LINEAR POISSON STRUCTURES AND HOM-LIE ALGEBROIDS

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ABSTRACT. Considering Hom-Lie algebroids in some special cases, we obtain some results of Lie algebroids for Hom-Lie algebroids. In particular, we introduce the local splitting theorem for Hom-Lie algebroids. Moreover, linear Hom-Poisson structure on the dual Hom-bundle will be introduced and a one-to-one correspondence between Hom-Poisson structures and Hom-Lie algebroids will be presented. Also, we introduce Hamiltonian vector fields by using linear Poisson structures and show that there exists a relation between these vector fields and the anchor map of a Hom-Lie algebroid.

1. INTRODUCTION

Hom-Lie algebroids were introduced by Laurent-Gengoux and Teles in [4] using the notion of Hom-Gerstenhaber algebras. Afterwards, in [1], Cai, Liu, and Sheng modified the definition of a Hom-Lie algebroid and gave its equivalent dual description. A Hom-Lie algebroid has its own geometric meaning and interesting examples, and it is more than a formal generalization of a Lie algebroid. Recently, many researchers have been interested in studying the algebraic and geometric concepts on Lie algebroids and Hom-Lie algebroids ([3, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16]).

In [1], the authors could fix the definition of Hom-Lie algebroid in a more suitable way by introducing the notion of Hom-bundle. In this sense, there is a fundamental example. Let $\varphi : M \rightarrow M$ be any diffeomorphism. Then the pull back bundle $\varphi^!TM$ of the tangent Lie algebroid TM is a Hom-Lie algebroid. This example is based on the concept of the Hom bundle, and using it the authors introduced the Hom-Poisson tensor structures. Therefore, we see the naturality of this definition of Hom-Lie algebroids. In this paper, we consider the definition of Hom-Lie algebroid given in [1].

Many results in Hamiltonian dynamics are indebted to Poisson geometry, where they serve as phase spaces. Though Poisson geometry was an outcome of symplectic geometry, it is a powerful theory in mathematics now. Lots of relations with other fields —like Hamiltonian dynamics, representation theory, quantum physics, dynamical and integrable systems— make Poisson geometry a simple to use and essential approach. Indeed, translating the concepts to Poisson literature, may

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reduce the amount of computations or may solve the problem in a more technical way. All of this paper's discussions can be reduced to the Lie algebroid case. Lie algebroids are just some fiber-wise linear Poisson structures. This kind of approach is different by considering them as infinitesimal versions of Lie groupoids. Here, one can see that Poisson makes a correlation between commutative and non-commutative objects [2].

The structure of this paper is organized as follows: In Section 2, we recall the notions of Hom-algebra and Hom-Lie algebroid and present some examples of these concepts. In Section 3, we obtain some results on Hom-Lie algebroids in special cases. In particular, we present the locally splitting theorem for Hom-Lie algebroids. Section 4 is devoted to the study of linear Poisson structures and Hamiltonian vector fields on Hom-Lie algebroids.

2. PRELIMINARIES

In this section, we present some notions and results about Hom-Lie algebras and Hom-Lie algebroids (see [1, 9, 10] for more details).

A triple $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ consisting of a linear space $\mathfrak{g}$, a bilinear map (bracket) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, and an algebra morphism $\phi_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

$$[u, v] = -[v, u] \quad \text{and} \quad \odot_{u,v,w} [\phi_{\mathfrak{g}}(u), [v, w]] = 0, \quad u, v, w \in \mathfrak{g},$$

is called a Hom-Lie algebra (the second equation is called Hom-Jacobi identity). The Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ is called regular (involutive), if $\phi_{\mathfrak{g}}$ is non-degenerate (satisfies $\phi_{\mathfrak{g}}^2 = 1$). A *representation* of $(\mathfrak{g}, [\cdot, \cdot], \phi_{\mathfrak{g}})$ is a triple (V, A, ρ) where V is a vector space, $A \in gl(V)$ and $\rho : \mathfrak{g} \rightarrow gl(V)$ is a linear map satisfying

$$\begin{cases} \rho(\phi_{\mathfrak{g}}(u)) \circ A = A \circ \rho(u), \\ \rho([u, v]_{\mathfrak{g}}) \circ A = \rho(\phi_{\mathfrak{g}}(u)) \circ \rho(v) - \rho(\phi_{\mathfrak{g}}(v)) \circ \rho(u), \end{cases}$$

for any $u, v \in \mathfrak{g}$.

Let A be a vector bundle of rank r over the manifold M . A *Hom-bundle* is a triple $(A \rightarrow M, \varphi, \phi_A)$ consisting of a vector bundle $A \rightarrow M$, a smooth map $\varphi : M \rightarrow M$, and an algebra morphism $\phi_A : \Gamma(A) \rightarrow \Gamma(A)$ satisfying

$$\phi_A(fX) = \varphi^*(f)\phi_A(X), \quad (2.1)$$

for any $X \in \Gamma(A)$ and $f \in C^\infty(M)$. If φ is a diffeomorphism and ϕ_A is an invertible map, then the Hom-bundle $(A \rightarrow M, \varphi, \phi_A)$ is called invertible. An example of a Hom-bundle is $(\Gamma(\varphi^!TM), \varphi, Ad_{\varphi^*})$, where $Ad_{\varphi^*} : \Gamma(\varphi^!TM) \rightarrow \Gamma(\varphi^!TM)$ is given by

$$Ad_{\varphi^*}(X) = \varphi^* \circ X \circ (\varphi^*)^{-1}, \quad \forall X \in \Gamma(\varphi^!TM).$$

Remark 2.1. If $\phi_A : \Gamma(A) \rightarrow \Gamma(A)$ satisfies (2.1), then we have the bundle morphism $\phi_A : \varphi^!A \rightarrow A$ given by $\phi_A(m, X_{\varphi(m)}) = (m, \phi_A(X)_m)$, for all $m \in M$. The converse of this assertion is also true.

Definition 2.2. Let $(A \rightarrow M, \varphi, \phi_A)$ and $(B \rightarrow M, \varphi, \phi_B)$ be two Hom-bundles. A bundle map $\rho : A \rightarrow B$ is called a morphism of Hom-bundles if the following condition holds:

$$\rho \circ \phi_A = \phi_B \circ \rho.$$

The linear map $\phi_A : \Gamma(A) \rightarrow \Gamma(A)$ given in a Hom-bundle $(A \rightarrow M, \varphi, \phi_A)$ can be extended to a linear map from $\Gamma(\wedge^k A)$ to $\Gamma(\wedge^k A)$ for which we use the same notation ϕ_A via

$$\phi_A(X) = \phi_A(X_1) \wedge \cdots \wedge \phi_A(X_k), \quad \forall X = X_1 \wedge \cdots \wedge X_k \in \Gamma(\wedge^k A).$$

If the Hom-bundle A is invertible, then the inverses of φ and ϕ_A are denoted by φ^{-1} and ϕ_A^{-1} , respectively. Also it is easy to see that

$$\phi_A^{-1}(fX) = (\varphi^*)^{-1}(f)\phi_A^{-1}(X), \quad \forall f \in C^\infty(M), X \in \Gamma(\wedge^k A).$$

So $(A \rightarrow M, \varphi^{-1}, \phi_A^{-1})$ is a Hom-bundle. We consider $\phi_A^\dagger : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^k A^*)$ defined by

$$(\phi_A^\dagger(\xi))(X) = \varphi^*\xi(\phi_A^{-1}(X)), \quad \forall X \in \Gamma(\wedge^k A), \xi \in \Gamma(\wedge^k A^*).$$

From the above equation, we can conclude that

$$\phi_A^\dagger(f\xi) = \varphi^*(f)\phi_A^\dagger(\xi). \quad (2.2)$$

Therefore $(\wedge^k A^*, \varphi, \phi_A^\dagger)$ is a Hom-bundle.

