CONFORMAL AND KILLING VECTOR FIELDS ON REAL
SUBMANIFOLDS OF THE CANONICAL COMPLEX
SPACE FORM $\mathbb{C}^m$

HANAN ALOHALI, HAILA ALODAN, AND SHARIEF DESHMUKH

Abstract. In this paper, we find a conformal vector field as well as a Killing vector field on a compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$. In particular, using immersion $\psi : M \to \mathbb{C}^m$ of a compact real submanifold $M$ and the complex structure $J$ of the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$, we find conditions under which the tangential component of $J\psi$ is a conformal vector field as well as conditions under which it is a Killing vector field. Finally, we obtain a characterization of $n$-spheres in the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$.

1. Introduction

Conformal vector fields and Killing vector fields play a vital role in geometry of a Riemannian manifold $(M, g)$ as well as in physics (cf. [13]). In geometry, these vector fields are used in characterizing spheres among compact or complete Riemannian manifolds (cf. [4]–[12]). A Killing vector field is said to be nontrivial if it is not parallel. The existence of a nontrivial Killing vector field on a compact Riemannian manifold constrains its geometry as well as its topology: it does not allow the Riemannian manifold $(M, g)$ to have nonpositive Ricci curvature and if $(M, g)$ is positively curved, its fundamental group has a cyclic subgroup (cf. [2]). In most of the cases, a conformal vector field or a Killing vector field on a Riemannian manifold $(M, g)$ is derived through treating it as a submanifold of a Euclidean space. For example, a unit sphere $S^n$ admits a conformal vector field that is tangential component of a constant vector field on the ambient Euclidean space $\mathbb{R}^{n+1}$. Similarly, an odd dimensional unit sphere $S^{2m-1}$ with unit normal vector field $N$ as a hypersurface of the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$ admits a Killing vector field $\xi = -JN$, where $J$ is the canonical complex structure on $\mathbb{C}^m$. Therefore it is an interesting question to find a conformal vector field as well as a Killing vector field on a real submanifold of a canonical complex space form.

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A similar study is taken up in [1] for submanifolds in a Euclidean space. Given an $n$-dimensional real submanifold $(M, g)$ of the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$ with immersion $\psi : M \to \mathbb{C}^m$, we treat $\psi$ as the position vector field of points on $M$ in $\mathbb{C}^m$, and consequently we have the expression $J\psi = v + N$, where $v$ is the tangential component and $N$ is the normal component of $J\psi$ on $M$. This gives a globally defined vector field $v$ on the real submanifold $M$.

In this paper, we study the above mentioned question for real submanifolds of the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$ and obtain conditions under which the vector field $v$ is a conformal vector field (Theorems 3.1, 3.2) or a Killing vector field (Theorems 4.1, 4.3). We also use this vector field $v$ to find a characterization of a sphere $S^n(c)$ of constant curvature $c$ in the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$ (cf. Theorem 5.1). It is worth noting that the existence of the Killing vector field $v$ not only restricts the geometry and topology of the real submanifold $M$ but also has an influence on the dimensions of both the real submanifold and the ambient canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$ (cf. Corollary 4.2). Finally, at the end of this paper, we give an example of a real submanifold on which $v$ is a nontrivial conformal vector field (that is, $v$ is not Killing) and another example of a real submanifold on which $v$ is nontrivial Killing vector field (that is, non-parallel).

2. Preliminaries

Let $M$ be an immersed $n$-dimensional real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)$, $J$ and $\langle \cdot, \cdot \rangle$ being the canonical complex structure and the Euclidean metric on $\mathbb{C}^m$ respectively. We denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on $M$, by $\Gamma(v)$ the space of sections of the normal bundle $v$ of $M$, and by $\nabla$ and $\nabla$ the Riemannian connections on $\mathbb{C}^m$ and on $M$ respectively. Then we have the following Gauss and Weingarten equations for the real submanifold $M$ (cf. [3]):

$$
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X N = -A_N X + \nabla_X^\perp N,
$$

(2.1)

