &= -w_1^{(2)}, \\
z \cdot w_1^{(2)} &= \frac{1}{2} (1 - i)(a + 1) w_1^{(2)}, \\
w \cdot w_1^{(2)} &= \frac{i}{\sqrt{2}} (1 - i)(a + 1) w_1^{(2)}, \\
x \cdot w_2^{(2)} &= -w_2^{(2)}, \\
y \cdot w_2^{(2)} &= w_2^{(2)}, \\
z \cdot w_2^{(2)} &= \frac{1}{2} (1 + i)(a + 1) w_2^{(2)}, \\
w \cdot w_2^{(2)} &= \frac{1}{2} (1 + i)(a + 1) w_2^{(2)},
\end{align*}
\]

and coactions given by

\[
\rho \left( w_1^{(2)} \right) = \frac{1}{2} (x + y) w \otimes w_1^{(2)} + \frac{\sqrt{2}}{2} (x - y) w \otimes w_2^{(2)},
\]

\[
\rho \left( w_2^{(2)} \right) = \frac{1}{2} (1 + xy) w \otimes w_2^{(2)} + \frac{\sqrt{2}}{2} (1 - xy) w \otimes w_2^{(2)}.
\]

Proof. It is straightforward to check by the definition of Yetter–Drinfel’d module. Actually, $M \simeq \bigoplus_{a + 1 = \pm \sqrt{2}} (W_1^a \oplus W_2^a)$ as Yetter–Drinfel’d modules over $H_8$, where $M = \mathbb{K}\{v_j \otimes z, v_j \otimes xz, v_j \otimes yz, v_j \otimes xyz \mid v_j \in V_2, j = 1, 2\}$.

Since $\sqrt{-1} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}, \frac{\sqrt{2}}{2} \sqrt{-1} (1 - i) = 1$. Denote $a + 1 = b \sqrt{2}, b = \pm 1, \quad p_1^{(1)} = \sqrt{i} w_1^{(1)} + w_2^{(1)}, p_2^{(1)} = -\sqrt{i} w_2^{(1)} + w_2^{(1)}$, then $W_1^a = \mathbb{K}p_1^{(1)} \oplus \mathbb{K}p_2^{(1)}$ with actions
Proposition 4.1. Given a simple Yetter–Drinfel’d module \( G \) with a simple Yetter–Drinfel’d module with Gelfand–Kirillov dimensions for some Nichols algebras. In particular, they constructed all simple Yetter–Drinfel’d modules of \( D \)-modules by using Radford’s construction \([47]\). In particular, they constructed all simple Yetter–Drinfel’d modules over \( D \) generated by Yetter–Drinfel’d modules over \( D \) with actions on the row vector \( (p_1^{(2)}, p_2^{(2)}) \) also given by \((3.14)\). Now we can observe that \( W_{1}^{-1+\sqrt{2}} \) is isomorphic to \( W_{1}^{-1-\sqrt{2}} \) as modules (or comodules) under a suitably chosen base, but they are not isomorphic as modules and comodules simultaneously. So \( W_{1}^{-1+\sqrt{2}} \neq W_{1}^{-1-\sqrt{2}} \) as Yetter–Drinfel’d modules. For the same reason, we have \( W_{2}^{-1+\sqrt{2}} \neq W_{2}^{-1-\sqrt{2}} \) and \( W_{1}^{a} \neq W_{2}^{a} \). \( \square \)

Obviously, any module in Lemma \( \ref{lem:3.9} \) is not isomorphic to any one of modules in Lemmas \( \ref{lem:3.10} \) and \( \ref{lem:3.12} \) as comodules. As \( H_{8} \)-modules, \( W_{1}^{b_{1},b_{2}} \cong V_{1}(b_{2}) \oplus V_{1}(-b_{1}b_{2}i) \), and \( W_{1}^{a} \cong W_{2}^{a} \cong V_{2} \). So Yetter–Drinfel’d modules in Lemmas \( \ref{lem:3.9} \), \( \ref{lem:3.10} \) and \( \ref{lem:3.12} \) are pairwise non-isomorphic. Keeping in mind that \( H_{8} \) is semisimple, now we are arriving at

**Theorem 3.13.** All the simple Yetter–Drinfel’d modules over \( H_{8} \) are classified by

- Eight pairwise non-isomorphic simple Yetter–Drinfel’d modules of one-dimension:
  \[ M(\langle b, g \rangle), \quad (b, g) \in \{ (\pm 1, 1), (\pm 1, xy), (\pm i, x), (\pm i, y) \} \, \]

- Fourteen pairwise non-isomorphic simple Yetter–Drinfel’d modules of two-dimension:
  \[ M(\langle 1, xy \rangle), \ M(\langle x, y \rangle), \ M(\langle g_{1}, g_{2} \rangle), \ W_{1}^{b_{1},b_{2}}, \ W_{1}^{a}, \ W_{2}^{a}, \]

where \( (g_{1}, g_{2}) \in \{ (1, y), (x, 1), (xy, x), (y, xy) \} \), \( b_{1}, b_{2} \in \{ \pm 1 \} \), \( a+1 = \pm \sqrt{2} \).

**Remark 3.14.** Jun Hu and Yinhau Zhang investigated \( D(H) \)-modules in \([37]\) and \([38]\) by using Radford’s construction \([47]\). In particular, they constructed all simple modules of \( D(H_{8}) \) under a different basis of \( H_{8} \).

### 4. Nichols algebras in \( H_{8} \)-YD

In this section, we try to determine all the finite-dimensional Nichols algebras generated by Yetter–Drinfel’d modules over \( H_{8} \). As a byproduct, we calculate Gelfand–Kirillov dimensions for some Nichols algebras.

We begin by studying the Nichols algebras of simple Yetter–Drinfel’d modules.

**Proposition 4.1.** Given a simple Yetter–Drinfel’d module \( M \) over \( H_{8} \), \( \dim \mathfrak{B}(M) \) (GKdim \( \mathfrak{B}(M) \) for some cases) is presented in Table \([2]\). Moreover,

\[
\mathfrak{B}(M(\langle b, g \rangle)) = \begin{cases} 
\mathbb{K}[p], & \text{if } (b, g) \in \{ (\pm 1, 1), (\pm 1, xy) \}, \\
\mathbb{K}[p]/p^{2} = \mathbb{K}[p], & \text{if } (b, g) \in \{ (\pm i, x), (\pm i, y) \}.
\end{cases}
\]

(2) Both braidings of $M\langle (g_1, g_2) \rangle$ for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$ and $W^{b_1, -1}$ for $b_1 = \pm 1$ are of Cartan type $A_2$, so their corresponding Nichols algebras are isomorphic to an algebra which is generated by $p_1, p_2$ satisfying relations $p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = 0$, $p_1^2 = p_2^2 = 0$.

<table>
<thead>
<tr>
<th>$M \in H_8 \mathcal{YD}$</th>
<th>condition</th>
<th>$\dim \mathfrak{B}(M)$</th>
<th>$\text{GKdim } \mathfrak{B}(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M\langle b, g \rangle$</td>
<td>$(b, g) \in { (\pm 1, 1), (\pm 1, x) }$</td>
<td>$\infty$</td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>$(b, g) \in { (\pm i, i), (\pm i, y) }$</td>
<td>$2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$M\langle (1, xy) \rangle$</td>
<td></td>
<td>$\infty$</td>
<td>$2$</td>
</tr>
<tr>
<td>$M\langle (x, y) \rangle$</td>
<td></td>
<td>$\infty$</td>
<td>$2$</td>
</tr>
<tr>
<td>$M\langle (g_1, g_2) \rangle$</td>
<td>$(g_1, g_2) \in { (1, y), (x, 1) }$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td></td>
<td>$(g_1, g_2) \in { (xy, x), (y, xy) }$</td>
<td>$8$</td>
<td>$0$</td>
</tr>
<tr>
<td>$W^{b_1, b_2}$</td>
<td>$b_1 = \pm 1, b_2 = -1$</td>
<td>$8$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$b_1 = \pm 1, b_2 = 1$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$W^a_1, W^a_2$</td>
<td>$a + 1 = \pm \sqrt{2}$</td>
<td></td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Table 2. Nichols algebras of simple Yetter–Drinfel’d modules over $H_8$.

**Proof.**

- Because $c(p \otimes p) = g \cdot p \otimes p = \begin{cases} p \otimes p, & \text{if } (b, g) \in \{ (\pm 1, 1), (\pm 1, xy) \} \\ -p \otimes p, & \text{if } (b, g) \in \{ (\pm i, i), (\pm i, y) \} \end{cases}$ under the assumption that $M\langle b, g \rangle = \mathbb{K} p$, part (1) is obvious.

- As for part (2), we only give a proof for the case $W^{b_1, -1}$ for $b_1 = \pm 1$. Let $p_1 = w_1^{b_1, b_2} + ib_1 w_2^{b_1, b_2}$ and $p_2 = w_1^{b_1, b_2} - ib_1 w_2^{b_1, b_2}$; then the braiding of $W^{b_1, b_2}$ is given by

\[
\begin{align*}
&c(p_1 \otimes p_1) = b_2 p_1 \otimes p_1, & c(p_2 \otimes p_2) = b_2 p_2 \otimes p_2, \\
&c(p_1 \otimes p_2) = -b_2 p_2 \otimes p_1, & c(p_2 \otimes p_1) = b_2 p_1 \otimes p_2.
\end{align*}
\]

When $b_2 = 1$, $\text{GKdim } \mathfrak{B}(W^{b_1, 1}) = \infty$ according to [6, Lemma 2.8]. When $b_2 = -1$, the braiding is of type $A_2$. As discussed in [10], the Nichols algebra $\mathfrak{B}(W^{b_1, -1})$ is generated by $p_1, p_2$ with relations $p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = 0$, $p_1^2 = p_2^2 = 0$. So dim $\mathfrak{B}(W^{b_1, -1}) = 8$.

- Let $p_1 = v \cdot 1, p_2 = v \cdot xy \in M\langle (1, xy) \rangle$; then $c(p_j \otimes p_k) = p_k \otimes p_j$, where $j, k = 1, 2$. If we view $M\langle (1, xy) \rangle = \mathbb{K} p_1 \oplus \mathbb{K} p_2$ as braided vector spaces, then $\text{GKdim } \mathfrak{B}(M\langle (1, xy) \rangle) = \text{GKdim } \mathfrak{B}(\mathbb{K} p_1) + \text{GKdim } \mathfrak{B}(\mathbb{K} p_2) = 2$ by Lemma 2.3. Similarly, $\text{GKdim } \mathfrak{B}(M\langle (x, y) \rangle) = 2$.

- Let $p_1 = (v_1 + v_2) \cdot 1, p_2 = (v_1 - v_2) \cdot xy \in M\langle (1, y) \rangle$. The braiding is given by

\[
\begin{align*}
&c(p_1 \otimes p_1) = p_1 \otimes p_1, & c(p_1 \otimes p_2) = p_2 \otimes p_1, \\
&c(p_2 \otimes p_1) = -p_1 \otimes p_2, & c(p_2 \otimes p_2) = p_2 \otimes p_2.
\end{align*}
\]
By [6, Lemma 2.8], \( \text{GKdim} \mathfrak{B}(M((1,y))) = \infty \). For the same reason, we obtain \( \text{GKdim} \mathfrak{B}(M((x,1))) = \infty \).