Remark 2.3. When $k = 1$, similar to Remark 2.1 we have the correspondence bundle map $\phi_A^\dagger(m, \omega_{\varphi(m)}) = (m, \phi_A^\dagger(\omega)_m)$, for all $m \in M$, for $\phi_A^\dagger : \Gamma(A^*) \rightarrow \Gamma(A^*)$ satisfying (2.2).

Definition 2.4 ([1]). A Hom-Lie algebroid is a multiple $(A, \varphi, \phi_A, [\cdot, \cdot]_A, a_A)$ such that $(A \rightarrow M, \varphi, \phi_A)$ is a Hom-bundle, $(\Gamma(A), [\cdot, \cdot]_A, \phi_A)$ is a Hom-Lie algebra on the section space $\Gamma(A)$, $a_A : A \rightarrow \varphi^!TM$ is a bundle map called the anchor map and moreover, we have

$$[X, fY]_A = \varphi^*(f)[X, Y]_A + a_A(\phi_A(X))(f)\phi_A(Y), \quad \forall X, Y \in \Gamma(A), \forall f \in C^\infty(M),$$

when $a_A : \Gamma(A) \rightarrow \Gamma(\varphi^!TM)$ is the representation of the Hom-Lie algebra $(\Gamma(A), [\cdot, \cdot]_A, \phi_A)$ on $C^\infty(M)$ with respect to φ^* induced by the anchor map.

It is easy to see that $(\varphi^!TM, \varphi, Ad_{\varphi^*}, [\cdot, \cdot]_{\varphi^*}, id)$ is a Hom-Lie algebroid [1], where $[X, Y]_{\varphi^*} = \varphi^* \circ X \circ (\varphi^*)^{-1} \circ Y \circ (\varphi^*)^{-1} - \varphi^* \circ Y \circ (\varphi^*)^{-1} \circ X \circ (\varphi^*)^{-1}$, $\forall X, Y \in \varphi^!TM$.

Let $(A, \varphi, \phi_A, [\cdot, \cdot]_A, a_A)$ and $(B, \varphi, \phi_B, [\cdot, \cdot]_B, a_B)$ be two Hom-Lie algebroids over M . A *base-preserving* morphism with the same base from A to B is a bundle map $\rho : A \rightarrow B$ such that

$$a_B \circ \rho = a_A, \quad \rho \circ \phi_A = \phi_B \circ \rho, \quad \rho([X, Y]_A) = [\rho(X), \rho(Y)]_B.$$

If φ is a diffeomorphism, it is easy to see that the map a_A is a morphism between Hom-Lie algebroids, i.e.,

$$a_A(\phi_A(X)) = Ad_{\varphi^*}(a_A(X)), \quad a_A([X, Y]_A) = [a_A(X), a_A(Y)]_{\varphi^*}.$$

3. LOCAL SPLITTING THEOREM

Let $x^i : U \subseteq M \rightarrow \mathbb{R}$ be a coordinate system on M , $\Psi_\alpha : U \subseteq M \rightarrow A$ and $\Psi^\alpha : U \subseteq M \rightarrow A^*$ be a local basis of sections of A and A^* , respectively. Let $((x^1, \dots, x^m, \xi_1, \dots, \xi_r), \pi^{-1}(U))$ be the coordinate system on A^* , where $\xi_\alpha : A^* \rightarrow \mathbb{R}$ is defined by $\xi_\beta(\xi_\alpha \Psi^{*\alpha}) = \xi_\beta$. So, the local vector fields on A^* associated to this coordinate system are denoted by $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi_\alpha}$.

Suppose that ϕ_A is diagonalizable on A , that is, there exists a basis of sections such that its matrix is diagonal with respect to it. Let $f_1, \dots, f_k$ be locally smooth functions on an open subset $U \subset M$ which are the eigenvalues of ϕ_A at every point. We denote the eigenspace of ϕ_A related to f_i at p by A_{ip} . Suppose $A_i = \cup_p A_{ip}$. Since $\phi_A : A \rightarrow A$ is a smooth function between two smooth manifolds, A_i has constant rank for all $i = 1, \dots, k$. Therefore, they are distributions of A . Now we can state the following theorem in order to classify ϕ_A .

Theorem 3.1. *Let $(A, \text{Id}_M, \phi_A = f\text{Id}_A, [\cdot, \cdot]_A, a_A)$ be a Hom-Lie algebroid where $f : M \rightarrow M$ is a smooth function. If the rank of a_A is greater than 0 then f must be constant 1 on M . Moreover, when $\varphi = \text{Id}_M$ and ϕ_A is arbitrary, if the rank of a_A is greater than zero on every A_i then ϕ_A is the identity isomorphism.*

Proof. Using the equation $a_A(\phi_A(X)) \circ \varphi^* = \varphi^* \circ a_A(X)$ and considering $\varphi = \text{Id}_M$ and $\phi_A = f\text{Id}_A$ one can get $fa_A(X) = a_A(X)$. Since a_A is a non-zero bundle morphism the result of the first part can be achieved, i.e., $f \equiv 1$. For the second part, let $X_i \in \Gamma(A_i)$ such that $a_A(X_i) \neq 0$. If $\phi_A(X_i) = f_i X_i$ then using the equation $a_A(\phi_A(X)) \circ \varphi^* = \varphi^* \circ a_A(X)$ we get $f_i a_A(X_i) = a_A(X_i)$. This shows that $f_i \equiv 1$ and so $\phi_A \equiv \text{Id}_A$. $\square$

Definition 3.2. An a_A -section S of $\varphi^!TM$ is a section of $\varphi^!TM$ such that there exists a section Ψ of A satisfying $a_A(\Psi) = S$. We denote the set of such sections by $\Gamma_A(\varphi^!TM)$. It is easy to see that $\Gamma_A(\varphi^!TM)$ is a $C^\infty(M)$ -module.

Theorem 3.3. *If $\phi_A = f\text{Id}_A$, then $\varphi_* : \Gamma_A(\varphi^!TM) \rightarrow \Gamma_A(\varphi^!TM)$ is a multiple of the identity.*