$X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(v)$, where $h$ is the second fundamental form, $A_N$ is the Weingarten map with respect to the normal $N \in \Gamma(v)$, which is related to the second fundamental form $h$ by

$$
g(A_N X, Y) = \langle h(X, Y), N \rangle, \quad X, Y \in \mathfrak{X}(M),
$$

and $\nabla^\perp$ is the connection in the normal bundle $v$. The curvature tensor field $R$ of the real submanifold $M$ is given by

$$
R(X, Y) Z = A_{h(Y, Z)} X - A_{h(X, Z)} Y, \quad X, Y, Z \in \mathfrak{X}(M).
$$

The Ricci tensor field of the real submanifold $M$ is given by

$$
\text{Ric}(X, Y) = n g(h(X, Y), H) - \sum_{i=1}^n g(h(X, e_i), h(Y, e_i)),
$$
where \( \{e_1, \ldots, e_n\} \) is a local orthonormal frame on \( M \) and
\[
H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)
\]
is the mean curvature vector field of the real submanifold \( M \).

The Ricci operator \( Q \) is a symmetric operator defined by
\[
\text{Ric}(X,Y) = g(Q(X), Y), \quad X,Y \in \mathcal{X}(M).
\]

Let \( \psi : M \to \mathbb{C}^m \) be the immersion of the real submanifold \( M \). Then we set
\[
J\psi = v + N,
\]
where \( v \) is the tangential component and \( N \) is the normal component of \( J\psi \).

Now, define skew symmetric tensors \( \varphi \) and \( G \), and the tensors \( \Psi \) and \( F \) as follows:
\[
JX = \varphi X + FX, \quad X \in \mathcal{X}(M),
JN = \Psi N + GN, \quad N \in \Gamma(v),
\]
where
\[
\varphi : \mathcal{X}(M) \to \mathcal{X}(M), \quad F : \mathcal{X}(M) \to \Gamma(v),
\Psi : \Gamma(v) \to \mathcal{X}(M), \quad G : \Gamma(v) \to \Gamma(v),
\]
that is, \( \varphi X \), \( \Psi N \) are the tangential components of \( JX \) and \( JN \) respectively and \( FX \), \( GN \) are the normal components of \( JX \) and \( JN \) respectively.

Define a symmetric tensor \( C \) of type \((1,1)\) by \( C(X) = A_N X \), \( X \in \mathcal{X}(M) \), and a smooth function \( E : M \to \mathbb{R} \) on the real submanifold \( M \) by \( E = \langle H, N \rangle \). Then we have
\[
\text{tr} C = nE.
\]

**Lemma 2.1.** Let \( M \) be an \( n \)-dimensional real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)\). Then
\[
\nabla_X v = \varphi X + C(X) \quad \text{and} \quad \nabla^\perp_X N = FX - h(X, v).
\]

**Proof.** As \( J \) is a complex structure, we have
\[
\nabla_X J\psi = J\nabla_X \psi,
\]
which in view of equation \ref{2.1} gives
\[
\nabla_X v + h(X, v) + \nabla^\perp_X N - C(X) = \varphi X + FX, \quad X \in \mathcal{X}(M).
\]
Equating the tangential and the normal components we get the result. \( \square \)

**Lemma 2.2.** Let \( M \) be an \( n \)-dimensional real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)\). Then for \( X, Y \in \mathcal{X}(M) \) and \( N \in \Gamma(v) \), we have
\[
(\nabla \varphi)(X,Y) = A_{F(Y)}X + \Psi(h(X,Y)), \quad \text{where} \quad (\nabla \varphi)(X,Y) = \nabla_X \varphi Y - \varphi \nabla_X Y
\]
\[
(D_X F)Y = G(h(X,Y)) - h(X, \varphi Y), \quad \text{where} \quad (D_X F)Y = \nabla_X FY - F(\nabla_X Y)
\]
\[
(D_X \Psi)N = A_{G(N)}X - \varphi A_N X, \quad \text{where} \quad (D_X \Psi)N = \nabla_X \Psi(N) - \Psi(\nabla^\perp_X N)
\]
\[
(\nabla^\perp_X G)N = F(A_N X) - h(X, \Psi(N)), \quad \text{where} \quad (\nabla^\perp_X G)N = \nabla^\perp_X GN - G(\nabla^\perp_X N).
\]
Proof. As $J$ is parallel, we have
\[ \nabla_X (\varphi Y + F(Y)) = J (\nabla_X Y + h(X,Y)), \]
which in view of equation (2.1) takes the form
\[ (\nabla \varphi) (X,Y) + (D_X F) Y = A_{F(Y)} X + \Psi (h(X,Y)) + G(h(X,Y)) - h(X, \varphi Y), \]
which on equating the tangential and the normal components gives the first two relations. Similarly, on using \( (\nabla_X J) N = 0 \), we get the remaining two.