Let \( \theta = \frac{1}{2}(i-1)(a+1) \). Then

\[
c\left( w_1^{(1)} \otimes w_1^{(1)} \right) = -\theta w_2^{(1)} \otimes w_2^{(1)}, \quad c\left( w_1^{(1)} \otimes w_2^{(1)} \right) = \theta w_1^{(1)} \otimes w_2^{(1)},
\]

\[
c\left( w_2^{(1)} \otimes w_1^{(1)} \right) = -\theta w_2^{(1)} \otimes w_1^{(1)}, \quad c\left( w_2^{(1)} \otimes w_2^{(1)} \right) = \theta w_2^{(1)} \otimes w_1^{(1)},
\]

\[
c\left( iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)} \right) = -i\theta \left( iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)} \right),
\]

\[
c\left( -iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)} \right) = i\theta \left( -iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)} \right).
\]

By induction,

\[
\mathfrak{B}_{2n-1,1} \left( \left( w_1^{(1)} \otimes w_2^{(1)} \right)^{\otimes n} \right) = \frac{(1 + \theta)[1 - (-\theta^2)^n]}{1 + \theta^2} \left( \left( w_1^{(1)} \otimes w_2^{(1)} \right)^{\otimes n} \right),
\]

\[
\mathfrak{B}_{2n,1} \left( \left( w_1^{(1)} \otimes w_2^{(1)} \right)^{\otimes n} \otimes w_1^{(1)} \right) = \frac{1 - \theta + (-1)^n \theta^{2n+1}(1 + \theta)}{1 + \theta^2} \left( \left( w_1^{(1)} \otimes w_2^{(1)} \right)^{\otimes n} \otimes w_1^{(1)} \right).
\]

It means that \( \left( w_1^{(1)} \otimes w_2^{(1)} \right)^{\otimes n} \) is an eigenvector of \( \mathfrak{B}_{2n-1} \) and

\( \left( w_1^{(1)} \otimes w_2^{(1)} \right)^{\otimes n} \otimes w_1^{(1)} \) is an eigenvector of \( \mathfrak{B}_{2n} \) both with nonzero eigenvalue. So dim \( \mathfrak{B}(W_1^1) = \infty \). And dim \( \mathfrak{B}(W_2^0) = \infty \) is similar to prove. \( \square \)

Proposition 4.2.

1. \( \mathfrak{B} [M(b, g) \oplus M(b', g')] \cong \mathfrak{B} (M(b, g)) \otimes \mathfrak{B} (M(b', g')) \) for \( (b, g), (b', g') \in \{(\pm 1, 1), (\pm 1, x, y), (\pm i, x), (\pm i, y)\} \).

2. When \( (b, g) \in \{(\pm 1, 1), (\pm 1, x, y)\} \) the following holds:

\[
\mathfrak{B} [M(b, g) \oplus M((1,x), y))] \cong \mathfrak{B} (M(b, g)) \otimes \mathfrak{B} (M((1, x), y))),
\]

\[
\mathfrak{B} [M(b, g) \oplus M((x, y))] \cong \mathfrak{B} (M(b, g)) \otimes \mathfrak{B} (M((x, y))).
\]

3. \( \mathfrak{B} [M(b, g) \oplus M((g_1, g_2))] \cong \mathfrak{B} (M(b, g)) \otimes \mathfrak{B} (M((g_1, g_2))) \) for the following cases:

\( \alpha \) \( (b, g) = (\pm i, x), (g_1, g_2) = (xy, x); \)

\( \beta \) \( (b, g) = (\pm i, y), (g_1, g_2) = (y, xy); \)

\( \gamma \) \( (b, g) = (\pm 1, 1), (g_1, g_2) \in \{(xy, x), (y, xy)\} \).

4. \( \mathfrak{B} [M(b, g) \oplus W^{b_1,-1}] \cong \mathfrak{B} (M(b, g)) \otimes \mathfrak{B} (W^{b_1,-1}) \) for the following cases:

\( \alpha \) \( (b, g) \in \{(1, 1), (1, xy)\}, b_1 = \pm 1; \)

\( \beta \) \( (b, g) \in \{(i, x), (i, y)\}, b_1 = 1; \)

\( \gamma \) \( (b, g) \in \{(-i, x), (-i, y)\}, b_1 = -1. \)

5. \( \mathfrak{B} [M((xy, x)) \oplus M((y, xy))] \cong \mathfrak{B} (M((xy, x))) \otimes \mathfrak{B} (M((y, xy))). \)

6. \( \mathfrak{B} (W^{1,-1} \oplus W^{-1,-1}) \cong \mathfrak{B} (W^{1,-1}) \otimes \mathfrak{B} (W^{-1,-1}). \)

7. \( \text{GKdim} \mathfrak{B} [M(b, g) \oplus M((g_1, g_2))] = \infty \) for \( (b, g) = (\pm i, x), (g_1, g_2) = (y, xy) \) or \( (b, g) = (\pm i, y), (g_1, g_2) = (xy, x) \).
(8) GKdim $\mathfrak{B} [M(b, g) \oplus W^{b_1,-1}] = \infty$ for $(b, g) \in \{(i, x), (i, y)\}$, $b_1 = -1$ or $(b, g) \in \{(-i, x), (-i, y)\}$, $b_1 = 1$.

(9) $\dim \mathfrak{B} (M\langle g_1, g_2 \rangle)_{\oplus 2} = \infty$ for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$.

(10) $\dim \mathfrak{B} (W^{b_1,-1} \oplus W^{b_1,-1}) = \infty$ for $b_1 = \pm 1$.

**Proof.** Parts (1)–(6) are direct results of Lemma 2.3. We only prove some cases as a byproduct in the following.

- Let $p_1 = (v_1 + v_2) \otimes g_1$, $p_2 = (v_1 - v_2) \otimes g_2 \in M\langle g_1, g_2 \rangle$, where $(g_1, g_2) \in \{(xy, x), (y, xy)\}$. Let $p = v \otimes g \in M(b, g)$. Then
  
  \[
  c(p \otimes p_1) = \begin{cases} 
  -p_1 \otimes p, & \text{if } g \in \{y, xy\} \\
  p_1 \otimes p, & \text{if } g \in \{1, x\},
  \end{cases} \quad c(p \otimes p_2) = \begin{cases} 
  -p_2 \otimes p, & \text{if } g \in \{x, xy\} \\
  p_2 \otimes p, & \text{if } g \in \{1, y\},
  \end{cases}
  \]

\[
  c(p_1 \otimes p) = \begin{cases} 
  b_2p \otimes p_1, & \text{if } g_1 = y \\
  p \otimes p_1, & \text{if } g_1 = xy,
  \end{cases} \quad c(p_2 \otimes p) = \begin{cases} 
  b_2p \otimes p_2, & \text{if } g_2 = x \\
  p \otimes p_2, & \text{if } g_2 = xy.
  \end{cases}
  \]

- When $(g_1, g_2) = (y, xy)$ and $(b, g) = (\pm i, x)$,
  \[
  c(p \otimes p_1) = p_1 \otimes p, \quad c(p \otimes p_2) = -p_2 \otimes p,
  \]
  \[
  c(p_1 \otimes p) = -p \otimes p_1, \quad c(p_2 \otimes p) = p \otimes p_2.
  \]

The generalized Dynkin diagram is given by Figure 1. According to 31, $\dim \mathfrak{B} [M(\pm i, x) \oplus M\langle y, xy \rangle] = \infty$.

\[\text{Figure 1}\]

- When $(g_1, g_2) = (xy, x)$ and $(b, g) = (\pm i, y)$, the generalized Dynkin diagram associated to the braiding is given by Figure 1. According to 31, $\dim \mathfrak{B} [M(\pm i, y) \oplus M\langle xy, xy \rangle] = \infty$. We thus finish part (7).

- As for cases listed in part (6), \(\mathfrak{B} [M(b, g) \oplus M\langle g_1, g_2 \rangle] \simeq \mathfrak{B} (M(b, g)) \otimes \mathfrak{B} (M\langle g_1, g_2 \rangle)\) by Lemma 2.3

- Let $p = v \otimes g \in M(b, g)$, where $(b, g) \in \{(\pm 1, 1), (\pm 1, xy), (\pm i, x), (\pm i, y)\}$. Then
  \[
  c(p \otimes w_1^{b_1,-1}) = \begin{cases} 
  w_1^{b_1,-1} \otimes p, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\
  ib_1 w_1^{b_1,-1} \otimes p, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\},
  \end{cases}
  \]

\[
  c(p \otimes w_2^{b_2,-1}) = \begin{cases} 
  w_2^{b_2,-1} \otimes p, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\
  -ib_1 w_2^{b_2,-1} \otimes p, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\},
  \end{cases}
  \]
This finishes (6). □

By Lemma 2.3, we have

\[ B \]

When (10).

As for

\[ W \]

since the generalized Dynkin diagram associated to the braiding is given by Figure 2. Now we finish parts (4) and (8).

As for \( (g_1, g_2) \in \{(x, y), (y, xy)\} \), \( \dim \mathfrak{B} \left( (M \langle (g_1, g_2) \rangle) ^{\otimes 2} \right) = \infty \) by [31], since the generalized Dynkin diagram associated to the braiding is given by Figure 2.

\[
\begin{array}{c}
-1 & -1 & -1 \\
-1 & & -1 \\
-1 & -1 & -1 \\
\end{array}
\]

\textbf{Figure 2}

As for \( W^{b_1, -1} \oplus W^{b'_1, -1} \) with \( b_1 \) and \( b'_1 \) in \( \{\pm 1\} \). Let

\[
p_1 = w_1^{b_1, -1} + ib'_1 w_2^{b_1, -1}, \quad p_2 = w_1^{b_1, -1} - ib'_1 w_2^{b_1, -1}, \quad p'_1 = w_1^{b_1, -1} + ib'_1 w_2^{b_1, -1} \quad \text{and} \quad p'_2 = w_1^{b_1, -1} - ib'_1 w_2^{b_1, -1}.
\]

Then

\[
c(p_1 \otimes p'_1) = -p'_1 \otimes p_1, \quad c(p_2 \otimes p'_2) = -p'_2 \otimes p_2, \quad c(p_1 \otimes p'_2) = p_2 \otimes p'_1, \quad c(p_2 \otimes p'_1) = -p'_1 \otimes p_2.
\]

When \( b_1 = b'_1 \), the generalized Dynkin diagram associated to the braiding is given by Figure 2. By [31], \( \dim \mathfrak{B} \left( W^{b_1, -1} \oplus W^{b_1, -1} \right) = \infty \). This finishes (10).

When \( b_1 = -b'_1 \), we have

\[
p_2 = w_1^{b_1, -1} + ib_1 w_2^{b_1, -1}, \quad p_1 = w_1^{b_1, -1} - ib_1 w_2^{b_1, -1}, \quad p'_2 = w_1^{b'_1, -1} + ib_1 w_2^{b'_1, -1}, \quad p'_1 = w_1^{b'_1, -1} - ib_1 w_2^{b'_1, -1},
\]

and

\[
c(p'_2 \otimes p_2) = -p_2 \otimes p'_2, \quad c(p'_1 \otimes p_1) = -p_1 \otimes p'_1, \quad c(p'_2 \otimes p_1) = p_1 \otimes p'_2, \quad c(p'_1 \otimes p_2) = -p_2 \otimes p'_1.
\]

By Lemma 2.3 we have

\[ \mathfrak{B} \left( W^{b_1, -1} \oplus W^{b_1, -1} \right) \simeq \mathfrak{B} \left( W^{b_1, -1} \right) \otimes \mathfrak{B} \left( W^{b_1, -1} \right). \]

This finishes (6). □
Proposition 4.3. The following equalities hold for $b_1 = \pm 1$:
\[
\dim \mathfrak{S}(M\langle(xy, x)\rangle \oplus W^{b_1, -1}) = \infty = \dim \mathfrak{S}(M\langle(y, xy)\rangle \oplus W^{b_1, -1}).
\]

Proof. We only prove \(\dim \mathfrak{S}(M\langle(xy, x)\rangle \oplus W^{b_1, -1}) = \infty\) because the rest is similar to prove. Let \(p'_1 = w_1^{b_1, -1} + ib_1 w_2^{b_1, -1}\) and \(p'_2 = w_1^{b_1, -1} - ib_1 w_2^{b_1, -1}\). Then
\[
c(p_1 \otimes p'_1) = p'_1 \otimes p_1, \quad c(p_1 \otimes p'_2) = p'_2 \otimes p_1,
\]
\[
c(p_2 \otimes p'_1) = p'_1 \otimes p_2, \quad c(p_2 \otimes p'_2) = -p'_2 \otimes p_2,
\]
\[
c(p'_1 \otimes p_1) = p_2 \otimes p'_1, \quad c(p'_1 \otimes p_2) = ib_1 p_1 \otimes p'_2,
\]
\[
c(p'_2 \otimes p_1) = p_2 \otimes p'_1, \quad c(p'_2 \otimes p_2) = -ib_1 p_1 \otimes p'_1.
\]
Suppose \(\mathfrak{S}(M\langle(xy, x)\rangle \oplus W^{b_1, -1})\) is finite-dimensional; then according to [33, Theorem 7.2(3)], \(\text{ad}(M\langle xy, x\rangle) (W^{b_1, -1}) = (id - c^2) (M\langle xy, x\rangle \otimes W^{b_1, -1})\) is irreducible. Denote
\[
A = (id - c^2)(p_1 \otimes p'_1) = p_1 \otimes p'_1 - p_2 \otimes p'_2;
\]
\[
B = (id - c^2)(p_1 \otimes p'_2) = p_1 \otimes p'_2 - p_2 \otimes p'_1;
\]
\[
C = (id - c^2)(p_2 \otimes p'_1) = p_2 \otimes p'_1 - ib_1 p_1 \otimes p'_2;
\]
\[
D = (id - c^2)(p_2 \otimes p'_2) = p_2 \otimes p'_2 + ib_1 p_1 \otimes p'_1.
\]
If \(a_1 A + a_2 B + a_3 C + a_4 D = 0\) for parameters \(a_j \in \mathbb{K}, j = 1, \ldots, 4\), then \(a_1 A + a_4 D = 0\) and \(a_2 B + a_3 C = 0\). Hence \(a_1 = a_4 = 0\), and \(a_2 = a_3 = 0\). So \(A, B, C, D\) are linearly independent. This is a contradiction since \((id - c^2) (M\langle xy, x\rangle \otimes W^{b_1, -1})\) is irreducible and there aren’t any 4-dimensional irreducible Yetter–Drinfel’d modules over \(H_8\).