Proof. Let X be a section of A . Using the relation $a_A(\phi_A(X)) \circ \varphi^* = \varphi^* \circ a_A(X)$ we get

$$f(p)(a_A(X))_p(g \circ \varphi) = a_A(X)_{\varphi(p)}(g),$$

for $g \in C^\infty(M)$ and $p \in M$. So, one can get

$$d_{\varphi(p)}g(f(p)\varphi_{*p}(a_A(X_p)) - a_A(X)_{\varphi(p)}).$$

Since g was an arbitrary smooth function on M we derive that

$$f(p)\varphi_{*p}(a_A(X_p)) - a_A(X)_{\varphi(p)} = 0.$$

Since ϕ_A is an isomorphism, $f(p) \neq 0$ and then we have

$$\varphi_{*p}(a_A(X_p)) = \frac{1}{f(p)}a_A(X)_{\varphi(p)},$$

and the proof is completed. $\square$

Proposition 3.4. *If φ is the identity mapping on M then we have*

$$a_A[X, Y]_A = [a_A(X), a_A(Y)]_M.$$

Proof. If φ is the identity then

$$a_A(\phi_A(X)) = a_A(X),$$

and

$$a_A[X, Y]_A = a_A(\phi_A(X))a_A(Y) - a_A(\phi_A(Y))a_A(X).$$

If we set $a_A(Y) = a_A(\phi(Y))$ and $a_A(X) = a_A(\phi(X))$ we get

$$a_A[X, Y]_A = a_A(X)a_A(Y) - a_A(Y)a_A(X),$$

but this is equivalent to

$$a_A[X, Y]_A = [a_A(X), a_A(Y)]_M. \quad \square$$

Theorem 3.5. *Let $(A, \varphi, \phi_A, [\cdot, \cdot]_A, a_A)$ be a Hom-Lie algebroid and x_0 be a point in M such that the rank of $a_A|_{A_t}$ at x_0 is r_t , the dimension of fibers of A_t is α_t , and A_t is the eigenspace of ϕ_A with respect to the eigenvalue f_t . Then there exist coordinates (x^i, y^j) , where $i = 1, \dots, r_t$ and $j = r_t + 1, \dots, m$ on a neighborhood U of x_0 and a basis of sections $\{\Psi_1, \dots, \Psi_{\alpha_t}\}$ of A_t over U satisfying*

$$a_A|_{A_t}(\Psi_i) = \frac{\partial}{\partial x^i}, \quad i = 1, \dots, r_t, \quad (3.1)$$

$$a_A|_{A_t}(\Psi_\alpha) = \sum_j a_\alpha^j \frac{\partial}{\partial y^j}, \quad \alpha = r_t + 1, \dots, \alpha_t, \quad (3.2)$$

where $a_\alpha^j \in C^\infty(U)$ are smooth functions depending only on the y 's and vanishing at x_0 ; namely $a_\alpha^j = a_\alpha^j(y^k)$ and $a_\alpha^j(0) = 0$. Moreover,

$$[\Psi_\alpha, \Psi_\beta] = \sum_\gamma c_{\alpha\beta}^\gamma \Psi_\gamma, \quad (3.3)$$

where $c_{\alpha\beta}^\gamma \in C^\infty(U)$ vanish if $\gamma \leq r_t$ and satisfy

$$\sum_{\gamma > r_t} a_\gamma^j \frac{\partial c_{\alpha\beta}^\gamma}{\partial x^i} = 0,$$

$$\sum_{\gamma > r_t} c_{\alpha\beta}^\gamma a_\gamma^j \frac{\partial f_t}{\partial y^j} = 0,$$

where f_t is the eigenvalue of ϕ_A related to the eigenspace A_t .

Proof. Our proof is based on the rank of $a_A|_{A_t}$. By assuming the rank of $a_A|_{A_t}$ to be r_t we give a proof by induction on r_t (indeed, we don't change any construction here but we can suppose that there exists some subspace of A_t so that the restriction of $a_A|_{A_t}$ to this subspace is of rank $k < r_t$). Suppose we have all results of the theorem for $0 \leq k < r_t$ and assume we have constructed coordinates $(x^i, \tilde{y}^j)$ where

$i \leq k$ and $k < j \leq m$ on $U \subset M$ and a basis of sections $\{\Psi_i, \tilde{\Psi}_\alpha\}$ for A_t over U , where $i \leq k$ and $k < \alpha \leq \alpha_t$ (α_t is the rank of A_t) such that

$$a_A(\Psi_i) = \frac{\partial}{\partial x^i}, \quad a_A(\tilde{\Psi}_\alpha) = a_\alpha^j \frac{\partial}{\partial \tilde{y}^j},$$

where a_α^j depend only on the $\tilde{y}$'s. Since $r_t > k$, there exists an α such that the section $a_A(\tilde{\Psi}_\alpha)$ does not vanish at x_0 (since the rank of $a_A|_{A_t}$ is supposed to be r_t , which is greater than k). We can assume that $\alpha = k+1$ and we set $\Psi_{k+1} = \tilde{\Psi}_{k+1}$. We can perform a change of coordinates

$$x^{k+1} = x^{k+1}(\tilde{y}^{k+1}, \dots, \tilde{y}^m), \quad y^j = y^j(\tilde{y}^{k+1}, \dots, \tilde{y}^m),$$

such that

$$a_A(\Psi_{k+1}) = \frac{\partial}{\partial x^{k+1}},$$

$$a_A(\tilde{\Psi}_\alpha) = a_\alpha^{k+1} \frac{\partial}{\partial x^{k+1}} + a_\alpha^{k+2} \frac{\partial}{\partial y^{k+2}} + \dots.$$

Note that such change of coordinates can happen as $a_A(\Psi_{k+1})(x_0) \neq 0$. If we change $\tilde{\Psi}_\alpha$ with $\tilde{\Psi}_\alpha - a_\alpha^{k+1} \Psi_{k+1}$ we get

$$a_A(\tilde{\Psi}_\alpha) = a_\alpha^l \frac{\partial}{\partial y^l},$$

where $a_\alpha^l = a_\alpha^l(x^{k+1}, y^{k+2}, \dots, y^m)$ and $l = k+2, \dots, m$. Since $r_t \geq 1$, according to Theorem 3.1 we have $f_t \equiv 1$. So, for all sections X, Y on A_t we have

$$a_A[X, Y]_A = [a_A(X), a_A(Y)]_M, \quad (3.4)$$

and this shows that

$$[\Psi_{k+1}, \tilde{\Psi}_\alpha]_A = \sum_{\gamma > k+1} c_{k+1, \alpha}^\gamma \tilde{\Psi}_\gamma. \quad (3.5)$$

By setting a_A in two sides of (3.5) and using (3.4) we get

$$\frac{\partial a_\alpha^l}{\partial x^{k+1}} \frac{\partial}{\partial y^l} = \sum_{\gamma > k+1} c_{k+1, \alpha}^\gamma a_\gamma^l \frac{\partial}{\partial y^l},$$

which yields

$$\frac{\partial a_\alpha^l}{\partial x^{k+1}} = \sum_{\gamma > k+1} c_{k+1, \alpha}^\gamma a_\gamma^l.$$

Similar to Fernandes' argument to solve this ODE we get a coordinate system (x^i, y^j) and sections $\{\Psi\}_{i=1}^{\alpha_t}$ satisfying the relations (3.1) and (3.2). Using the equation

$$a_A[X, Y]_A = [a_A(X), a_A(Y)]_M,$$

one can easily check that $c_{\alpha\beta}^\gamma \in C^\infty(U)$ vanish if $\gamma \leq r_t$ in (3.3). Using the Hom-Jacobi identity, one can get the following for $i \leq r_t$ and $r_t \leq \alpha, \beta \leq \alpha_t$:

$$\begin{aligned} a_A[f_t\Psi_i, [\Psi_\alpha, \Psi_\beta]_A]_A &= f_t[f_t[\frac{\partial}{\partial x^i}, a_\alpha^l \frac{\partial}{\partial y^l}]_A, f_t a_A \Psi_\beta]_A \\ &\quad + f_t[f_t a_A \Psi_\alpha, f_t[\frac{\partial}{\partial x^i}, a_\beta^l \frac{\partial}{\partial y^l}]_A]_A = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} a_A[f_t\Psi_i, [\Psi_\alpha, \Psi_\beta]_A]_A &= a_A\left[f_t\Psi_i, \sum_{\gamma > r_t} c_{\alpha\beta}^\gamma \Psi_\gamma\right]_A, \\ f_t\left[f_t\frac{\partial}{\partial x^i}, \sum_{\gamma > r_t} c_{\alpha\beta}^\gamma a_\gamma^j \frac{\partial}{\partial y^j}\right]_A &= (f_t)^2 \sum_{\gamma > r_t} a_\gamma^j \frac{\partial c_{\alpha\beta}^\gamma}{\partial x^i} \frac{\partial}{\partial y^j} + f_t \sum_{\gamma > r_t} c_{\alpha\beta}^\gamma a_\gamma^j \frac{\partial f_t}{\partial y^j} \frac{\partial}{\partial x^i} = 0. \end{aligned}$$