Using Lemma 2.1, we find the divergence of the vector field $v$ as $\text{div} \, v = nE$ and consequently, we have the following:

**Lemma 2.3.** Let $M$ be an $n$-dimensional compact real submanifold of the canonical complex space form $\langle \mathbb{C}^m, J, \langle \cdot, \cdot \rangle \rangle$. Then
\[ \int_M E \, dV = 0. \]

The following lemma is an immediate consequence of Lemma 2.1.

**Lemma 2.4.** Let $M$ be an $n$-dimensional real submanifold of the canonical complex space form $\langle \mathbb{C}^m, J, \langle \cdot, \cdot \rangle \rangle$. Then the tensor $C$ satisfies
\[ (i) \quad (\nabla C)(X,Y) - (\nabla C)(Y,X) = R(X,Y)v + (\nabla \varphi)(Y,X) - (\nabla \varphi)(X,Y), \]
\[ (ii) \quad \sum_{i=1}^n (\nabla C)(e_i, e_i) = n\text{div} \, v + \sum_{i=1}^n (\nabla \varphi)(e_i, e_i), \]
where $\langle \nabla C \rangle (X,Y) = \nabla_X C(Y) - C(\nabla_X Y)$, $X,Y \in \mathfrak{X}(M)$, and \( \{e_1, \ldots, e_n\} \) is a local orthonormal frame of $M$.

**Lemma 2.5.** Let $M$ be an $n$-dimensional real submanifold of the canonical complex space form $\langle \mathbb{C}^m, J, \langle \cdot, \cdot \rangle \rangle$. Then the skew symmetric tensor $\varphi$ satisfies
\[ (i) \quad (\nabla \varphi)(X,Y) - (\nabla \varphi)(Y,X) = A_{F_Y} X - A_{F_X} Y, \]
\[ (ii) \quad \sum_{i=1}^n (\nabla \varphi)(e_i, e_i) = n\Psi(H) + \sum_{i=1}^n A_{Fe_i} e_i, \]
where $X,Y \in \mathfrak{X}(M)$ and \( \{e_1, \ldots, e_n\} \) is a local orthonormal frame of $M$.

**Proof.** (i) Using Lemma 2.2, we get
\[ (\nabla \varphi)(X,Y) - (\nabla \varphi)(Y,X) = A_{F_Y} X + \Psi(h(X,Y)) - A_{F_X} Y - \Psi(h(Y,X)) \]
\[ = A_{F_Y} X - A_{F_X} Y, \quad X,Y \in \mathfrak{X}(M). \]

(ii) As $\text{tr} \, \varphi = 0$, we have
\[ \sum_{i=1}^n g(\nabla \varphi(X,e_i), e_i) = 0, \]
which gives
\[ \sum_{i=1}^n \{ g(\nabla \varphi(e_i, X), e_i) + g(A_{Fe_i} X, e_i) - g(A_{FXe_i}, e_i) \} = 0, \]
that is,
\[ \sum_{i=1}^n \{ g(-\nabla \varphi(e_i, X) + A_{Fe_i} e_i, X) + g(n\Psi(H), X) \} = 0. \]
Hence,
\[ \sum_{i=1}^{n} (\nabla \varphi) (e_i, e_i) = n \Psi (H) + \sum_{i=1}^{n} A_{Fe_i} e_i. \]

\[ \nabla \varphi \]

**Lemma 2.6.** Let \( M \) be an \( n \)-dimensional compact real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot , \cdot \rangle)\). Then
\[ \int_{M} \left( \text{Ric} (v,v) + \| C \|^2 - \| \varphi \|^2 - n^2 E^2 \right) dV = 0. \]