Remark 4.4. According to Propositions 4.1, 4.2, and 4.3 we calculate Nichols algebras over direct sum of two simple objects of \(H_8^{\mathcal{YD}}\) in Table 3.

Proof of Theorem [A] Firstly, we recall the fact that for any submodule \(M_1 \subset M_2 \in H_8^{\mathcal{YD}}\), \(\mathfrak{S}(M_1) \subset \mathfrak{S}(M_2)\). Then \(\dim \mathfrak{S}(M_2) = \infty\) if \(\dim \mathfrak{S}(M_1) = \infty\). The Nichols algebras \(\mathfrak{S}(M)\) associated with \(M\) listed in Theorem [A] are finite-dimensional according to Lemma 2.3 and 31. In fact, \(\Omega_1(n_1, n_2, n_3, n_4)\) for \(k = 2, 3, 4, 5\) is of Cartan type \(A_1 \times \cdots \times A_1\); \(\Omega_k(n_1, n_2)\) for \(k = 2, 3, 4, 5\) is of Cartan type \(A_1 \times \cdots \times A_1 \times A_2\); \(n_1 + n_2 + n_3 + n_4\) \(\Omega_k\) for \(k = 6, 7\) is of Cartan type \(A_2 \times A_2\). So let \(M \in H_8^{\mathcal{YD}}\); then \(\dim \mathfrak{S}(M) < \infty\) if and only if \(M\) is isomorphic to one of the modules in the list of Theorem [A] according to Table 2, Table 3, Propositions 4.1 & 4.2.

5. Hopf Algebras over \(H_8\)

In this section, according to the lifting method, we determine the finite-dimensional Hopf algebra \(H\) with coradical \(H_8\) such that its infinitesimal braiding is isomorphic to a Yetter–Drinfel’d module \(M\) over \(H_8\). We begin by proving that \(H\) is generated by elements of degree one in Theorem 5.1. That is, \(\text{gr } H \simeq \mathfrak{S}(M) \# H_8\).
<table>
<thead>
<tr>
<th>$M \in \mathcal{H}^0 \mathcal{YD}$</th>
<th>condition</th>
<th>$\dim \mathfrak{B}(M)$</th>
<th>GKdim $\mathfrak{B}(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M\langle b_1, g_1 \rangle \oplus M\langle b_2, g_2 \rangle$</td>
<td>$(b_1, g_1), (b_2, g_2) \in {\pm 1, 1, \pm 1, xy}$</td>
<td>$\infty$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$(b_1, g_1) \in {(\pm 1, 1), (\pm 1, xy)}$</td>
<td>$\infty$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$(b_2, g_2) \in {(\pm i, x), (\pm i, y)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(b_1, g_1), (b_2, g_2) \in {(\pm i, x), (\pm i, y)}$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$M\langle b, g \rangle \oplus M\langle 1, xy \rangle$</td>
<td>$(b, g) \in {(\pm 1, 1), (\pm 1, xy)}$</td>
<td>$\infty$</td>
<td>3</td>
</tr>
<tr>
<td>$M\langle b, g \rangle \oplus M\langle x, y \rangle$</td>
<td>$(b, g) \in {(\pm 1, 1), (\pm 1, xy)}$</td>
<td>$\infty$</td>
<td>3</td>
</tr>
<tr>
<td>$M\langle (g_1, g_2) \rangle \oplus M\langle (g_1', g_2') \rangle$</td>
<td>$(g_1, g_2) = (g_1', g_2') = (xy, x)$ or $(y, xy)$</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(g_1, g_2) = (xy, x)$, $(g_1', g_2') = (y, xy)$</td>
<td>64</td>
<td>0</td>
</tr>
<tr>
<td>$W^{b_1, -1} \oplus W^{b_1', -1}$</td>
<td>$b_1 = b_1' = \pm 1$</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b_1 = 1, b_1' = -1$</td>
<td>64</td>
<td>0</td>
</tr>
<tr>
<td>$M\langle (g_1, g_2) \rangle \oplus W^{b_1, -1}$</td>
<td>$(g_1, g_2) = (xy, x)$</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>or $(y, xy), b_1 = \pm 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M\langle b, g \rangle \oplus M\langle (g_1, g_2) \rangle$</td>
<td>$(b, g) = (\pm i, x), (g_1, g_2) = (xy, x)$</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(b, g) = (\pm i, x), (g_1, g_2) = (y, xy)$</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(b, g) = (\pm i, y), (g_1, g_2) = (xy, x)$</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(b, g) = (\pm i, y), (g_1, g_2) = (y, xy)$</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(b, g) \in {(\pm 1, 1)}$</td>
<td>$\infty$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$(g_1, g_2) \in {(xy, x), (y, xy)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M\langle b, g \rangle \oplus W^{b_1, -1}$</td>
<td>$(b, g) \in {(1, 1), (1, xy)}, b_1 = \pm 1$</td>
<td>$\infty$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$(b, g) \in {(i, x), (i, y)}, b_1 = 1$</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(b, g) \in {(i, x), (i, y)}, b_1 = -1$</td>
<td>$\infty$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(b, g) \in {(-i, x), (-i, y)}, b_1 = -1$</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(b, g) \in {(-i, x), (-i, y)}, b_1 = 1$</td>
<td>$\infty$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Nichols algebras over the direct sum of two simple objects in $\mathcal{H}^0 \mathcal{YD}$. 

Theorem 5.1. Let $H$ be a finite-dimensional Hopf algebra over $H_8$ such that its infinitesimal braiding is isomorphic to a Yetter–Drinfel’d module over $H_8$. Then the diagram of $H$ is a Nichols algebra, and consequently $H$ is generated by the elements of degree one with respect to the coradical filtration.

Proof. Since $\text{gr} H \simeq R\# H_8$, with $R = \bigoplus_{n \geq 0} R(n)$ the diagram of $H$, we need to prove that $R$ is a Nichols algebra. Actually we only need to prove that $R \simeq \mathfrak{S}(M)$ for some $M$ in the list of Theorem A since $R$ is finite-dimensional. Let $\mathcal{J} = \bigoplus_{n \geq 0} R(n)^*$ be the graded dual of $R$; then $\mathcal{J}$ is a graded Hopf algebra in $H^*_8 \mathcal{YD}$ with $J(0) = \mathbb{K}$. According to [13] Lemma 5.5], $R(1) = \mathcal{P}(R)$ if and only if $\mathcal{J}$ is generated as an algebra by $J(1)$, that is, if $\mathcal{J}$ is itself a Nichols algebra.

Considering $\mathfrak{S}(M) \in H^*_8 \mathcal{YD}$ for $M$ in the list of Theorem A since $\mathfrak{S}(M) = T(M)/I$, in order to show that $\mathcal{P}(\mathcal{J}) = J(1)$ it is enough to prove that the relations that generate the ideal $I$ hold in $\mathcal{J}$. This can be done by a case-by-case computation. We perform here three cases, and leave the rest to the reader.

Suppose $M = \Omega_1(n_1, n_2, n_3, n_4)$. A direct computation shows that the elements $r$ in $\mathcal{J}$ representing the quadratic relations are primitive and they satisfy $c(r \otimes r) = r \otimes r$. As $\dim \mathcal{J} < \infty$, it must be $r = 0$ in $\mathcal{J}$ and hence there exists a projective algebra map $\mathfrak{S}(M) \to \mathcal{J}$, which implies that $\mathcal{P}(\mathcal{J}) = J(1)$.

Suppose $M = \Omega_2(n_1, n_2)$. Then $M$ is generated by elements $p_1 = (v_1 + v_2) \otimes x, y, p_2 = (v_1 - v_2) \otimes x, y$, and the ideal defining the Nichols algebra is generated by the elements $p_1^2, p_2^2, p_1p_2, p_1^2p_2^2, p_1^2p_2p_1^2, p_1^2p_2^2p_1^2, p_1^2p_2p_1^2, p_1^2p_2^2, p_1^2p_2p_1, p_1^2p_2^2, p_1^2p_2p_1, p_1^2p_2p_1^2$. We can check directly that all those generators of the defining ideal of $\mathfrak{S}(M)$ are primitive elements, or by using [21] Theorem 6. It is enough to show that $c(r \otimes r) = r \otimes r$ for all generators given above for the defining ideal. Since $\rho(p_1) = xy \otimes p_1, \rho(p_2) = x \otimes p_2, \rho(p_1^2) = y \otimes p_1^2, \rho(p_1p_2p_1p_2) = 1 \otimes (p_1p_2p_1p_2), \rho(p_1p_2 + p_2p_1p_2) = (x \otimes p_1p_2 + p_2p_1p_2).$ It is easy to see that $c(r \otimes r) = r \otimes r$ holds for $r = p_1^2, p_1p_2p_1p_2 + p_2p_1p_2p_1, p_1p_1^2 + p_1p_1p_2$. We leave the rest to the reader.