So, we have

$$\sum_{\gamma > r_t} a_\gamma^j \frac{\partial c_{\alpha\beta}^\gamma}{\partial x^i} = 0, \quad \sum_{\gamma > r_t} c_{\alpha\beta}^\gamma a_\gamma^j \frac{\partial f_t}{\partial y^j} = 0.$$

Note that, in general, r_t can be zero and so it is not necessary that we have $f_t \equiv 1$. $\square$

Remark 3.6. Using the above theorem and the equation

$$\rho[X, Y]_{\mathfrak{g}} = \rho(\phi_{\mathfrak{g}}(X)) \circ \rho(Y) - \rho(\phi_{\mathfrak{g}}(Y)) \circ \rho(X),$$

one can deduce a Fernandes-like result. Indeed, if $\{\Psi_1^i, \dots, \Psi_{\alpha_i}^i\}$ is a basis of sections for A_i and $(x_1^1, \dots, x_i^{r_i}, y_i^{r_i+1}, \dots, y_i^m)$ is a coordinate system about $x_0 \in M$ for each A_i satisfying the last theorem and if the rank of a_A is q at x_0 then there exist coordinates (x^i, y^j) and a basis of sections $\{\Psi^\alpha\}_1^r$ of A over U satisfying

$$\begin{aligned} a_A(\Psi_i) &= \frac{\partial}{\partial x^i}, \quad i = 1, \dots, r, \\ a_A(\Psi_\alpha) &= \sum_j a_\alpha^j \frac{\partial}{\partial y^j}, \quad \alpha > r, \end{aligned}$$

where r_i is the rank of $a_A|_{A_i}$ and α_i is the rank of A_i . Indeed, we choose a linearly independent subset of

$$\left\{ \frac{\partial}{\partial x_1^1}, \frac{\partial}{\partial x_1^2}, \dots, \frac{\partial}{\partial x_1^{r_1}}, \frac{\partial}{\partial x_2^1}, \dots, \frac{\partial}{\partial x_2^{r_2}}, \dots, \frac{\partial}{\partial x_k^1}, \dots, \frac{\partial}{\partial x_k^{r_k}} \right\},$$

where k is the number of eigenspaces of a_A at x_0 .

Let $A \xrightarrow{\pi} M$ and $B \xrightarrow{\nu} M$ be vector bundles and let $T : A \rightarrow B$ be a bundle map which satisfies

$$\nu \circ T = \pi.$$

Now, let $K_x = \frac{A_x}{\ker_x T}$ and let $K = \cup_x K_x$ for every $x \in M$, where A_x is the fiber at x and $\ker_x T$ is the kernel of T . It is easy to see that if the rank of T on every A_x is the same number then K is a subbundle of A . Now, we suppose that A is

a Hom-Lie algebroid and let $B = TM$. We can suppose that $a_A|_{A_x}$ is a surjective linear map. The result is that K is a subbundle of A .

Now, let $\rho : K \rightarrow TM$ and $\Delta : K \rightarrow K$ be bundle maps defined by

$$\rho(X_x + K_x) = a_A(X_x)$$

and

$$\Delta(X_x + K_x) = \phi_A(X_x) + K_x.$$

Then using $\rho(\Delta X) = \rho(\phi_A(X) + K) = a_A(\phi_A(X)) = a_A(X) = \rho(X + K)$ we get

$$\rho(\Delta X) = \rho(X).$$

Proposition 3.7. *Using the above notations one can prove that Δ is an identity bundle map.*

Proof. It is easy to see that $\rho \circ \Delta = \rho$. If $\Delta(X_x + K_x) = \lambda(X_x + K_x)$ then using that relation we get $\rho(X_x + K_x) = \lambda\rho(X_x + K_x)$. Since ρ is one-to-one, we have $\lambda = 1$. $\square$

Lemma 3.8. *ρ and Δ are well-defined and one-to-one mappings.*

Proof. Let $X_x - Y_x = Z_x \in K_x$. It is easy to see that $a_A(X_x - Y_x) = 0$ and so $a_A(X_x) = \rho(X_x + K_x) = \rho(Y_x + K_x) = a_A(Y_x)$. Now, we show that $\Delta(X_x + K_x) = \Delta(Y_x + K_x)$, i.e., $\phi_A(X_x - Y_x) \in K_x$. Since, $a_A \circ \phi_A = a_A$, so $a_A(\phi_A(X_x - Y_x)) = a_A(X_x - Y_x) = 0$.

Now, let $\rho(X_x + K_x) = a_A(X_x) = 0$. This shows that $X_x \in K_x$ and the result is that $X_x + K_x = K_x$ and a_A is one-to-one. The equation $a_A \circ \phi_A = a_A$ shows that if $\Delta(X_x + K_x) = K_x$ then $X_x \in K_x$ and so Δ is one-to-one. $\square$

Theorem 3.9. *$(K, \rho, [\cdot, \cdot]_K, M)$ is a Lie algebroid, where $[\cdot, \cdot]_K : K \times K \rightarrow K$ is defined by*

$$[X + K, Y + K]_K = \rho^{-1}[a_A(X), a_A(Y)]_M.$$

Proof. It is obvious that $[\cdot, \cdot]_K$ is skew-symmetric. We show that it satisfies the Jacobi identity. Let

$$\begin{aligned} L = [X + K, [Y + K, Z + K]_K]_K &+ [Z + K, [X + K, Y + K]_K]_K \\ &+ [Y + K, [Z + K, X + K]_K]_K. \end{aligned}$$

Then we have

$$\begin{aligned} \rho L = [a_A(X), [a_A(Y), a_A(Z)]_M]_M &+ [a_A(Z), [a_A(X), a_A(Y)]_M]_M \\ &+ [a_A(Y), [a_A(Z), a_A(X)]_M]_M. \end{aligned}$$

Since $[\cdot, \cdot]_M$ satisfies the Jacobi identity, $\rho L = 0$ and so $L = 0$, because ρ is one-to-one. Now, we check the Leibniz rule for $[\cdot, \cdot]_K$. We have

$$\rho[X + K, fY + K]_K = [a_A(X), f a_A(Y)]_M = a_A(X)(f) a_A(Y) + f[a_A(X), a_A(Y)]_M.$$

Then

$$[X + K, fY + K]_K = a_A(X)(f) \rho^{-1}(a_A(Y)) + f \rho^{-1}[a_A(X), a_A(Y)]_M,$$

which is equivalent to

$$[X + K, fY + K]_K = \rho(X + K)(f)(Y + K) + f[X + K, Y + K]_M. \quad \square$$

4. LINEAR HOM-POISSON STRUCTURES

In this section, we introduce the notion of linear Hom-Poisson structure that is a generalization of the notion of linear Hom-Poisson structure introduced in [6]. Also, we show that there exists a one-to-one correspondence between linear Hom-Poisson structures and Hom-Lie algebroids.