**Proof.** Using Lemmas 2.4 and 2.5, we get
\[ \text{div} \varphi v = -\sum_{i=1}^{n} g (A_{Fe_i} e_i, v) - ng (\Psi (H), v) - \| \varphi \|^2, \tag{2.2} \]
\[ \text{div} C v = \text{Ric} (v,v) + n v (E) + ng (\Psi (H), v) + \| C \|^2 + \sum_{i=1}^{n} g (A_{Fe_i} e_i, v), \]
and
\[ \text{div} E v = v (E) + n E^2. \tag{2.3} \]
Using these equations, we conclude that
\[ \text{div} C v = \text{Ric} (v,v) + n \text{div} E v - n^2 E^2 - \text{div} \varphi v - \| \varphi \|^2 + \| C \|^2, \]
which on integration gives the result. \( \square \)

**Lemma 2.7.** Let \( M \) be an \( n \)-dimensional compact real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot , \cdot \rangle)\). If \( v \) satisfies \( \triangle v = -\lambda v \) for a constant \( \lambda > 0 \), where \( \triangle \) is the Laplace operator acting on smooth vector fields on \( M \), then
\[ \int_{M} \left\{ \text{Ric} (v,v) + \lambda \| v \|^2 - 2 \| \varphi \|^2 - n^2 E^2 \right\} dV = 0. \]

**Proof.** Using the definition of the operator \( C \) and Lemma 2.1, we have
\[ (\nabla C) (X,Y) = \nabla_X CY - C \nabla_X Y \]
\[ = \nabla_X (\nabla_Y v - \varphi Y) - \nabla_X \varphi v + \varphi \nabla_X Y \]
\[ = \nabla_X \nabla_Y v - \nabla_X \varphi v - (\nabla \varphi) (X,Y), \quad X,Y \in \mathfrak{X}(M). \]
Taking a local orthonormal frame \( \{ e_1, \ldots, e_n \} \), the above equation leads to
\[ \sum_{i=1}^{n} (\nabla C) (e_i, e_i) = \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} v - \nabla \nabla_{e_i} e_i, v) - \sum_{i=1}^{n} (\nabla \varphi) (e_i, e_i) \]
\[ = \triangle v - \sum_{i=1}^{n} (\nabla \varphi) (e_i, e_i) \]
\[ = -\lambda v - \sum_{i=1}^{n} (\nabla \varphi) (e_i, e_i), \]
where we used the definition of the Laplace operator acting on smooth vector fields.
Now, using Lemma 2.4 (ii) and Lemma 2.5 we conclude
\[-\lambda \|v\|^2 = \text{Ric} (v, v) + nv (E) + 2g \left( \sum_{i=1}^{n} A_{F(e_i)} e_i, v \right) + 2ng (\Psi (H), v),\]
and this equation together with equations (2.2) and (2.3) by integration gives
\[\int_{M} \left\{ \text{Ric} (v, v) + \lambda \|v\|^2 - 2 \|\varphi\|^2 - n^2 E^2 \right\} dV = 0.\]

3. Submanifolds with \(v\) as a conformal vector field

Recall that a smooth vector field \(\xi\) on a Riemannian manifold \((M, g)\) is said to be a conformal vector field if the flow of \(\xi\) consists of conformal transformations of the Riemannian manifold \((M, g)\). Equivalently, a smooth vector field \(\xi\) on a Riemannian manifold \((M, g)\) is a conformal vector field if there exists a smooth function \(\rho\) on \(M\) that satisfies \(\mathcal{L}_\xi g = 2\rho g\), where \(\mathcal{L}_\xi g\) is the Lie derivative of \(g\) with respect to \(\xi\). The smooth function \(\rho\) is called the potential function of the conformal vector field \(\xi\). A conformal vector field \(\xi\) is said to be a non trivial conformal vector field if the potential function \(\rho\) is not a constant. In this section, we find conditions under which the vector field \(v\) on the real submanifold \(M\) of the canonical complex space form \((\mathbb{C}^m, J, \langle , \rangle)\) is a conformal vector field.

**Theorem 3.1.** Let \(M\) be an \(n\)-dimensional compact real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle , \rangle)\). If the Ricci curvature \(\text{Ric}(v, v)\) of \(M\) satisfies
\[\text{Ric} (v, v) \geq n (n - 1) E^2 + \|\varphi\|^2,\]
then \(v\) is a conformal vector field on \(M\).