Suppose $M = \Omega_4(n_1, n_2)$. Then $M$ is generated by elements $p_1 = w_1^1 - 1 + iw_2^1 - 1, p_2 = w_1^1 - 1 - iw_2^1 - 1, \{X_j\} j = 1, \ldots, n_1, \{Y_k\} k = 1, \ldots, n_2$ with $\mathbb{K}X_j \simeq M(i, x), \mathbb{K}Y_k \simeq M(i, y)$, and the ideal defining the Nichols algebra is generated by the elements $p_1^2, p_2^2, p_1p_2p_1p_2 + p_2p_1p_2, X_j^2, \{X_jX_j + X_jX_j\}_{1 \leq j < j \leq n_1, \{Y_kY_k + Y_kY_k\}_{1 \leq k_1, k_2 \leq n_2, p_1Y_k - Y_kp_1, p_2Y_k + Y_kp_2, p_1X_j - X_jp_1, p_2X_j + X_jp_2, X_j^2, X_j Y_j + X_j Y_j\}}$. We can check directly that all those generators of the defining ideal of $\mathfrak{S}(M)$ are primitive elements, or by using [21] Theorem 6. It is enough to show that $c(r \otimes r) = r \otimes r$ for all generators given above for the defining ideal. Since $\rho(p_1) = (f_{00} - if_{11})z \otimes p_1 + (f_{00} + if_{11})z \otimes p_2, \rho(p_2) = (f_{00} + if_{11})z \otimes p_2 + (f_{00} - if_{11})z \otimes p_1, \rho(X_j) = x \otimes X_j,$

\[
\rho(p_1p_2p_1p_2 + p_2p_1p_2p_1) = [((f_{00} - if_{11})z(f_{00} + if_{11})z)^2 \otimes p_1p_2p_1p_2] + [(f_{00} + if_{11})z(f_{00} + if_{11})z]^2 \otimes p_2p_1p_2p_1 + [(f_{00} + if_{11})z(f_{00} + if_{11})z]^2 \otimes p_2p_1p_2p_1 + [(f_{00} + if_{11})z(f_{00} + if_{11})z]^2 \otimes p_1p_2p_1p_2
\]

\[= xy \otimes (p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1),\]
\[
\rho(\rho_1 X_j - X_j p_1) = (f_{00} + i f_{11}) z \otimes (p_1 X_j - X_j p_1) + (f_{01} - i f_{01}) z \otimes (p_2 X_j + X_j p_2).
\]
Because
\[
(f_{01} - i f_{01}) z \otimes (p_1 X_j - X_j p_1) = \frac{f_{01} - i f_{01}}{2} \cdot \left[ ((1 + y) z \cdot p_1) (z \cdot X_j) - ((1 - y) z \cdot X_j) (xz \cdot p_1) \right]
\]
\[
= (-i) (f_{01} - i f_{01}) \cdot (p_1 X_j - X_j p_1) = 0,
\]
\[
(f_{00} + i f_{11}) z \cdot (p_1 X_j - X_j p_1) = (-i) (f_{00} + i f_{11}) \cdot (p_1 X_j - X_j p_1) = p_1 X_j - X_j p_1,
\]
\[xy \cdot (p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1),\]
\[c(r \otimes r) = r \otimes r \text{ holds for } r = p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 \text{ and } p_1 X_j - X_j p_1. \]
We leave the rest to the reader. \(\square\)

**Lemma 5.2** (\cite{13} Lemma 6.1). Let \(H\) be a Hopf algebra, \(\psi : H \to H\) an automorphism of Hopf algebras, \(V, W\) Yetter–Drinfeld module over \(H\).

1. Let \(V^\psi\) be the same space underlying \(V\) but with action and coaction
\[\delta \cdot v = \psi(h) \cdot v, \quad \rho^\psi(v) = (\psi^{-1} \otimes \mathrm{id}) \rho(v), \quad h \in H, v \in V.\]

Then \(V^\psi\) is also a Yetter–Drinfeld module over \(H\). If \(T : V \to W\) is a morphism in \(H_{YD}\), then \(T^\psi : V^\psi \to W^\psi\) also is. Moreover, the braiding \(c : V^\psi \otimes W^\psi \to W^\psi \otimes V^\psi\) coincides with the braiding \(c : V \otimes W \to W \otimes V\).

2. If \(R\) is an algebra (resp., a coalgebra, a Hopf algebra) in \(H_{YD}\), then \(R^\psi\) also is, with the same structural maps.

3. Let \(R\) be a Hopf algebra in \(H_{YD}\). Then the map \(\Psi : R^\psi \# H \to R \# H\) given by \(\Psi(r \# h) = r \# \psi(h)\) is an isomorphism of Hopf algebras.

**Corollary 5.3.**

1. \([M(b i, x)]^\tau = M(-b i, y), b = \pm 1.\]

2. \([M((x i, y))]^\tau = M((y, x i))\]
\([W_{b_1}]^\tau = W_{b_1^{-1}} \text{ with } b_1 = \pm 1.\]

3. \(\mathcal{B}(\Omega_2(n_1, n_2)) \# H_8 \cong \mathcal{B}(\Omega_3(n_2, n_1)) \# H_8,\]
\(\mathcal{B}(\Omega_4(n_1, n_2)) \# H_8 \cong \mathcal{B}(\Omega_5(n_2, n_1)) \# H_8.\)

Let \(H\) be a lifting of \(\mathcal{B}(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8.\) Then there exists an epimorphism of Hopf algebras \(\phi : T(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8 \to H\) \cite{16} Proposition 2.4. Denote
\[
X_j = (v \circ x) \# 1, \quad v \circ x \in M(i, x), \quad j = 1, \ldots, n_1,
\]
\[
Y_k = (v \circ x) \# 1, \quad v \circ x \in M(-i, x), \quad k = 1, \ldots, n_2,
\]
\[
p_s = (v \circ y) \# 1, \quad v \circ y \in M(i, y), \quad s = 1, \ldots, n_3,
\]
\[
q_t = (v \circ y) \# 1, \quad v \circ y \in M(-i, y), \quad t = 1, \ldots, n_4.
\]

**Definition 5.4.** For \(n_1, n_2, n_3, n_4 \in \mathbb{N}^{\geq 0}\) with \(n_1 + n_2 + n_3 + n_4 \geq 1, \) and \(I_1 = \{(\lambda_j, s)_{n_1 \times n_3}, (\theta_{k, t})_{n_2 \times n_4}\} \) with entries in \(\mathbb{K},\) we denote by \(\mathcal{A}_1(n_1, n_2, n_3, n_4; I_1)\) the algebra \([T(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8] / I(I_1),\) where \(I(I_1)\) is the ideal generated by
\[
X_j^2 = 0, \quad Y_k^2 = 0, \quad p_s^2 = 0, \quad q_t^2 = 0, \quad (5.2)
\]

A Remark

5.5 It is easy to see that \( \mathcal{I}(I_1) \) is a Hopf ideal, hence \( \mathfrak{U}_1(n_1, n_2, n_3, n_4; I_1) \) is a Hopf algebra. In particular, when parameters in \( I_1 \) are all equal to zero, then \( \mathfrak{U}_1(n_1, n_2, n_3, n_4; I_1) \simeq \mathfrak{H}((\Omega_1(n_1, n_2, n_3, n_4)) \# H_8).

Proposition 5.6. (1) Let \( H \) be a finite-dimensional Hopf algebra with coradical \( H_8 \) such that its infinitesimal braiding is isomorphic to \( \Omega_1(n_1, n_2, n_3, n_4) \). Then \( H \simeq \mathfrak{U}_1(n_1, n_2, n_3, n_4; I_1) \).

(2) \( \mathfrak{U}_1(n_1, n_2, n_3, n_4; I_1) \simeq \mathfrak{U}_1(n_1, n_2, n_3, n_4; I_1') \) iff there exist invertible matrices \( (\alpha_{jj'})_{n_1 \times n_1}, (\beta_{kk'})_{n_2 \times n_2}, (\gamma_{ss'})_{n_3 \times n_3}, (\eta_{tt'})_{n_4 \times n_4} \) such that

\[
\sum_{j'=1}^{n_1} \sum_{k'=1}^{n_2} \alpha_{jj'} \beta_{kk'} = 0, \quad \sum_{j'=1}^{n_1} \sum_{t'=1}^{n_4} \alpha_{jj'} \eta_{tt'} = 0, \quad \sum_{j'=1}^{n_1} \sum_{s'=1}^{n_3} \alpha_{jj'} \gamma_{ss'} \lambda_{j,s} = \lambda_{j,s},
\]

\[
\sum_{k'=1}^{n_2} \sum_{s'=1}^{n_3} \beta_{kk'} \gamma_{ss'} = 0, \quad \sum_{k'=1}^{n_2} \sum_{t'=1}^{n_4} \gamma_{ss'} \eta_{tt'} = 0, \quad \sum_{k'=1}^{n_2} \sum_{s'=1}^{n_3} \beta_{kk'} \eta_{tt'} \theta_{k,t} = \theta_{k,t};
\]

or \( n_1 = n_4, n_2 = n_3 \) and there exist invertible matrices \( (\alpha'_{jj'})_{n_1 \times n_1}, (\beta'_{kk'})_{n_2 \times n_2}, (\gamma'_{ss'})_{n_3 \times n_3}, (\eta'_{tt'})_{n_4 \times n_4} \) such that

\[
\sum_{t'=1}^{n_2} \sum_{s'=1}^{n_1} \alpha'_{tt'} \beta'_{kk'} = 0, \quad \sum_{k'=1}^{n_2} \sum_{j'=1}^{n_1} \gamma'_{sk} \gamma'_{tt'} = 0, \quad \sum_{t'=1}^{n_2} \sum_{k'=1}^{n_1} \alpha'_{tt'} \gamma'_{sk} \lambda'_{j,s} = \lambda_{j,s},
\]

\[
\sum_{s'=1}^{n_3} \sum_{j'=1}^{n_1} \alpha'_{tt'} \eta'_{tt'} = 0, \quad \sum_{s'=1}^{n_3} \sum_{k'=1}^{n_2} \beta'_{kk'} \gamma'_{ss'} = 0, \quad \sum_{s'=1}^{n_3} \sum_{j'=1}^{n_1} \beta'_{kk'} \eta'_{tt'} \theta'_{k,t} = \theta_{k,t}.
\]

Proof. (1) According to \ref{5.1}, the extra relations of generators in \( T(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8 \) besides \( H_8 \) are given by

\[
\begin{align*}
&x X_j = -X_j x, & y X_j = -X_j y, & z X_j = i X_j x, \\
&x Y_k = -Y_k x, & y Y_k = -Y_k y, & z Y_k = -i Y_k x, \\
&x p_s = -p_s x, & y p_s = -p_s y, & z p_s = i p_s x, \\
&x q_t = -q_t x, & y q_t = -q_t y, & z q_t = -i q_t x,
\end{align*}
\]

and their coproducts are given by

\[
\begin{align*}
\Delta(X_j) &= X_j \otimes 1 + x \otimes X_j, & \Delta(Y_k) &= Y_k \otimes 1 + x \otimes Y_k, \\
\Delta(p_s) &= p_s \otimes 1 + y \otimes p_s, & \Delta(q_t) &= q_t \otimes 1 + y \otimes q_t.
\end{align*}
\]
Let $p = (v \otimes g)\#1 \in [M(b, g)]\#1$, $p' = (v' \otimes g')\#1 \in [M(b', g')]\#1$; then $\Delta[\phi(pp' + p'p)] = \phi(pp' + p'p) \circ 1 + gg' \otimes \phi(pp' + p'p)$. So $\phi(pp' + p'p) = 0$ (when $gg' = 1$) or $\phi(pp' + p'p) = \lambda(1 - gg')$ (when $gg' \neq 1$) for some $\lambda \in \mathbb{K}$ related with $p$ and $p'$. So $\phi$ keeps relations (5.2)–(5.7), and there exist $(\lambda_{j,s})_{n_1 \times n_3}$, $(\mu_{j,t})_{n_1 \times n_3}$, $(\zeta_{k,s})_{n_2 \times n_3}$, $(\theta_{k,t})_{n_2 \times n_4}$, and entries in $\mathbb{K}$ such that $\phi$ keeps relations (5.9) and $\phi(X_jq_t + q_tX_j) = \mu_{j,t}(1 - xy)$, $\phi(Ykp_t + p_tY_k) = \zeta_{k,s}(1 - xy)$. Since $z(1 - xy) = (1 - xy)z$, so $\phi[z(X_jq_t + q_tX_j)] = \phi([X_jq_t + q_tX_j]z]$). By direct calculation, we have $\phi[z(X_jq_t + q_tX_j)] = -\phi([X_jq_t + q_tX_j]z)$, so $\mu_{j,t} = 0$. Similarly, $\zeta_{k,s} = 0$. Now we have $\mathcal{I}(I_1) \subseteq ker \phi$. So there is a surjective Hopf algebra map from $\mathfrak{A}_1(n_1, n_2, n_3; I_1)$ to $H$. We can observe that any element of $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ can be expressed by a linear sum of $\{X_1^\alpha \cdots X_{n_1}^\alpha Y_1^\beta \cdots Y_{n_2}^\beta p_1^\eta_1 \cdots p_{n_3}^\eta_3 q_1^\mu_1 \cdots q_{n_4}^\mu_4 x^c y^d z^e\}$ for all parameters $\alpha_1, \ldots, \alpha_{n_1}$, $\beta_1, \ldots, \beta_{n_2}$, $\gamma_1, \ldots, \gamma_{n_3}$, $\kappa_1, \ldots, \kappa_{n_4}$, $c$, $d$, $e$ in $\{0, 1\}$. In fact, the set is a basis of $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ according to the Diamond Lemma [17]. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. That is, the ambiguities can be reduced to the same expression by different substitution rules. Calculating the ambiguities and showing that these are resoluble is a tedious but straightforward computation. By Theorem 5.1 we have $\text{gr} H \simeq \mathfrak{B}(\Omega_1(n_1, n_2, n_3, n_4))\#H_8$. That is to say that $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ has the same dimension as $H$. So $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ is a lifting for all $n_1, n_2, n_3, n_4$.