Definition 4.1. Let $(A \rightarrow M, \varphi, \phi_A)$ be a Hom-bundle. A linear Hom-Poisson structure on a Hom-bundle $(A^* \rightarrow M, \varphi, \phi_A^\dagger)$ is a pair that consists of a bracket of functions

$$\{\cdot, \cdot\}_{A^*} : C^\infty(A^*) \times C^\infty(A^*) \rightarrow C^\infty(A^*)$$

and the map $(\phi_A^\dagger)^* : C^\infty(A^*) \rightarrow C^\infty(A^*)$ (a morphism of the function ring $C^\infty(A^*)$), such that the following conditions are satisfied:

- (i) $(C^\infty(A^*), \{\cdot, \cdot\}_{A^*}, (\phi_A^\dagger)^*)$ is a Hom-Lie algebra,
- (ii) $\{\psi_1, \psi_2 \psi_3\}_{A^*} = (\phi_A^\dagger)^*(\psi_2)\{\psi_1, \psi_3\}_{A^*} + \{\psi_1, \psi_2\}_{A^*}(\phi_A^\dagger)^*(\psi_3)$,
- (iii) $\{\cdot, \cdot\}_{A^*}$ is linear, that is, if ψ_1 and ψ_2 are linear functions on A^* then $\{\psi_1, \psi_2\}_{A^*}$ is a linear function.

There exists a one-to-one correspondence between the section space $\Gamma(A)$ and the space of linear functions on A^* . In fact, if $X \in \Gamma(A)$, then the corresponding linear function $\hat{X}$ is defined by

$$\hat{X}(\omega_m) = \phi_A^\dagger(\omega)(\phi_A(X))(\varphi^{-1}(m)), \quad (4.1)$$

where $\omega_m \in A_m^*$. We have

$$\begin{aligned} \hat{X}(\omega_\alpha(m)e^\alpha(m)) &= (\phi_A^\dagger(\omega_\alpha e^\alpha)(\phi_A(X)))(\varphi^{-1}(m)) \\ &= \varphi^*(\omega_\alpha)(\varphi^{-1}(m))(\phi_A^\dagger(e^\alpha)(\phi_A(X)))(\varphi^{-1}(m)) = \omega_\alpha(m)\hat{X}(e^\alpha(m)). \end{aligned}$$

Also

$$\begin{aligned} \widehat{fX}(\omega_m) &= (\phi_A^\dagger(\omega)(\phi_A(fX)))(\varphi^{-1}(m)) \\ &= (\phi_A^\dagger(\omega)(\varphi^*(f)\phi_A(X)))(\varphi^{-1}(m)) = \varphi^*(f)(\varphi^{-1}(m))\hat{X}(\omega_m) \\ &= f(m)\hat{X}(\omega_m) = f(\tau_{A^*}(\omega(m)))\hat{X}(\omega_m) = (f \circ \tau_{A^*})(\omega_m)\hat{X}(\omega_m). \end{aligned}$$

Therefore $\widehat{fX} = (f \circ \tau_{A^*})\hat{X}$. We consider $(\phi_A^\dagger)^* : C^\infty(A^*) \rightarrow C^\infty(A^*)$ given by

$$((\phi_A^\dagger)^*(F))(\omega_m) = F((\phi_A^\dagger)^{-1}(\omega)(\varphi(m))), \quad \forall F \in C^\infty(A^*), \quad (4.2)$$

which gives us

$$((\phi_A^\dagger)^*(\hat{X}))(\omega_m) = (\omega(\phi_A(X)))_m. \quad (4.3)$$

For any $f \in C^\infty(M)$, we conclude

$$\begin{aligned} (\phi_A^\dagger)^*(\widehat{fX})(\omega_m) &= \omega(\phi_A(fX))_m = (\varphi^*(f)\omega(\phi_A(X)))_m \\ &= \varphi^*(f)(m)(\phi_A^\dagger)^*(\hat{X})(\omega_m) = \varphi^*(f)(\tau_{A^*}(\omega_m))(\phi_A^\dagger)^*(\hat{X})(\omega_m), \end{aligned}$$

i.e.,

$$(\phi_A^\dagger)^*(\widehat{fX}) = (\varphi^*(f) \circ \tau_{A^*})(\phi_A^\dagger)^*(\hat{X}). \quad (4.4)$$

Proposition 4.2. *Let $(\{\cdot, \cdot\}_{A^*}, (\phi_A^\dagger)^*)$ be a linear Hom-Poisson structure on a Hom-bundle $(A^* \rightarrow M, \varphi, \phi_A^\dagger)$. If we consider $X \in \Gamma(A)$ and $f, g \in C^\infty(M)$, then*

- (i) $\{\hat{X}, \hat{f}\}_{A^*}$ is a basic function with respect to the projection τ_{A^*} ;
- (ii) $\{\hat{f}, \hat{g}\}_{A^*} = 0$.

Proof. Using (4.4) and property (ii) in Definition 4.1, we get

$$\{\hat{X}, (f \circ \tau_{A^*})\hat{Y}\}_{A^*} = (\varphi^*(f) \circ \tau_{A^*})\{\hat{X}, \hat{Y}\}_{A^*} + \{\hat{X}, (f \circ \tau_{A^*})\}_{A^*}(\phi_A^\dagger)^*(\hat{Y}). \quad (4.5)$$

Since $\{\hat{X}, (f \circ \tau_{A^*})\hat{Y}\}_{A^*}$ and $\{\hat{X}, \hat{Y}\}_{A^*}$ are $\mathbb{R}$ -linear functions, we conclude that $\{\hat{X}, (f \circ \tau_{A^*})\}_{A^*}$ is a basic function with respect to τ_{A^*} , which implies that (i) holds.

Using (i) and Definition 4.1, we have

$$\{(f \circ \tau_{A^*}), \hat{Y}(g \circ \tau_{A^*})\}_{A^*} = (\phi_A^\dagger)^*(\hat{Y})\{(f \circ \tau_{A^*}), (g \circ \tau_{A^*})\}_{A^*} + \{(f \circ \tau_{A^*}), \hat{Y}\}_{A^*}\widehat{\varphi^*(g)},$$

which is a basic function with respect to τ_{A^*} . Therefore we deduce that $\{(f \circ \tau_{A^*}), (g \circ \tau_{A^*})\}_{A^*} = 0$. This proves (ii). $\square$

Proposition 4.3. *If we consider $\phi_A(\Psi_\lambda) = \phi_\lambda^\mu \Psi_\mu$, then*

$$\widehat{\Psi}_\alpha = \xi_\alpha, \quad (\phi_A^\dagger)^*(\xi_\alpha) = (\phi_\alpha^\beta \circ \tau_{A^*})\xi_\beta, \quad ((\phi_A^\dagger)^*)^{-1}\widehat{\phi_A(\Psi_\alpha)} = \widehat{\Psi}_\alpha.$$

Proof. Using (4.1), we obtain

$$\widehat{\Psi}_\alpha((\Psi^\beta)_m) = ((\phi_A^\dagger)(\Psi^\beta)\phi_A(\Psi_\alpha))_{\varphi^{-1}(m)} = (\varphi^*((\Psi^\beta)(\Psi_\alpha)))_{\varphi^{-1}(m)} = \delta_\alpha^\beta(m). \quad (4.6)$$