**Proof.** Using Lemma 2.6, we have
\[\int_{M} \left( \text{Ric} (v, v) - n (n - 1) E^2 - \|\varphi\|^2 + \|C\|^2 - nE^2 \right) dV = 0,
which together with the condition in the hypothesis and Schwarz’s inequality \(\|C\|^2 \geq nE^2\) gives
\[\text{Ric} (v, v) = n (n - 1) E^2 + \|\varphi\|^2 \quad \text{and} \quad \|C\|^2 = nE^2.\]
The second equality holds if and only if \(C = EI\), and consequently, the first equation in Lemma 2.1 reads
\[\nabla_X v = EX + \varphi X, \quad X \in \mathfrak{X}(M).\]
This equation proves that
\[(\mathcal{L}_v g) (X, Y) = 2 Eg(X, Y), \quad X, Y \in \mathfrak{X} (M),\]
that is, \(v\) is a conformal vector field with potential function \(E\). \(\square\)
Theorem 3.2. Let \( M \) be an \( n \)-dimensional compact real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)\). If the vector field \( v \) is an eigenvector of the Laplace operator, \( \Delta v = -\lambda v \), and the Ricci curvature \( \text{Ric}(v,v) \) satisfies
\[
\text{Ric}(v,v) \geq n(n - 2)E^2 + \lambda \|v\|^2 ,
\]
then \( v \) is a conformal vector field.

Proof. Lemma 2.6 implies
\[
-\int_M \|\varphi\|^2 \,dv = \int_M \left( -\text{Ric}(v,v) - \|C\|^2 + n^2E^2 \right) \,dV ,
\]
which in view of Lemma 2.7, gives
\[
\int_M \left( \text{Ric}(v,v) - \lambda \|v\|^2 + 2\|C\|^2 - n^2E^2 \right) \,dV = 0 ,
\]
that is,
\[
\int_M \left( \text{Ric}(v,v) - \lambda \|v\|^2 - n(n - 2)E^2 + 2(\|C\|^2 - nE^2) \right) \,dV = 0 .
\]
Thus, using the hypothesis and Schwarz’s inequality \( \|C\|^2 \geq nE^2 \), we get
\[
\text{Ric}(v,v) = n(n - 2)E^2 + \lambda \|v\|^2 \quad \text{and} \quad \|C\|^2 = nE^2 ,
\]
that is, \( C = EI \). Hence, by Lemma 2.1 we get that \( v \) is a conformal vector field.

\[ \square \]

4. Submanifolds with \( v \) as a Killing vector field

Recall that a smooth vector field \( \xi \) on a Riemannian manifold \((M, g)\) is said to be a Killing vector field if the flow of \( \xi \) consists of isometries of the Riemannian manifold \((M, g)\). Equivalently, a smooth vector field \( \xi \) on a Riemannian manifold \((M, g)\) is a Killing vector field if \( \mathcal{L}_\xi g = 0 \). In this section, we find conditions under which the vector field \( v \) on the real submanifold \( M \) of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)\) is a Killing vector field.

Theorem 4.1. Let \( M \) be an \( n \)-dimensional compact real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot , \cdot \rangle)\). Suppose that \( v \) satisfies

(i) \( v \) is an eigenvector of the Laplace operator with eigenvalue \(-\lambda\),
(ii) \( \text{Ric}(v,v) \geq n(n-1)E^2 + \|\varphi\|^2 \),
(iii) \( \|\varphi\|^2 \geq \lambda \|v\|^2 \).

Then \( v \) is a Killing vector field.

Proof. The condition (ii), in view of Theorem 3.1 implies that \( v \) is a conformal vector field with \( C = EI \) and
\[
\text{Ric}(v,v) = n(n-1)E^2 + \|\varphi\|^2 .
\]
Now, the condition (i), $\Delta v = -\lambda v$, combined with Lemma 2.7 and the above conclusion, gives
\[
\int_M \left( n(n-1)E^2 + \|\varphi\|^2 + \lambda \|v\|^2 - 2\|\varphi\|^2 - n^2E^2 \right) dV = 0,
\]
that is,
\[
\int_M \left( (\|\varphi\|^2 - \lambda \|v\|^2) + nE^2 \right) dV = 0. \tag{4.2}
\]
Using condition (iii), we conclude that $E = 0$ and consequently $C = 0$. Thus, Lemma 2.1 gives
\[
\nabla_X v = \varphi X, \quad X \in \mathfrak{X}(M),
\]
that is,
\[
(\mathcal{L}_v g)(X,Y) = 0, \quad X,Y \in \mathfrak{X}(M).
\]
Hence, $v$ is a Killing vector field. \hfill \Box