(2) Suppose that $\Phi : \mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1) \to \mathfrak{A}_1(n_1, n_2, n_3, n_4; I'_1)$ is an isomorphism of Hopf algebras, where $I'_1 = \{(\lambda'_{j,s})_{n_1 \times n_3}, (\theta'_{k,t})_{n_2 \times n_4}\}$ and $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I'_1)$ is generated by $x, y, z, X_j', Y'_k, p'_s, q'_t$.

When $\Phi|_{H_8} = id$ or $\tau_1$, $\Phi(X_j)$ is $(x, 1)$-skew primitive, so $\Phi(X_j) \in \oplus_{j=1}^{n_1} \mathbb{K}X'_j + \oplus_{k=1}^{n_2} \mathbb{K}Y'_k \otimes \mathbb{K}(1 - x)$. $xX_j = -X_jx$ implies that $\Phi(X_j)$ doesn’t contain the term of $1 - x$. And $zX_j = iX_jxz$, $zY'_k = -iY'_kxz$ implies that $\Phi(X_j)$ doesn’t contain the terms of $\oplus_{k=1}^{n_2} \mathbb{K}Y'_k$. So there exists an invertible matrix $(\alpha_{j,j'})_{n_1 \times n_1}$ such that $\Phi(X_j) = \sum_{j'=1}^{n_1} \alpha_{j,j'} X'_j$. Similarly, $\Phi(Y_k) = \sum_{k'=1}^{n_2} \beta_{k,k'} Y'_k$, $\Phi(p_s) = \sum_{s'=1}^{n_3} \gamma_{s,s'} p'_s$, $\Phi(q_t) = \sum_{t'=1}^{n_4} \eta_{t,t'} q'_t$ with $(\alpha_{j,j'})_{n_2 \times n_3}$, $(\beta_{k,k'})_{n_3 \times n_2}$, $(\gamma_{s,s'})_{n_2 \times n_3}$, $(\eta_{t,t'})_{n_3 \times n_4}$ invertible. In this case, $\Phi$ is an isomorphism of Hopf algebras if and only if the relations (5.10) hold.

When $\Phi|_{H_8} = \tau_3$ or $\tau_4$, $\Phi(X_j)$ is $(y, 1)$-skew primitive, so $\Phi(X_j) \in \oplus_{s'=1}^{n_3} \mathbb{K}p'_s + \oplus_{t'=1}^{n_4} \mathbb{K}q'_t \otimes \mathbb{K}(1 - y)$. Since $xX_j = -\Phi(X_jx)$, $\Phi(X_j)$ doesn’t contain the term of $1 - y$. And $\Phi(X_j)$ doesn’t contain the terms of $\oplus_{s'=1}^{n_3} \mathbb{K}p'_s$, because of $\Phi(zX_j) = i\Phi(X_jxz)$. So $n_1 = n_4$, and there exists an invertible matrix $(\alpha'_{j,j'})_{n_1 \times n_1}$ such that $\Phi(X_j) = \sum_{j'=1}^{n_1} \alpha'_{j,j'} q'_t$. Similarly, we have $n_2 = n_3$ and $\Phi(Y_k) = \sum_{k'=1}^{n_2} \beta'_{k,k'} p'_s$, $\Phi(p_s) = \sum_{s'=1}^{n_3} \gamma'_{s,s'} p'_s$, $\Phi(q_t) = \sum_{t'=1}^{n_4} \eta'_{t,t'} q'_t$ with $(\beta'_{k,k'})_{n_2 \times n_2}$, $(\gamma'_{s,s'})_{n_2 \times n_3}$, $(\eta'_{t,t'})_{n_3 \times n_4}$ invertible. In this case, $\Phi$ is an isomorphism of Hopf algebras if and only if the relations (5.11) hold.

Lemma 5.7. Suppose $H$ is a finite-dimensional Hopf algebra with coradical $H_8$ such that its infinitesimal braiding is isomorphic to $M((y, xy))$. Then $H \simeq \mathfrak{B}[M((y, xy))]\#H_8$.

Proof. Let $H$ be a lifting of $\mathfrak{B}(\Omega_2(n_1, n_2))\#H_8$. Then there exists an epimorphism of Hopf algebras $\phi : T(M((y, xy)))\#H_8 \to H$ [16] Proposition 2.4]. Denote $p_1 =
It is easy to see that \( \ker \phi \) such that its infinitesimal braiding is isomorphic to \( \mathcal{M}(xy, x) \). Suppose Lemma 5.8.

Proposition 5.10. \( B \) is isomorphic to \( \mathcal{M}(y, x) \) by Corollary 5.3.

Lemma 5.8. Suppose \( H \) is a finite-dimensional Hopf algebra with coradical \( H_8 \) such that its infinitesimal braiding is isomorphic to \( \Omega_2(n_1, n_2) \). Then \( H \cong \mathcal{M}(xy, x) \). For Lemma 5.7.

Remark 5.9. Let \( p_1 = [(v_1 + v_2) \otimes xy] \# 1 \), \( p_2 = [(v_1 - v_2) \otimes x] \# 1 \) be a basis of \( \mathcal{M}(xy, x) \) \# 1. It is easy to see that \( p_1^2 \), \( p_2^2 \), and \( p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 \) are primitive. The proof of the lemma is similar to that of Proposition 2.4. Denote

\[
\begin{align*}
p_1 &= [(v_1 + v_2) \otimes xy] \# 1, \\
p_2 &= [(v_1 - v_2) \otimes x] \# 1, \\
v_1, v_2 &\in V_2, \\
X_j &= (v \otimes x) \# 1, \\
v &\in V_1(i), \\
j &\in \{1, \ldots, n_1\}, \\
Y_k &= (v' \otimes x) \# 1, \\
v' &\in V_1(-i), \\
k &\in \{1, \ldots, n_2\}.
\end{align*}
\]

Let \( I \) be the ideal of relations of \( \mathcal{M}(xy, x) \) \# \( H_8 \); then \( I \) is generated by the relations

\[
\begin{align*}
p_1^2 &= 0, \\
p_2^2 &= 0, \\
p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 &= 0, \\
X_{j_1} X_{j_2} + X_{j_2} X_{j_1} &= 0, \\
j_1, j_2 &\in \{1, \ldots, n_1\}, \\
Y_{k_1} Y_{k_2} + Y_{k_2} Y_{k_1} &= 0, \\
k_1, k_2 &\in \{1, \ldots, n_2\}, \\
X_j^2 &= 0, \\
Y_k^2 &= 0, \\
X_j Y_k + Y_k X_j &= 0, \\
p_2 X_j + X_j p_2 &= 0, \\
p_2 Y_k + Y_k p_2 &= 0, \\
p_1 X_j - X_j p_1 &= 0, \\
p_1 Y_k - Y_k p_1 &= 0.
\end{align*}
\]

We have the following formulae by direct calculation:

\[
\begin{align*}
x(p_1 X_j - X_j p_1) &= -(p_1 X_j - X_j p_1), \\
x(p_1 Y_k - Y_k p_1) &= -(p_1 Y_k - Y_k p_1), \\
\Delta[\phi(p_1 X_j - X_j p_1)] &= \phi(p_1 X_j - X_j p_1), \\
\Delta[\phi(p_1 Y_k - Y_k p_1)] &= \phi(p_1 Y_k - Y_k p_1), \\
\Delta[\phi(p_2 X_j + X_j p_2)] &= \phi(p_2 X_j + X_j p_2), \\
\Delta[\phi(p_2 Y_k + Y_k p_2)] &= \phi(p_2 Y_k + Y_k p_2).
\end{align*}
\]

\[
\Delta[\phi(p_1 X_j - X_j p_1)] = \phi(p_1 X_j - X_j p_1), \\
\Delta[\phi(p_1 Y_k - Y_k p_1)] = \phi(p_1 Y_k - Y_k p_1), \\
\Delta[\phi(p_2 X_j + X_j p_2)] = \phi(p_2 X_j + X_j p_2), \\
\Delta[\phi(p_2 Y_k + Y_k p_2)] = \phi(p_2 Y_k + Y_k p_2).
\]
Together with Lemma 5.8 and Proposition 5.10, we can see $I \subseteq \ker \phi$, so there is a surjective map from $\mathcal{B}[\Omega_2(n_1, n_2)] \# H_8$ to $H$. By Theorem 5.1, $\text{gr} H \simeq \mathcal{B}[\Omega_2(n_1, n_2)] \# H_8$, so $H \simeq \text{gr} H$. 

**Definition 5.11.** For $\lambda \in \mathbb{K}$, denote $\mathcal{A}_6(\lambda)$ by the algebra $[T(\Omega_6) \# H_8]/I(\lambda)$, where $I(\lambda)$ is the ideal generated by the relations

\begin{align*}
p_1^2 &= 0, \quad p_2^2 = 0, \quad p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = 0, \quad (5.12) \\
q_1^2 &= 0, \quad q_2^2 = 0, \quad q_1 q_2 q_1 q_2 + q_2 q_1 q_2 q_1 = 0, \quad (5.13) \\
p_1 q_1 + q_1 p_1 &= \lambda (1 - x), \quad p_2 q_2 + q_2 p_2 = \lambda (1 - y), \\
p_1 q_2 - q_2 p_1 &= 0, \quad p_2 q_1 + q_1 p_2 = 0.
\end{align*}

**Remark 5.12.** In fact, $I(\lambda)$ is a Hopf ideal, so $\mathcal{A}_6(\lambda)$ is a Hopf algebra. In particular, when $\lambda = 0$, $\mathcal{A}_6(0) \simeq \mathcal{B}(\Omega_6) \# H_8$.

**Proposition 5.13.** (1) Suppose $H$ is a finite-dimensional Hopf algebra with coradical $H_8$ such that its infinitesimal braiding is isomorphic to $\Omega_6$. Then $H \simeq \mathcal{A}_6(\lambda)$.

(2) $\mathcal{A}_6(\lambda) \simeq \mathcal{A}_6(1)$ for $\lambda \neq 0$, and $\mathcal{A}_6(1) \neq \mathcal{A}_6(0)$.

**Proof.** (1) Let $H$ be a lifting of $\mathcal{B}(\Omega_6) \# H_8$. Then there exists an epimorphism of Hopf algebras $\phi: T(\Omega_6) \# H_8 \to H$. Denote $p_1 = [(v_1 + v_2) \otimes y] \# 1$, $p_2 = [(v_1 - v_2) \otimes x y] \# 1$, $q_1 = [(v_1 + v_2) \otimes xy] \# 1$, $q_2 = [(v_1 - v_2) \otimes x] \# 1$ in $[M((y, x y)) \oplus M((x y, x))] \# 1$. By Lemmas 5.7 and 5.8, the map $\phi$ keeps relations (5.12) and (5.13).