Thus we get

$$\widehat{\Psi}_\alpha(\omega_m) = \widehat{\Psi}_\alpha(\omega_\beta(m)(\Psi^\beta)_m) = \omega_\beta(m)\delta_\alpha^\beta = \omega_\alpha(m) = \xi_\alpha(\omega_m).$$

We have

$$\begin{aligned} (\phi_A^\dagger)^*(\xi_\alpha)(\omega_m) &= \xi_\alpha((\phi_A^\dagger)^{-1}(\omega)_{\varphi(m)}) = \xi_\alpha((\phi_A^\dagger)^{-1}(\omega_\beta \Psi^\beta)_{\varphi(m)}) \\ &= \xi_\alpha((\varphi^*)^{-1}(\omega_\beta)_{\varphi(m)}(\phi_A^\dagger)^{-1}(\Psi^\beta)_{\varphi(m)}). \end{aligned} \quad (4.7)$$

On the other hand,

$$\begin{aligned} (\phi_A^\dagger)^{-1}(\Psi^\beta)(\Psi_\alpha) &= (\varphi^*)^{-1}(\Psi^\beta(\phi_A(\Psi_\alpha))) = (\varphi^*)^{-1}(\Psi^\beta(\phi_\alpha^\gamma \Psi_\gamma)) \\ &= (\phi_\alpha^\gamma \circ \varphi^{-1})\delta_\gamma^\beta = (\phi_\alpha^\beta \circ \varphi^{-1}), \end{aligned}$$

which gives

$$(\phi_A^\dagger)^{-1}(\Psi^\beta) = (\phi_\alpha^\beta \circ \varphi^{-1})\Psi^\alpha.$$

Setting the above equation in (4.7) implies

$$(\phi_A^\dagger)^*(\xi_\alpha)(\omega_m) = \xi_\alpha(\omega_\beta(m)\phi_\gamma^\beta(m)(e^\gamma)_{\varphi(m)}) = \phi_\alpha^\beta(m)\omega_\beta(m) = \phi_\alpha^\beta(m)\xi_\beta(\omega_m).$$

The third equation is obtained as follows:

$$\begin{aligned} \widehat{\phi_A(\Psi_\alpha)}(\omega_m) &= (\phi_A^\dagger(\omega)(\phi_A^2(\Psi_\alpha)))_{\varphi^{-1}(m)} = (\varphi^*(\omega(\phi_A(\Psi_\alpha))))_{\varphi^{-1}(m)} = \omega(\phi_A(\Psi_\alpha))_m \\ &= (\phi_A^\dagger)^*(\widehat{\Psi}_\alpha)(\omega_m). \end{aligned} \quad \square$$

Corollary 4.4. *We have*

$$(\phi_A^\dagger)^*(\widehat{\Psi}_\alpha) = (\phi_\alpha^\beta \circ \tau_{A^*})\widehat{\Psi}_\beta.$$

Theorem 4.5. *Let $(A, \varphi, \phi_A, [\cdot, \cdot]_A, a_A)$ be a Hom-Lie algebroid. Then there exists a one-to-one correspondence between the Hom-Lie algebroid and its linear Hom-Poisson structure. Also, if $(\{\cdot, \cdot\}_{A^*}, (\phi_A^\dagger)^*)$ is a linear Hom-Poisson structure on a Hom-bundle $(A^* \rightarrow M, \varphi, \phi_A^\dagger)$, then the corresponding Hom-Lie algebroid structure $([\cdot, \cdot]_A, a_A)$ on A is characterized by the following conditions:*

$$\begin{aligned} \widehat{[X, Y]}_A &= -\{\hat{X}, \hat{Y}\}_{A^*}, \\ a_A(\phi_A(X))(f) \circ \tau_{A^*} &= \{(f \circ \tau_{A^*}), \hat{X}\}_{A^*} = \varphi^*(a_A(X)((\varphi^*)^{-1}(f))) \circ \tau_{A^*}. \end{aligned} \quad (4.8)$$

Proof. We consider $\{\cdot, \cdot\}_{A^*}$ as a linear Hom-Poisson structure on A^* . Since $\{\cdot, \cdot\}_{A^*}$ is an $\mathbb{R}$ -bilinear skew-symmetric bracket, we conclude that $[\cdot, \cdot]_A$ is also $\mathbb{R}$ -bilinear skew-symmetric. As $\{\cdot, \cdot\}_{A^*}$ satisfies the Hom-Leibniz rule, we conclude that $a_A(X)$ is a vector field on M , for any $X \in \Gamma(A)$. Using (4.8) and the Hom-Leibniz rule we get

$$\begin{aligned} a_A(\phi_A(gX))(f) \circ \tau_{A^*} &= \{(f \circ \tau_{A^*}), (g \circ \tau_{A^*})\hat{X}\}_{A^*} \\ &= (\phi_A^\dagger)^*((g \circ \tau_{A^*}))\{(f \circ \tau_{A^*}), \hat{X}\}_{A^*} \\ &= (\phi_A^\dagger)^*((g \circ \tau_{A^*}))a_A(\phi_A(X))(f) \circ \tau_{A^*}, \end{aligned}$$

which gives us

$$a_A(gX)(f) = ga_A(X)(f),$$

that is $a_A : \Gamma(\tau_A) \rightarrow \Gamma(\varphi^!TM)$ is a morphism of $C^\infty(M)$ -modules. Also using (4.5) and (4.8) we obtain

$$\begin{aligned} \widehat{[X, fY]}_A &= -\{\hat{X}, \widehat{fY}\}_{A^*} \\ &= -(\phi_A^\dagger)^*((f \circ \tau_{A^*}))\{\hat{X}, \hat{Y}\}_{A^*} + a_A(\phi_A(X))(f)(\phi_A^\dagger)^*(Y) \\ &= (\varphi^*(f) \circ \tau_{A^*})\widehat{[X, Y]}_A + a_A(\phi_A(X))(f)(\widehat{\phi_A(Y)}). \end{aligned}$$

Therefore we have

$$[X, fY]_A = \varphi^*(f)[X, Y]_A + a_A(\phi_A(X))(f)\phi_A(Y).$$

Since $\{\cdot, \cdot\}_{A^*}$ satisfies the Hom-Jacobi identity, it is easy to see that the Hom-Jacobi identity holds for $[\cdot, \cdot]_A$. Therefore $([\cdot, \cdot]_A, a_A)$ is a Hom-Lie algebroid structure on $(\tau_A : A \rightarrow M, \varphi, \phi_A)$.