**Corollary 4.2.** Let $M$ be an $n$-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle , \rangle)$, with positive sectional curvature. Suppose that $v$ satisfies

(i) $v$ is an eigenvector of the Laplace operator with eigenvalue $-\lambda$, that is, $\Delta v = -\lambda v$,

(ii) $\text{Ric} \ (v,v) \geq n(n-1)E^2 + \|\varphi\|^2$,

(iii) $\|\varphi\|^2 \geq \lambda \|v\|^2$.

Then either $n$ is odd or $m \geq n$.

**Proof.** Notice that $n < 2m$. Suppose the conditions (i)–(iii) hold. Then equation (4.2) implies $E = 0$, $\lambda \|v\|^2 = \|\varphi\|^2$, and combining these with equation (4.1), we get
\[
\text{Ric} \ (v,v) = \lambda \|v\|^2 = \|\varphi\|^2. \tag{4.3}
\]
Now, consider the smooth function $f = \frac{1}{2} \|v\|^2$, which by Lemma 2.1 and $E = 0$, gives the gradient $\nabla f = -\varphi v$, and we compute
\[
\Delta f = -\sum_{i=1}^n g(\nabla e_i \varphi v, e_i) = -\sum_{i=1}^n g(\nabla e_i \nabla_v v, e_i). \tag{4.4}
\]
Note that $E = 0$, as in the proof of Theorem 4.1, we get $C = 0$ and thus, an easy computation on using Lemma 2.1 with $E = 0$ gives
\[
R(X,v) v = \nabla_X \nabla_v v - \varphi^2 X,
\]
that is,
\[
R(X,v,v,X) = g(\nabla_X \nabla_v v, X) + \|\varphi X\|^2.
\]
This equation in view of equation (4.4) implies
\[
\text{Ric} \ (v,v) = -\Delta f + \|\varphi\|^2,
\]
which together with equation (4.3) gives $\Delta f = 0$. Hence, $f$ is a constant, that is, $v$ has constant length and consequently, $\varphi v = 0$.\hfill \Box
If \( v = 0 \), then Lemma 2.1 implies \( \varphi = 0 \), that is, \( J\varphi = N \), which on taking covariant derivative and using Lemma 2.1 gives \( JX = FX \), \( X \in \mathfrak{X}(M) \), and we get that \( M \) is a totally real real submanifold of \( \mathbb{C}^m \). Hence, in this case we have \( 2n \leq 2m \).

If \( v \neq 0 \), as \( v \) is a Killing vector field of constant length \( v(p) \neq 0 \) for each \( p \in M \), and as \( M \) is compact connected with positive sectional curvature, then \( M \) is odd-dimensional (for on an even-dimensional compact connected manifold of positive sectional curvature a Killing vector field has a zero).

**Theorem 4.3.** Let \( M \) be an \( n \)-dimensional compact real submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)\). Suppose that \( v \neq 0 \) is not closed and satisfies \( \varphi v = 0 \), with Ricci curvature

\[
\text{Ric} (v, v) \geq n (n - 1) E^2 + \| \varphi \|^2.
\]

Then \( v \) is a Killing vector field of constant length.

**Proof.** As in Theorem 3.1, the condition \( \text{Ric} (v, v) \geq n (n - 1) E^2 + \| \varphi \|^2 \) implies that \( v \) is a conformal vector field and the following hold:

\[
\nabla_X v = \varphi X + E X, \quad X \in \mathfrak{X}(M) \quad \text{and} \quad \text{Ric} (v, v) = n (n - 1) E^2 + \| \varphi \|^2. \quad (4.5)
\]

Using the first equation in (4.5), we get

\[
R(X, Y)v = X(E)Y - Y(E)X + (\nabla \varphi)(X, Y) - (\nabla \varphi)(Y, X),
\]

which gives

\[
\text{Ric}(Y, v) = -(n - 1)Y(E) - g \left( Y, \sum_{i=1}^{n} (\nabla \varphi)(e_i, e_i) \right),
\]

that is,

\[
\text{Ric}(v, v) = -(n - 1)v(E) - g \left( v, \sum_{i=1}^{n} (\nabla \varphi)(e_i, e_i) \right). \quad (4.6)
\]