Since $\text{gr} H \simeq \mathcal{B}(\Omega_6) \# H_8$ and

\[
\Delta[\phi(p_1 q_1 + q_1 p_1)] = \phi(p_1 q_1 + q_1 p_1) \otimes 1 + x \otimes \phi(p_1 q_1 + q_1 p_1), \\
\Delta[\phi(p_1 q_2 + q_2 p_2)] = \phi(p_1 q_2 + q_2 p_2) \otimes 1 + y \otimes \phi(p_1 q_2 + q_2 p_2), \\
\Delta[\phi(p_1 q_2 - q_2 p_1)] = \phi(p_1 q_2 - q_2 p_1) \otimes 1 + x y \otimes \phi(p_1 q_2 - q_2 p_1), \\
\Delta[\phi(p_2 q_1 + q_1 p_2)] = \phi(p_2 q_1 + q_1 p_2) \otimes 1 + 1 \otimes \phi(p_2 q_1 + q_1 p_2),
\]

we have $\phi(p_1 q_1 + q_1 p_1) = \lambda (1 - x)$, $\phi(p_1 q_2 + q_2 p_2) = \lambda (1 - y)$, $\phi(p_1 q_2 - q_2 p_1) = \lambda (1 - x y)$, $\phi(p_2 q_1 + q_1 p_2) = 0$ for some $\lambda_1$, $\lambda_2$ and $\lambda_3$ in $\mathbb{K}$. Since $z(p_1 q_1 + q_1 p_1) = (p_2 q_2 + q_2 p_2) z$ and $z(p_1 q_2 - q_2 p_1) = (p_2 q_1 + q_1 p_2) z$, we have $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$. That is to say $I(\lambda) \subseteq \ker \phi$. So there is a surjective map from $\mathcal{A}_6(\lambda)$ to $H$. Now we only need to prove that $\dim \mathcal{A}_6(\lambda) = \dim H$. In fact, $\mathcal{A}_6(\lambda) \simeq \mathcal{B}(\Omega_6) \# H_8$ as vector space by the Diamond Lemma. It suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resolvable is a tedious but straightforward computation.

(2) When $\lambda \neq 0$, $\Phi: \mathcal{A}_6(\lambda) \simeq \mathcal{A}_6(1)$, by $\Phi|_{H_8} = \text{id}$, $p_i \mapsto \frac{p_i}{\sqrt{\lambda}}$, $q_i \mapsto \frac{q_i}{\sqrt{\lambda}}$ for $i = 1, 2$. $\mathcal{A}_6(1) \neq \mathcal{A}_6(0)$ can be proved similarly to the proof of the second part of Proposition 5.17. 

**Lemma 5.14.** Suppose $H$ is a finite-dimensional Hopf algebra with coradical $H_8$ such that its infinitesimal braiding is isomorphic to $W^{b_1, -1}$, where $b_1 = \pm 1$. Then

\[ H \simeq \left[ T \left( W^{b_1, -1} \right) \# H_8 \right] / \mathcal{I}(\lambda_1, \lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{K}, \]
where $\mathcal{I}(\lambda_1, \lambda_2)$ is a Hopf ideal generated by the relations

$$p_1^2 = \lambda_1(1 - xy), \quad p_2^2 = ib_1 \lambda_1(1 - xy), \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = \lambda_2(1 - xy). \quad (5.14)$$

**Proof.** Let $H$ be a lifting of $\mathfrak{g}(W^{b_1,-1}) \# H_8$. Then there exists an epimorphism of Hopf algebras $\phi : T(\mathfrak{g}(W^{b_1,-1}) \# H_8) \to H$. $p_1 = \left( w_1^{b_1,-1} + ib_1 w_2^{b_1,-1} \right) \# 1$, $p_2 = \left( w_1^{b_1,-1} - ib_1 w_2^{b_1,-1} \right) \# 1$,

$$\Delta(p_1) = [f_{00} - ib_1 f_{11}] z \otimes p_1 + [f_{10} + ib_1 f_{01}] z \otimes p_2 + p_1 \otimes 1, \quad (5.15)$$

$$\Delta(p_2) = [f_{00} + ib_1 f_{11}] z \otimes p_2 + [f_{10} - ib_1 f_{01}] z \otimes p_1 + p_2 \otimes 1. \quad (5.16)$$

By a straightforward computation, we have

$$\Delta[\phi(p_1^2)] = \frac{1}{2} (1 + xy) \otimes \phi(p_1^2) + \frac{ib_1}{2} (1 - xy) \otimes \phi(p_2^2) + \phi(p_1^2) \otimes 1,$$

$$\Delta[\phi(p_2^2)] = \frac{1}{2} (1 + xy) \otimes \phi(p_2^2) - \frac{ib_1}{2} (1 - xy) \otimes \phi(p_1^2) + \phi(p_2^2) \otimes 1.$$

So there exists a parameter $\lambda_1 \in \mathbb{K}$ such that $\phi(p_1^2) = \lambda_1(1 - xy)$ and $\phi(p_2^2) = ib_1 \lambda_1(1 - xy)$.

$$\Delta[\phi(p_1p_2)] = \frac{1}{2} (x + y) \otimes \phi(p_1p_2) + \frac{ib_1}{2} (x - y) \otimes \phi(p_2p_1) + \phi(p_1p_2) \otimes 1$$

$$+ \phi(p_2) \left[ f_{00} - ib_1 f_{11} \right] z \otimes \phi(p_1) + \phi(p_1) \left[ f_{10} + ib_1 f_{01} \right] z \otimes \phi(p_2)$$

$$+ \phi(p_1) \left[ f_{00} + ib_1 f_{11} \right] z \otimes \phi(p_2) + \phi(p_2) \left[ f_{10} - ib_1 f_{01} \right] z \otimes \phi(p_1),$$

$$\Delta[\phi(p_2p_1)] = \frac{1}{2} (x + y) \otimes \phi(p_2p_1) + \frac{ib_1}{2} (y - x) \otimes \phi(p_1p_2) + \phi(p_2p_1) \otimes 1$$

$$+ [f_{00} + ib_1 f_{11}] z \phi(p_1) \otimes \phi(p_2) + [f_{10} - ib_1 f_{01}] z \phi(p_1) \otimes \phi(p_1)$$

$$+ \phi(p_2) \left[ f_{00} - ib_1 f_{11} \right] z \otimes \phi(p_1) + \phi(p_2) \left[ f_{10} + ib_1 f_{01} \right] z \otimes \phi(p_2).$$

Denote $\Delta[\phi(p_1p_2)] = B - A + E_1$ and $\Delta[\phi(p_2p_1)] = B + A + E_2$, where

$$A = \left[ f_{00} + ib_1 f_{11} \right] z \phi(p_1) \otimes \phi(p_2) + \left[ f_{10} - ib_1 f_{01} \right] z \phi(p_1) \otimes \phi(p_1),$$

$$B = \phi(p_2) \left[ f_{00} - ib_1 f_{11} \right] z \otimes \phi(p_1) + \phi(p_2) \left[ f_{10} + ib_1 f_{01} \right] z \otimes \phi(p_2),$$

$$E_2 = \frac{1}{2} (x + y) \otimes \phi(p_2p_1) + \frac{ib_1}{2} (y - x) \otimes \phi(p_1p_2) + \phi(p_2p_1) \otimes 1,$$

$$E_1 = \frac{1}{2} (x + y) \otimes \phi(p_1p_2) + \frac{ib_1}{2} (x - y) \otimes \phi(p_2p_1) + \phi(p_1p_2) \otimes 1.$$

We can obtain $A^2 + B^2 = 0$, since

$$A^2 = -\frac{1}{2} (x + y) \phi(p_1^2) \otimes \phi(p_2^2) + \frac{ib_1}{2} (1 - xy) \phi(p_1^2) \otimes \phi(p_1^2)$$

$$= ib_1 \lambda_1^2 (1 - xy) \otimes (1 - xy),$$

$$B^2 = -\frac{1}{2} (x + y) \phi(p_2^2) \otimes \phi(p_2^2) + \frac{ib_1}{2} (1 - xy) \phi(p_2^2) \otimes \phi(p_2^2)$$

$$= -ib_1 \lambda_1^2 (1 - xy) \otimes (1 - xy).$$
Keeping in mind that
\[ p_1(p_1p_2 + p_2p_1) = (p_2p_1 + p_1p_2)p_1, \quad p_2(p_1p_2 + p_2p_1) = (p_2p_1 + p_1p_2)p_2, \]
\[ p_1(p_1p_2 - p_2p_1) = (p_2p_1 - p_1p_2)p_1, \quad p_2(p_1p_2 - p_2p_1) = (p_2p_1 - p_1p_2)p_2, \]
\[ (x + y)p_2(f_00 - ib_1f_{11})z = -p_2(f_00 - ib_1f_{11})z(x + y), \]
\[ (x - y)p_2(f_00 - ib_1f_{11})z = p_2(f_00 - ib_1f_{11})z(x - y), \]
\[ (x + y)p_2(f_{10} + ib_1f_{01})z = -p_2(f_{10} + ib_1f_{01})z(x + y), \]
\[ (x - y)p_2(f_{10} + ib_1f_{01})z = p_2(f_{10} + ib_1f_{01})z(x - y), \]
\[ (p_1p_2 + p_2p_1)f_2(f_{10} - ib_1f_{01})z = -p_2(f_{10} - ib_1f_{01})z(p_1p_2 + p_2p_1), \]
\[ (p_1p_2 + p_2p_1)f_2(f_{10} + ib_1f_{01})z = -p_2(f_{10} + ib_1f_{01})z(p_1p_2 + p_2p_1), \]
we deduce \( B(E_1 + E_2) + (E_1 + E_2)B = 0 \). Similarly, we have \( A(E_2 - E_1) + (E_2 - E_1)A = 0 \).

\[
\Delta[\phi(p_1p_2p_1p_2 + p_2p_1p_2p_1)] = (B - A + E_1)^2 + (B + A + E_2)^2
\]
\[
= 2(A^2 + B^2) + B(E_1 + E_2) + (E_1 + E_2)B
\]
\[
+ A(E_2 - E_1) + (E_2 - E_1)A + E_1^2 + E_2^2
\]
\[
= E_1^2 + E_2^2
\]
\[
= \left( \frac{1}{2}(x + y) \otimes \phi(p_1p_2) + \frac{ib_1}{2}(x - y) \otimes \phi(p_2p_1) + \phi(p_1p_2) \otimes 1 \right)^2
\]
\[
+ \left( \frac{1}{2}(x + y) \otimes \phi(p_2p_1) + \frac{ib_1}{2}(y - x) \otimes \phi(p_1p_2) + \phi(p_2p_1) \otimes 1 \right)^2
\]
\[
= \frac{1}{2}(1 + xy) \otimes [\phi(p_1p_2)]^2 - \frac{1}{2}(1 - xy) \otimes [\phi(p_2p_1)]^2 + [\phi(p_1p_2)]^2 \otimes 1
\]
\[
+ \frac{1}{2}(1 + xy) \otimes [\phi(p_2p_1)]^2 - \frac{1}{2}(1 - xy) \otimes [\phi(p_1p_2)]^2 + [\phi(p_2p_1)]^2 \otimes 1
\]
\[
= xy \otimes \phi \left[ (p_1p_2)^2 + (p_2p_1)^2 \right] + \phi \left[ (p_1p_2)^2 + (p_2p_1)^2 \right] \otimes 1.
\]

So there exists a parameter \( \lambda_2 \in \mathbb{K} \) such that \( \phi(p_1p_2p_1p_2 + p_2p_1p_2p_1) = \lambda_2(1 - xy) \).

Hence the map \( \phi \) keeps relations \([5, 14] \), so there exists a surjective map from \([T(W^{b_1, -1})\#H_8]/\mathcal{I}(\lambda_1, \lambda_2)\) to \( H \). Now we only need to prove that

\[
\dim \left[ T(W^{b_1, -1})\#H_8 \right]/\mathcal{I}(\lambda_1, \lambda_2) = \dim H.
\]

In fact, \([T(W^{b_1, -1})\#H_8]/\mathcal{I}(\lambda_1, \lambda_2) \simeq \mathfrak{B}(W^{b_1, -1}) \otimes H_8\) as vector space by the Diamond Lemma. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resolvable is a tedious but straightforward computation. ∎

**Definition 5.15.** Let \( I_7 = (\lambda_1, \ldots, \lambda_5) \) with \( \lambda_i \in \mathbb{K} \) \((i = 1, \ldots, 5)\). Denote by \( \mathfrak{A}_7(I_7) \) the algebra \([T(O_7)\#H_8]/\mathcal{I}(I_7)\), where \( \mathcal{I}(I_7) \) is the ideal generated by the
relations
\[ p_1^2 = \lambda_1(1 - xy), \quad p_2^2 = i\lambda_1(1 - xy), \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = \lambda_2(1 - xy), \quad (5.17) \]
\[ q_1^2 = \lambda_3(1 - xy), \quad q_2^2 = -i\lambda_3(1 - xy), \quad q_1q_2q_1q_2 + q_2q_1q_2q_1 = \lambda_4(1 - xy), \quad (5.18) \]
\[ p_1q_2 + q_2p_1 = 0, \quad p_2q_1 + q_1p_2 = 0, \quad (5.19) \]
\[ p_1q_1 + q_1p_1 = \lambda_5(x + y - 2), \quad p_2q_2 - q_2p_2 = -i\lambda_5(x - y). \quad (5.20) \]

Remark 5.16. In fact, \( \mathcal{I}(I_7) \) is a Hopf ideal, so \( \mathfrak{A}_7(I_7) \) is a Hopf algebra. In particular, when \( \lambda_i = 0 \) for \( i = 1, 2, \ldots, 5 \), \( \mathfrak{A}_7(I_7) \simeq \mathfrak{A}(\Omega_7) \# H_8 \simeq [\mathfrak{A}(W^{-1,-1}) \otimes \mathfrak{A}(W^{-1,-1})] \# H_8 \).