Conversely, let $([\cdot, \cdot]_A, a_A)$ be a Hom-Lie algebroid structure on a Hom-bundle $(\tau_A : A \rightarrow M, \varphi, \phi_A)$. If we consider $x \in M$, then there exists an open subset U of M and a unique linear Poisson structure on the Hom-bundle $(\tau_{A^*}^{-1}(U) \rightarrow U, \varphi, \phi_A^\dagger)$ such that

$$\begin{aligned} \{\hat{X}, \hat{Y}\}_{\tau_{A^*}^{-1}(U)} &= -\widehat{[X, Y]}_{\tau_A^{-1}(U)}, \\ \{(f \circ \tau_{A^*}), \hat{X}\}_{\tau_{A^*}^{-1}(U)} &= a_{\tau_A^{-1}(U)}(\phi_A(X))(f), \\ \{(f \circ \tau_{A^*}), (g \circ \tau_{A^*})\}_{\tau_{A^*}^{-1}(U)} &= 0, \end{aligned}$$

for any $X, Y \in \Gamma(\tau_{\tau_A^{-1}(U)})$ and $f, g \in C^\infty(M)$. As $[\cdot, \cdot]_A$ satisfies the Hom-Jacobi identity, then the Hom-Jacobi identity holds for $\{\cdot, \cdot\}_{A^*}$. So $([\cdot, \cdot]_{\tau_A^{-1}(U)}, a_{\tau_A^{-1}(U)})$ is a Hom-Lie algebroid structure on Hom-bundle $(\tau_{\tau_A^{-1}U} : \tau_A^{-1}(U) \rightarrow M, \varphi, \phi_{\tau_A^{-1}(U)})$. Therefore, there exists a unique linear Hom-Poisson structure $(\{\cdot, \cdot\}_{A^*}, (\phi_A^\dagger)^*)$ on A^* that satisfies (4.8). $\square$

Let x^i denote the local coordinates over a neighborhood U of M . If e_α is a basis of sections of the Hom-bundle $(\tau_A^{-1}(U) \rightarrow U, \varphi, \phi_A)$, then (x^i, ξ_α) are the corresponding local coordinates on the Hom-bundle $(A^* \rightarrow M, \varphi, \phi_A^\dagger)$. By Proposition 4.2 one can easily see that

$$\{\xi_\alpha, \xi_\beta\}_{A^*} = -(C_{\alpha\beta}^\gamma \circ \tau_{A^*})\xi_\gamma, \quad \{x^i, \xi_\alpha\}_{A^*} = (\phi_\alpha^\beta a_\beta^i) \circ \tau_{A^*}, \quad \{x^i, x^j\}_{A^*} = 0,$$

where $a_\beta^i, C_{\alpha\beta}^\mu$ are smooth maps on U .

Proposition 4.6. *Let $(\{\cdot, \cdot\}_{A^*}, (\phi_A^\dagger)^*)$ be a linear Hom-Poisson structure on the Hom-bundle A^* and $([\cdot, \cdot]_A, a_A)$ be the corresponding Hom-Lie algebroid structure on the Hom-bundle $(\tau_A : A \rightarrow M, \varphi, \phi_A)$. If x^i are local coordinates on an open subset U of M and e_α is a basis of sections of the Hom-bundle $\tau_A^{-1}(U) \rightarrow U$, then*

$$[\Psi_\alpha, \Psi_\beta]_A = C_{\alpha\beta}^\lambda \Psi_\lambda, \quad a_A(\phi_A(\Psi_\alpha)) = \phi_\alpha^\beta a_\beta^i \frac{\partial}{\partial x^i},$$

where $C_{\alpha\beta}^\lambda, \phi_\alpha^\beta a_\beta^i$ are called the local structure functions of the Hom-Lie algebroid structure $([\cdot, \cdot]_A, a_A)$ on the Hom-bundle A .

Proof. We have

$$[\widehat{\Psi_\alpha}, \widehat{\Psi_\beta}]_A = -\{\widehat{\Psi_\alpha}, \widehat{\Psi_\beta}\}_{A^*} = (C_{\alpha\beta}^\gamma \circ \tau_{A^*})\widehat{\Psi_\gamma}.$$

Contracting the above equation by Ψ^ν and using (4.6) we obtain

$$\Psi^\nu([\Psi_\alpha, \Psi_\beta]_A) = C_{\alpha\beta}^\nu.$$

Also we get

$$a_A(\phi_A(\Psi_\alpha))(f) \circ \tau_{A^*} = a_A(\phi_A(\Psi_\alpha))(f) \circ \tau_{A^*} = \{f \circ \tau_{A^*}, \widehat{\Psi_\alpha}\}_{A^*} = (\phi_\alpha^\beta a_\beta^i) \circ \tau_{A^*}.$$

$\square$

Now we shall compute the Hamiltonian vector fields X_{x^i}, X_{ξ_α} on A^* . Using the definition we get

$$X_{x^i}(x^j) = \{x^i, x^j\} = 0, \quad X_{x^i}(\xi_\alpha) = \{x^i, \xi_\alpha\} = (a_\beta^i \phi_\alpha^\beta) \circ \tau_{A^*}.$$

So, one can get

$$X_{x^i} = (a_\beta^i \phi_\alpha^\beta) \circ \tau_{A^*} \frac{\partial}{\partial \xi_\alpha}.$$

For X_{ξ_α} we have

$$X_{p_\alpha}(x^i) = \{\xi_\alpha, x^i\} = -(a_\beta^i \phi_\alpha^\beta) \circ \tau_{A^*}, \quad X_{p_\alpha}(\xi_\beta) = \{\xi_\alpha, \xi_\beta\} = -(C_{\alpha\beta}^\gamma \circ \tau_{A^*})\xi_\gamma.$$

So

$$X_{\xi_\alpha} = -(a_\beta^i \phi_\alpha^\beta) \circ \tau_{A^*} \frac{\partial}{\partial x^i} - (C_{\alpha\beta}^\gamma \circ \tau_{A^*}) \xi_\gamma \frac{\partial}{\partial \xi_\beta}.$$

Let $v = v^\alpha \Psi_\alpha \in \Gamma(A)$ be a local section and define $f_v : A^* \rightarrow \mathbb{R}$ in the usual way

$$f_v(x, \xi) = v^\alpha(x) \xi_\alpha(x, \xi) \quad \text{or} \quad f_v = (v^\alpha \circ \tau_{A^*}) \xi_\alpha,$$

where $\xi = \sum_\alpha \xi_\alpha \Psi^{*\alpha}$. Now, we compute X_{f_v} as follows:

$$X_{f_v}(x^i) = -X_{x^i}(f_v) = -(a_\beta^i \phi_\alpha^\beta \varphi^*(v^\alpha)) \circ \tau_{A^*},$$

and

$$\begin{aligned} X_{f_v}(\xi_\alpha) &= \{f_v, p_\alpha\} = \{(v^\lambda \circ \tau_{A^*}) \xi_\lambda, \xi_\alpha\} \\ &= (\phi_\lambda^\gamma \circ \tau_{A^*}) \xi_\gamma \{v^\lambda \circ \tau_{A^*}, \xi_\alpha\} + (\varphi^*(v^\lambda) \circ \tau_{A^*}) \{\xi_\lambda, \xi_\alpha\} \\ &= ((\phi_\lambda^\gamma a_\beta^i \phi_\alpha^\beta \frac{\partial v^\lambda}{\partial x^i}) \circ \tau_{A^*}) \xi_\gamma - ((\varphi^*(v^\lambda) C_{\lambda\alpha}^\gamma) \circ \tau_{A^*}) \xi_\gamma. \end{aligned}$$

So we have

$$\begin{aligned} X_{f_v} &= -(a_\beta^i \phi_\alpha^\beta \varphi^*(v^\alpha)) \circ \tau_{A^*} \frac{\partial}{\partial x^i} \\ &\quad + \left(\left(\phi_\lambda^\gamma a_\beta^i \phi_\alpha^\beta \frac{\partial v^\lambda}{\partial x^i} \right) \circ \tau_{A^*} \xi_\gamma - (\varphi^*(v^\lambda) C_{\lambda\alpha}^\gamma) \circ \tau_{A^*} \xi_\gamma \right) \frac{\partial}{\partial \xi_\alpha}. \end{aligned}$$