Now, taking divergence on both sides of the equation \( \varphi v = 0 \), in view of equation (4.5), we have

\[
- \| \varphi \|^2 - g \left( v, \sum_{i=1}^{n} (\nabla \varphi)(e_i, e_i) \right) = 0, \quad (4.7)
\]

and inserting this equation in (4.6) leads to

\[
\text{Ric}(v, v) = -(n - 1)v(E) + \| \varphi \|^2,
\]

which on comparing with the second equation in (4.5) implies

\[
v(E) = -nE^2. \quad (4.8)
\]

Also, using \( \varphi v = 0 \) in the first equation in (4.5) gives

\[
\nabla_v v = E v, \quad (4.9)
\]

which in view of equations (4.5) and (4.8) leads to

\[
R(X, v)v = X(E) v + nE^2 X - (\nabla \varphi)(v, X) - E\varphi X - \varphi^2 X,
\]
which on taking the inner product with \( v \) and using \( \varphi(\nabla v) = 0 \) (outcome of equation (4.9)), gives \( X(E)\|v\|^2 + nE^2g(X,v) = 0 \), that is,

\[
\|v\|^2 \nabla E = -nE^2v. \tag{4.10}
\]

Hence, as \( v \neq 0 \), we get \( \varphi(\nabla E) = 0 \), and taking divergence on both sides of this equation leads to \( \text{div} (\varphi(\nabla E)) = 0 \), that is,

\[
g \left( \nabla E, \sum_{i=1}^{n} (\nabla \varphi)(e_i, e_i) \right) = 0,
\]

which in view of equation (4.10) implies

\[
-nE^2 g \left( v, \sum_{i=1}^{n} (\nabla \varphi)(e_i, e_i) \right) = 0.
\]

Using (4.7) in the above equation, we get

\[
nE^2 \|\varphi\|^2 = 0,
\]

and as \( v \) is not closed, from above equation, we conclude that \( E = 0 \), and thus equation (4.5) reads, \( \nabla X = \varphi X \), \( X \in \mathfrak{X}(M) \), which proves that \( v \) is a Killing vector field.

Moreover, if \( f = \frac{1}{2} \|v\|^2 \), then we have

\[
X(f) = g(\varphi X, v) = 0, \quad X \in \mathfrak{X}(M),
\]

that is, \( v \) has constant length. \( \square \)

5. A CHARACTERIZATION OF SPHERES

In this section we consider an \( n \)-dimensional compact real submanifold \( M \) of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)\), and prove the following characterization for the spheres.

**Theorem 5.1.** Let \( M \) be an \( n \)-dimensional compact Einstein submanifold of the canonical complex space form \((\mathbb{C}^m, J, \langle \cdot, \cdot \rangle)\), \( n > 2 \). Suppose that \( v \) satisfies

(i) \( v \) is an eigenvector of the Laplace operator with eigenvalue \(-\lambda < \frac{S}{n}\),

(ii) \( \text{Ric}(v,v) \geq n(n-1)E^2 + \|\varphi\|^2 \), where \( S \) is the constant scalar curvature.

Then \( M \) is isometric to the sphere \( S^n(c) \), for a constant \( c > 0 \).

**Proof.** Using Theorem 3.1 we get that \( v \) is a conformal vector field on \( M \) and equation (4.5) holds. Thus, using the first equation in (4.5), we conclude

\[
(\nabla \varphi)(X,Y) = \nabla_X \nabla_Y v - \nabla_{\nabla_X Y} v - X(E)Y, \quad X, Y \in \mathfrak{X}(M), \tag{5.1}
\]

where \( (\nabla \varphi)(X,Y) = \nabla_X \varphi y - \varphi \nabla_X Y \). Taking sum in the above equation over a local orthonormal frame \( \{e_1, \ldots, e_n\} \) on \( M \) and using \( \Delta v = -\lambda v \), we get