Proposition 5.17. (1) Suppose \( H \) is a finite-dimensional Hopf algebra with coradical \( H_8 \) such that its infinitesimal braiding is isomorphic to \( \Omega_7 \). Then \( H \simeq \mathfrak{A}_7(I_7) \).

(2) \( \mathfrak{A}_7(\lambda_1, \ldots, \lambda_5) \simeq \mathfrak{A}_7(\lambda_1', \ldots, \lambda_5') \) iff there exist nonzero parameters \( a, b, \alpha, \beta \in \mathbb{K} \) such that
\[ a^2\lambda_1' = \lambda_1, \quad b^2\lambda_1' = \lambda_1, \quad a^2b^2\lambda_2' = \lambda_2, \quad a\alpha\lambda_3' = \lambda_5, \quad (5.21) \]
or there exist nonzero parameters \( \alpha_1, \beta_1 \) in \( \mathbb{K} \) such that
\[ \alpha_1^2\lambda_3' = \lambda_1, \quad \beta_1^2\lambda_1' = \lambda_3, \quad -\alpha_1^2\lambda_4' = \lambda_2, \quad -\beta_1^2\lambda_2' = \lambda_4, \quad \alpha_1\beta_1\lambda_5' = \lambda_5. \quad (5.22) \]

Proof. (1) Let \( H \) be a lifting of \( \mathfrak{A}(\Omega_7) \# H_8 \). Then there exists an epimorphism of Hopf algebras \( \phi : T(\Omega_7) \# H_8 \to H \). Denote \( p_1 = (w_1^{-1,1} + i w_2^{-1,1}) \# 1, \quad p_2 = (w_1^{-1,1} - i w_2^{-1,1}) \# 1, \quad q_1 = (w_1^{-1,1} - i w_2^{-1,1}) \# 1, \quad q_2 = (w_1^{-1,1} + i w_2^{-1,1}) \# 1 \).

The coproducts of \( p_1, p_2, q_1, q_2 \) are given by \((5.15)\) and \((5.16)\). The relations of generators in \( T(\Omega_7) \# H_8 \) are given by
\[ xp_1 = p_1x, \quad yp_1 = p_1y, \quad xp_2 = -p_2x, \quad yp_2 = -p_2y, \]
\[ xq_1 = q_1x, \quad yq_1 = q_1y, \quad xq_2 = -q_2x, \quad yq_2 = -q_2y, \]
\[ zp_1 = -p_1z, \quad zp_2 = ip_2xz, \quad zq_1 = -q_1z, \quad zq_2 = -iq_2xz. \]

By Lemma \(5.14\), the map \( \phi \) keeps relations \((5.17)\) and \((5.18)\). It is only possible for \( \phi(p_1q_2 + q_2p_1) = 0, \phi(p_2q_1 + q_1p_2) = 0 \), since \( x(p_1q_2 + q_2p_1) = -(p_1q_2 + q_2p_1)x, \)
\[ x(p_2q_1 + q_1p_2) = -(p_2q_1 + q_1p_2)x, \]
and
\[ \Delta[\phi(p_1q_2 + q_2p_1)] = \frac{1}{2} [(1 + xy) + i(1 - xy)] \otimes \phi(p_1q_2 + q_2p_1) + \phi(p_1q_2 + q_2p_1) \otimes 1, \]
\[ \Delta[\phi(p_2q_1 + q_1p_2)] = \frac{1}{2} [(1 + xy) - i(1 - xy)] \otimes \phi(p_2q_1 + q_1p_2) + \phi(p_2q_1 + q_1p_2) \otimes 1. \]

Similarly, the map \( \phi \) keeps relation \((5.20)\), since
\[ z(p_1q_1 + q_1p_1) = (p_1q_1 + q_1p_1)z, \quad z(p_2q_2 - q_2p_2) = -(p_2q_2 - q_2p_2)z, \]
\[ \Delta[\phi(p_1q_1 + q_1p_1)] = \frac{x+y}{2} \otimes \phi(p_1q_1 + q_1p_1) + \phi(p_1q_1 + q_1p_1) \otimes 1 \\
+ \frac{i(x-y)}{2} \otimes \phi(p_2q_2 - q_2p_2), \]
\[ \Delta[\phi(p_2q_2 - q_2p_2)] = \frac{x+y}{2} \otimes \phi(p_2q_2 - q_2p_2) + \phi(p_2q_2 - q_2p_2) \otimes 1 \\
- \frac{i(x-y)}{2} \otimes \phi(p_1q_1 + q_1p_1). \]

That is to say \( I(I_7) \subseteq \ker \phi \), so there exists a surjective map from \( \mathcal{A}_7(I_7) \) to \( H \). By Theorem 5.11, \( \text{gr} \ H \simeq \mathcal{B}(\Omega_7) \# H_8 \). Now we only need to prove that \( \dim \mathcal{A}_7(I_7) = \dim H \). In fact, \( \mathcal{A}_7(I_7) \simeq \mathcal{B}(W^{1,-1}) \otimes \mathcal{B}(W^{-1,-1}) \otimes H_8 \) as vector spaces by the Diamond Lemma. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resolvable is a tedious but straightforward computation.

(2) Let \( A = \mathcal{A}_7(I_7) \), and \( \Phi \in \text{Aut}_{\text{Hopf}} A \); then \( \Phi \mid H_8 \) is given by Table I. Denote \( A'_0 = Kx \oplus Kx \oplus Ky \oplus Ky, A''_0 = Kx \oplus Kx \oplus Ky \oplus Ky \), and \( A'_1 = p_1 H_8 \oplus p_2 H_8 \oplus q_1 H_8 \oplus q_2 H_8 \). Then the first two terms of coradical filtration of \( A \) are \( A_0 = H_8 \) and \( A_1 = A'_1 \oplus H_8 \).

\[ \Delta \Phi(p_1) = \Phi ((f_{00} - i f_{11})z) \otimes \Phi(p_1) + \Phi ((f_{10} + i f_{01})z) \otimes \Phi(p_2) + \Phi(p_1) \otimes 1, \]
\[ \in A''_0 \otimes [\Phi(p_1) + \Phi(p_2)] + \Phi(p_1) \otimes 1; \]
\[ \Delta \Phi(p_2) = \Phi ((f_{00} + i f_{11})z) \otimes \Phi(p_2) + \Phi ((f_{10} - i f_{01})z) \otimes \Phi(p_1) + \Phi(p_2) \otimes 1, \]
\[ \in A''_0 \otimes [\Phi(p_1) + \Phi(p_2)] + \Phi(p_2) \otimes 1. \]

Denote \( \Phi(p_1) = a_0 + a_1 + a_2, \Phi(p_2) = b_0 + b_1 + b_2 \), where \( a_0, b_0 \in A'_0, a_1, b_1 \in A''_0, \) and \( a_2, b_2 \in A'_1 \). Then \( \Delta(a_0) = a_0 \otimes 1 \), which implies \( a_0 \in K1 \), and similarly \( b_0 \in K1 \). Hence \( \Delta(a_1) = \Phi ((f_{00} - i f_{11})z) \otimes (a_0 + a_1) + \Phi ((f_{10} + i f_{01})z) \otimes (b_0 + b_1) + \Phi(a_1) \otimes 1 \), which implies \( a_1 = -a_0 \Phi ((f_{00} - i f_{11})z) - b_0 \Phi ((f_{10} + i f_{01})z) \).

And similarly, \( b_1 = -b_0 \Phi ((f_{00} + i f_{11})z) - a_0 \Phi ((f_{10} - i f_{01})z) \).

\[ \Phi(xp_1) = \Phi(p_1 x) \Rightarrow \Phi(x) a_1 = a_1 \Phi(x) \Rightarrow b_0 \Phi ((f_{10} + i f_{01})z) = 0 \Rightarrow b_0 = 0, \]
\[ \Phi(zp_1) = -\Phi(p_1 z) \Rightarrow \Phi(z) a_1 = -a_1 \Phi(z) \Rightarrow a_0 \Phi(f_{00} + i f_{11}) = 0 \Rightarrow a_0 = 0. \]

So \( a_1 = b_1 = 0 \), hence \( \Phi(p_1) = a_2 \in A'_1 \) and \( \Phi(p_2) = b_2 \in A'_1 \).

Now suppose \( \Phi : \mathcal{A}_7(\lambda_1, \ldots, \lambda_5) \to \mathcal{A}_7(\lambda'_1, \ldots, \lambda'_5) \) is an isomorphism. When \( \Phi \mid H_8 = \text{id} \), then by \( 5.15 \) and \( 5.16 \), \( \Phi(p_1) = ap'_1 + cq'_2 \) for some \( a, c \in K \) and \( \Phi(p_2) = bp'_2 + dq'_1 \) for some \( b, d \in K \), where \( p'_1, \ldots, p'_5 \) are generators of \( \mathcal{A}_7(\lambda'_1, \ldots, \lambda'_5) \). Then \( \Phi(xp_1) = x(ap'_1 + cq'_2) = (ap'_1 - cq'_2)x = \Phi(p_1 x) = (ap'_1 + cq'_2)x \), so \( c = 0 \) and similarly \( b = 0 \). Similarly, we have \( \Phi(q_1) = \alpha q'_1, \Phi(q_2) = \beta q'_2 \) for some nonzero parameters \( \alpha \) and \( \beta \). In this case, \( \Phi \) is an isomorphism of Hopf algebras if and only if the relations \( 5.21 \) hold.

When \( \Phi \mid H_8 = \gamma_2 \), suppose \( \Phi(p_1) = \alpha_1 p'_1 + \alpha_2 p'_2 + \alpha_3 q'_1 + \alpha_4 q'_2 \); then \( a_2 = a_4 = 0 \) since \( \Phi(xp_1) = \Phi(p_1 x) \). By \( 5.15 \) and \( 5.16 \), we can obtain \( \alpha_3 = 0 \) and \( \Phi(p_2) = \gamma_2 = 0 \).
\(-\alpha_1 p'_2\). Similarly, \(\Phi(q_1) = \beta_1 q'_1\), \(\Phi(q_2) = -\beta_1 q'_2\) for some parameter \(\beta_1\). \(\Phi\) respects relations of \(A_1\).

According to the defining relations of \(I(I_7)\), we have

\[
\alpha_2^* \lambda'_1 = \lambda_1, \quad \alpha_2^* \lambda'_2 = \lambda_2, \quad \beta_1^* \lambda'_3 = \lambda_3, \quad \beta_1^* \lambda'_4 = \lambda_4, \quad \alpha_1 \beta_1 \lambda'_5 = \lambda_5.
\]

This is just a special case of (5.21).

When \(\Phi|_{H_8} = \tau_3\), then \(\Phi(p_1) \in \mathbb{K} q'_1\), \(\Phi(p_2) \in \mathbb{K} q'_2\), \(\Phi(q_1) \in \mathbb{K} p'_1\), \(\Phi(q_2) \in \mathbb{K} p'_2\). According to (5.15) and (5.16), \(\Phi(p_1) = \alpha_1 q'_1\), \(\Phi(p_2) = -i \alpha_1 q'_2\), \(\Phi(q_1) = \beta_1 p'_1\), \(\Phi(q_2) = i \beta_1 p'_2\) for nonzero parameters \(\alpha_1\) and \(\beta_1\). \(\Phi\) respects relations of \(A_1\). In this case, \(\Phi\) is an isomorphism of Hopf algebras if and only if the relations (5.22) hold.