Since $(\tau_{A^*})_*(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}$ and $(\tau_{A^*})_*(\frac{\partial}{\partial \xi_\alpha}) = 0$, from the above equation we get

$$(\tau_{A^*})_*(X_{f_v}) = -(a_\beta^i \phi_\alpha^\beta \varphi^*(v^\alpha)) \circ \tau_{A^*} \frac{\partial}{\partial x^i} = -a_A(\phi_A(v)).$$

Let $v = v^\alpha \Psi_\alpha$ and $w = w^\beta \Psi_\beta$. Then one can compute $\{f_v, f_w\}$ as follows:

$$\begin{aligned} \{f_v, f_w\} &= \{v^\alpha \xi_\alpha, w^\beta \xi_\beta\} \\ &= (\phi_\alpha^\lambda \circ \tau_{A^*}) \xi_\lambda \{v^\alpha, w^\beta \xi_\beta\} + \varphi^*(v^\alpha) \circ \tau_{A^*} \{\xi_\alpha, w^\beta \xi_\beta\} \\ &= \left((\phi_\alpha^\lambda \varphi^*(w^\beta)) \circ \tau_{A^*} \right) \xi_\lambda \{v^\alpha, \xi_\beta\} + ((\phi_\alpha^\lambda \phi_\beta^\gamma) \circ \tau_{A^*}) \xi_\lambda \xi_\gamma \{v^\alpha, w^\beta\} \\ &\quad + (\varphi^*(v^\alpha) \varphi^*(w^\beta)) \circ \tau_{A^*} \{\xi_\alpha, \xi_\beta\} + ((\varphi^*(v^\alpha) \phi_\beta^\gamma) \circ \tau_{A^*}) \xi_\gamma \{\psi_\alpha, w^\beta\} \\ &= ((\phi_\alpha^\lambda a_\mu^i \phi_\beta^\mu \varphi^*(w^\beta) \frac{\partial v^\alpha}{\partial x^i}) \circ \tau_{A^*}) \xi_\lambda - ((\varphi^*(v^\alpha) \varphi^*(w^\beta) C_{\alpha\beta}^\gamma) \circ \tau_{A^*}) \xi_\gamma \\ &\quad - \left((\varphi^*(v^\alpha) \phi_\beta^\gamma a_\mu^i \phi_\alpha^\mu \frac{\partial w^\beta}{\partial x^i}) \circ \tau_{A^*} \right) \xi_\gamma. \end{aligned} \tag{4.9}$$

It is a straightforward computation to see that

$$\begin{aligned} [v, w] &= - \left((\phi_\alpha^\lambda a_s^i \phi_\beta^s \varphi^*(w^\beta) \frac{\partial v^\alpha}{\partial x^i} - \varphi^*(v^\alpha) \varphi^*(w^\beta) C_{\alpha\beta}^\lambda \right. \\ &\quad \left. - \varphi^*(v^\alpha) \phi_\beta^\lambda a_u^i \phi_\alpha^u \frac{\partial w^\beta}{\partial x^i} \right) \circ \tau_{A^*} \Psi_\lambda, \end{aligned}$$

and so

$$f_{[v,w]} = - \left((\phi_\alpha^\lambda a_s^i \phi_\beta^s \varphi^*(w^\beta) \frac{\partial v^\alpha}{\partial x^i} - \varphi^*(v^\alpha) \varphi^*(w^\beta) C_{\alpha\beta}^\lambda - \varphi^*(v^\alpha) \phi_\beta^\lambda a_u^i \phi_\alpha^u \frac{\partial w^\beta}{\partial x^i}) \circ \tau_{A^*} \right) \xi_\lambda. \quad (4.10)$$

(4.9) and (4.10) imply that $-f_{[v,w]} = \{f_v, f_w\} = \{-f_v, -f_w\}$. Using the above calculations we have the following theorem.

Theorem 4.7. *The assignment $v \mapsto -f_v$ defines a Lie algebra homomorphism $(\Gamma(A), [\cdot, \cdot]_A) \rightarrow (C^\infty(A^*), \{\cdot, \cdot\}_a)$. Moreover, if X_{f_v} denotes the Hamiltonian vector field associated with f_v , then X_{f_v} is τ_{A^*} -related to $-a_A(\phi_A(v))$.*

REFERENCES

- [1] L. Cai, J. Liu and Y. Sheng, *Hom-Lie algebroids, Hom-Lie bialgebroids and Hom-Courant algebroids*, J. Geom. Phys. **121** (2017), 15–32. MR 3705378.
- [2] J.-P. Dufour and N. T. Zung, *Poisson structures and their normal forms*, Progress in Mathematics, vol. 242, Birkhäuser Verlag, Basel, 2005. MR 2178041.
- [3] R.L. Fernandes, *Lie algebroids, holonomy and characteristic classes*, Adv. Math. **170** (2002), no. 1, 119–179. MR 1929305.
- [4] C. Laurent-Gengoux and J. Teles, *Hom-Lie algebroids*, J. Geom. Phys. **68** (2013), 69–75. MR 3035115.
- [5] J. Liu, Y. Sheng, C. Bai and Z. Chen, *Left-symmetric algebroids*, Math. Nachr. **289** (2016), no. 14–15, 1893–1908. MR 3563909.
- [6] M. de León, J. C. Marrero, D. Martín de Diego, *Linear almost Poisson structures and Hamilton–Jacobi equation. Applications to nonholonomic mechanics*, J. Geom. Mech. **2** (2010), no. 2, 159–198. MR 2660714.
- [7] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Math. Soc. Lecture notes series, vol. 213, Cambridge University Press, Cambridge, 2005. MR 2157566.
- [8] S. Merati, M. R. Farhangdoost, *Representation up to homotopy of Hom-Lie algebroids*, Int. J. Geom. Methods Mod. Phys. **15** (2018), no. 5, 1850074, 13 pp. MR 3786506.
- [9] E. Peyghan and L. Nourmohammadifar, *Para-Kähler hom-Lie algebras*, J. Algebra Appl. **18** (2019), no. 3, 1950044, 24 pp. MR 3924822.
- [10] E. Peyghan and L. Nourmohammadifar, *Hom-left symmetric algebroids*, Asian-Eur. J. Math. **11** (2018), no. 2, 1850027, 24 pp. MR 3786367.
- [11] Y. Sheng and C. Bai, *A new approach to hom-Lie bialgebras*, J. Algebra **399** (2014), 232–250. MR 3144586.
- [12] S. Vacaru, *Clifford-Finsler algebroids and nonholonomic Einstein-Dirac structures*, J. Math. Phys. **47** (2006), no. 9, 093504, 20 pp. MR 2263658.
- [13] S. Vacaru, *Finsler and Lagrange geometries in Einstein and string gravity*, Int. J. Geom. Methods Mod. Phys. **5** (2008), no. 4, 473–511. MR 2428807.
- [14] L. Bubuianu and S. I. Vacaru, *Dynamical equations and Lagrange–Ricci flow evolution on prolongation Lie algebroids*, Canad. J. Phys. **97** (2019), no. 2, 145–154.

- [15] S. Vacaru, *Nonholonomic algebroids, Finsler geometry, and Lagrange-Hamilton spaces*, Math. Sci. (Springer) **6** (2012), Art. 18, 33 pp. MR 3064449.
- [16] S. Vacaru, *Almost Kähler Ricci flows and Einstein and Lagrange-Finsler structures on Lie algebroids*, Mediterr. J. Math. **12** (2015), no. 4, 1397–1427. MR 3416867.

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