When \(\Phi|_{H_8} = \tau_4\), then \(\Phi(p_1) = \alpha_1 q'_1\), \(\Phi(p_2) = i \alpha_1 q'_2\), \(\Phi(q_1) = \beta_1 p'_1\), \(\Phi(q_2) = -i \beta_1 p'_2\) for nonzero parameters \(\alpha_1\) and \(\beta_1\). \(\Phi\) respects relations of \(A_1\). According to the defining relations of \(I(I_7)\), we obtain relations of parameters which exactly coincide with (5.22).

\(\square\)

**Definition 5.18.** For a set of parameters \(I_4 = \{\lambda_1, \lambda_2, (\lambda_{j,k})_{n_1 \times n_2}\}\), denote by \(\mathfrak{A}_4(n_1, n_2; I_4)\) the algebra \(T[\Omega(4(n_1, n_2)]##H_8/I(I_4)\), where \(I(I_4)\) is the ideal generated by the relations

\[
p_1^2 = \lambda_1(1 - xy), \quad p_2^2 = i \lambda_1(1 - xy), \quad p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = \lambda_2(1 - xy),
\]

\[
X_{j_1}^2 = 0, \quad X_{j_1} X_{j_2} + X_{j_2} X_{j_1} = 0, \quad j_1, j_2 \in \{1, \ldots, n_1\},
\]

\[
Y_{j_1}^2 = 0, \quad Y_{k_1} Y_{k_2} + Y_{k_2} Y_{k_1} = 0, \quad k_1, k_2 \in \{1, \ldots, n_2\},
\]

\[
X_j Y_k + Y_k X_j = \lambda_{j,k}(1 - xy),
\]

\[
p_1 Y_k - Y_k p_1 = 0, \quad p_2 Y_k + Y_k p_2 = 0, \quad p_1 X_j - X_j p_1 = 0, \quad p_2 X_j + X_j p_2 = 0.
\]

**Remark 5.19.** In fact, \(I(I_4)\) is a Hopf ideal, so \(\mathfrak{A}_4(n_1, n_2; I_4)\) is a Hopf algebra. In particular, when all the parameters in \(I_4\) are zero, then \(\mathfrak{A}_4(n_1, n_2; I_4) \simeq \mathfrak{B}[\Omega(4(n_1, n_2)]##H_8\).

**Proposition 5.20.** (1) Suppose \(H\) is a finite-dimensional Hopf algebra with coradical \(H_8\) such that its infinitesimal braiding is isomorphic to \(\Omega(4(n_1, n_2))\). Then \(H \simeq \mathfrak{A}_4(n_1, n_2; I_4)\).

(2) \(\mathfrak{A}_4(n_2, n_2; I_4) \simeq \mathfrak{A}_4(n_1, n_2; I'_4)\) iff there exist two invertible matrices \((\alpha_{j,s})_{n_1 \times n_1}\), \((\beta_{k,t})_{n_2 \times n_2}\) and two nonzero parameters \(a, b\), such that

\[
\sum_{s=1}^{n_1} \sum_{t=1}^{n_2} \alpha_{j,s} = \beta_{k,t} \lambda'_{j,k} = \lambda_{j,k}, \quad j \in \{1, \ldots, n_1\}, \quad k \in \{1, \ldots, n_2\}.
\]

\[
a^2 \lambda'_1 = \lambda_1, \quad b^2 \lambda'_2 = \lambda_2, \quad a^2 b^2 \lambda'_2 = \lambda_2.
\]

**Proof.** (1) Let \(H\) be a lifting of \(\mathfrak{B}[\Omega(4(n_1, n_2)]##H_8\). Then there exists an epimorphism of Hopf algebras \(\phi : T(\Omega(4(n_1, n_2)]##H_8 \to H\). Denote

\[
p_1 = \left( w_1^{1,-1} + i w_2^{1,-1} \right) \#1, \quad p_2 = \left( w_1^{1,-1} - i w_2^{1,-1} \right) \#1,
\]

\[
X_j = (v \otimes x) \#1, \quad Y_k = (v \otimes y) \#1, \quad v \in V_1(i), \quad j \in \{1, \ldots, n_1\}, \quad k \in \{1, \ldots, n_2\}.
\]
Then the relations of generators in $T[\Omega_4(n_1, n_2)]\# H_8$ are given by

$$ xp_1 = p_1 x, \quad yp_1 = p_1 y, \quad xp_2 = -p_2 x, \quad yp_2 = -p_2 y, $$

$$ zp_1 = -p_1 z, \quad xX_j = -X_j x, \quad yX_j = -X_j y, \quad zX_j = iX_j z, $$

$$ zp_2 = i p_2 xz, \quad xY_k = -Y_k x, \quad yY_k = -Y_k y, \quad zY_k = iY_k z, $$

and the coproducts of generators are given by

$$ \Delta(X_j) = X_j \otimes 1 + x \otimes X_j, \quad \Delta(Y_k) = Y_k \otimes 1 + y \otimes Y_k, $$

$$ \Delta(p_1) = (f_{00} - if_{11}) z \otimes p_1 + (f_{10} + if_{01}) z \otimes p_2 + p_1 \otimes 1, $$

$$ \Delta(p_2) = (f_{00} + if_{11}) z \otimes p_2 + (f_{10} - if_{01}) z \otimes p_1 + p_2 \otimes 1. $$

As similarly proved in Proposition 5.6 and Lemma 5.14, the map $\phi$ keeps relations (5.23)-(5.26). Since $r = 0$ in $\text{gr } H$ for $r = \phi(p_1 Y_k - Y_k p_1)$ and $\phi(p_2 Y_k + Y_k p_2)$, $r$ is an element of at most degree one. It is only possible for

$$ \phi(p_1 Y_k - Y_k p_1) = -\mu_k (-f_{10} + if_{01}) z, \quad \phi(p_2 Y_k + Y_k p_2) = -\mu_k (f_{00} - if_{11}) z + \mu_k 1, $$

because of the following relations:

$$ x (p_1 Y_k - Y_k p_1) = -(p_1 Y_k - Y_k p_1) x, \quad z (p_1 Y_k - Y_k p_1) = -i (p_1 Y_k - Y_k p_1) xz, $$

$$ x (p_2 Y_k + Y_k p_2) = (p_2 Y_k + Y_k p_2) x, \quad z (p_2 Y_k + Y_k p_2) = (p_2 Y_k + Y_k p_2) z, $$

$$ \Delta(p_1 Y_k - Y_k p_1) = (p_1 Y_k - Y_k p_1) \otimes 1 + (f_{00} + if_{11}) z \otimes (p_1 Y_k - Y_k p_1) $$

$$ \Delta(p_2 Y_k + Y_k p_2) = (p_2 Y_k + Y_k p_2) \otimes 1 + (f_{00} - if_{11}) z \otimes (p_2 Y_k + Y_k p_2) $$

$$ - (f_{10} + if_{01}) z \otimes (p_1 Y_k - Y_k p_1). $$

Similarly, we get

$$ \phi(p_1 X_j - X_j p_1) = -\mu'_j (f_{10} - if_{01}) z, \quad \phi(p_2 X_j + X_j p_2) = -\mu'_j (f_{00} - if_{11}) z + \mu'_j 1, $$

from the following formulae:

$$ x (p_1 X_j - X_j p_1) = -(p_1 X_j - X_j p_1) x, \quad z (p_1 X_j - X_j p_1) = -i (p_1 X_j - X_j p_1) xz, $$

$$ x (p_2 X_j + X_j p_2) = (p_2 X_j + X_j p_2) x, \quad z (p_2 X_j + X_j p_2) = (p_2 X_j + X_j p_2) z, $$

$$ \Delta(p_1 X_j - X_j p_1) = (f_{00} + if_{11}) z \otimes (p_1 X_j - X_j p_1) + (p_1 X_j - X_j p_1) \otimes 1 $$

$$ \Delta(p_2 X_j + X_j p_2) = (f_{00} - if_{11}) z \otimes (p_2 X_j + X_j p_2) + (p_2 X_j + X_j p_2) \otimes 1 $$

$$ + (f_{10} + if_{01}) z \otimes (p_1 X_j - X_j p_1). $$

Since $\phi(X_j Y_k + Y_k X_j) = \lambda_{j,k} (1 - xy), \phi[p_1 (X_j Y_k + Y_k X_j)] = \phi[(X_j Y_k + Y_k X_j) p_1] \Rightarrow \mu'_j = \mu_k = 0$. So $\phi$ keeps relations (5.27). Now we have $I(I_7) \subseteq \ker \phi$, so there exists a surjective map from $\mathfrak{A}_4(n_1, n_2; I_4)$ to $H$. By Theorem 5.1, $\text{gr } H \cong \mathfrak{S}[\Omega_4(n_1, n_2)]\# H_8$. Now we only need to prove that $\dim \mathfrak{A}_4(n_1, n_2; I_4) = \dim H$. In fact, $\mathfrak{A}_4(n_1, n_2; I_4) \cong \mathfrak{S}(W^{1,-1}) \otimes \mathfrak{S}(M(i, x))^0 \otimes \mathfrak{S}(M(i, y))^0 \otimes H_8$ as vector spaces by the Diamond Lemma. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resoluble is a tedious but straightforward computation.
(2) Suppose \( \Phi : \mathfrak{A}_4(n_1,n_2;I_4) \to \mathfrak{A}_4(n_1,n_2;I'_4) \) is an isomorphism of Hopf algebras. Similarly to the proof of Proposition 5.17, it is easy to see that \( \Phi|_{H_8} \in \{\text{id}, \tau_2\} \) and \( \Phi(p_1) = ap_1', \Phi(p_2) = bp_2' \). Since \( xX_j \) is \((x,1)\)-skew primitive and \( xX_j = -X_jx \), we have \( \Phi(X_j) \in \bigoplus_{s=1}^{n_1} \mathbb{K}X'_s \). Similarly, we have \( \Phi(Y_k) \in \bigoplus_{t=1}^{n_2} \mathbb{K}Y'_t \). Let \( \Phi(X_j) = \sum_{s=1}^{n_1} \alpha_{js}X'_s \), \( \Phi(Y_k) = \sum_{t=1}^{n_2} \beta_{kt}Y'_t \), where the matrices \((\alpha_{js})_{n_1 \times n_1}\) and \((\beta_{kt})_{n_2 \times n_2}\) are invertible. Then \( \Phi \) is an isomorphism of Hopf algebras if and only if the relations (5.28) hold. □

Proof of Theorem B. Let \( H \) be a finite-dimensional Hopf algebra over \( H_8 \) such that its infinitesimal braiding \( M \in H_8^H \mathbb{K}YD \); then \( M \) is in the list of Theorem A. We need to give a construction for any finite-dimensional Hopf algebra \( H \) over \( H_8 \) up to isomorphism such that its infinitesimal braiding is isomorphic to \( M \). By Theorem 5.1, \( \text{gr} H \simeq \mathfrak{B}(M) \# H_8 \). According to Corollary 5.3, \( \text{gr} H \simeq \mathfrak{B}(M) \# H_8 \) for \( M = \Omega_1(n_1,n_2,n_3,n_4), \Omega_2(n_1,n_2), \Omega_4(n_1,n_2), \Omega_6, \Omega_7 \). Propositions 5.6, 5.10, 5.20, 5.13 and 5.17 finish the proof.

Acknowledgements

The author thanks Prof. Naihong Hu for his encouragement and constructive suggestions which push him to consider the classification of finite-dimensional Hopf algebras. The author would like to thank Prof. Wenxue Huang, Yunnan Li, and Rongchuan Xiong for useful discussions, and also, thank the referees for careful reading and helpful comments on the writing of this paper. Special thanks to Prof. Nicolás Andruskiewitsch for recommending the Revista de la Unión Matemática Argentina to the author and pointing out references [11] and [45].

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*Received: February 25, 2017*
*Accepted: September 13, 2018*