\[
\sum_{i=1}^{n} (\nabla \varphi)(e_i, e_i) = \Delta v - \nabla E = -\lambda v - \nabla E. \tag{5.2}
\]
Also, using equation (5.1), we find

\[
(\nabla \varphi) (X, Y) - (\nabla \varphi) (Y, X) = R(X, Y) v + Y(E) X - X(E) Y,
\]

which on choosing \( X = e_i \) and taking the inner product with \( e_i \) and adding these \( n \) equations corresponding to a local orthonormal frame \( \{e_1, \ldots, e_n\} \) on \( M \), we get

\[
- g \left( \sum_{i=1}^{n} (\nabla \varphi) (e_i, e_i), Y \right) = \text{Ric} (Y, v) + (n - 1) Y(E),
\]

where we used the fact that \( \varphi \) is skew-symmetric and consequently \( \sum g(\varphi e_i, e_i) = 0 \), and that \( g((\nabla \varphi) (X, Y), Z) = -g((\nabla \varphi) (X, Z), Y) \). Combining equations (5.2) and (5.3), we arrive at

\[
Q(v) = \lambda v - (n - 2) \nabla E.
\]

Moreover, \( M \) being an Einstein manifold, \( Q(v) = S \frac{v}{n} \), and thus using equation (5.4) we get

\[
\nabla E = -S - n\lambda \frac{n}{n(n - 2)} v,
\]

and as \( S \) is a constant, we have \( \nabla E = -cv \) for a constant \( c \). This leads to

\[
\nabla_X (\nabla E) = -c \nabla_X v = -c(EX + \varphi X),
\]

that is, the Hessian of the smooth function \( E \) is given by

\[
H_E (X, Y) = -cEg(X, Y) - cg(\varphi X, Y) a, \quad X, Y \in \mathfrak{X}(M),
\]

\[
H_E (X, Y) - H_E (Y, X) = 2cg(\varphi Y, X).
\]

Since the Hessian is symmetric, we get \( cg(\varphi Y, X) = 0 \), \( X, Y \in \mathfrak{X}(M) \). However, condition (i) in the hypothesis does not allow \( c = 0 \) (as \( c = 0 \) implies \( S = n\lambda \)); consequently we get \( \varphi = 0 \), which changes equation (5.5) to

\[
\nabla_X (\nabla E) = -cEX, \quad X \in \mathfrak{X}(M),
\]

where \( c \) is a positive constant by condition (i). Hence, by Obata’s Theorem (cf. [11]), we get that \( M \) is isometric to \( S^n(c) \). \[ \square \]

6. Examples

In this section, we give two examples of real submanifolds of a canonical complex space form \( (\mathbb{C}^m, J, \langle \cdot , \cdot \rangle) \), one admitting a conformal vector field that is not Killing and other admitting a Killing vector field that is not parallel.

(i) Consider

\[
S^{2n}(c) = \left\{ x = (x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1} : \|x\| = \frac{1}{\sqrt{c}}, \; c > 1 \right\}
\]

and an immersion \( \psi : S^{2n}(c) \to \mathbb{C}^{m+1} \) defined by

\[
\psi(x) = \left( x_1, \ldots, x_{2n+1}, \sqrt{1 - \frac{1}{c}} \right),
\]

where the image of \( S^{2n}(c) \) under \( \psi \) is a real submanifold of \( \mathbb{C}^{m+1} \) that admits a conformal vector field that is not Killing.
which is clearly a smooth immersion. Observe that
\[ T_p(S^{2n}(c)) = \{ X \in \mathbb{R}^{2n+1} : \langle X, p \rangle = 0 \} . \]
The two orthogonal unit normals \( N_1, N_2 \) for the real submanifold \( S^{2n}(c) \) in \( C^{n+1} \) are given by
\[ N_1 = \left( -\sqrt{c-1}x_1, \ldots, -\sqrt{c-1}x_{2n+1}, \frac{1}{\sqrt{c}} \right) \]
and
\[ N_2 = \left( x_1, \ldots, x_{2n+1}, \sqrt{1 - \frac{1}{c}} \right) . \]
Also, the standard complex structure \( J \) on \( C^{n+1} \) gives
\[ J\psi = \left( -x_{n+2}, \ldots, -x_{2n+1}, -\sqrt{1 - \frac{1}{c}}, x_1, \ldots, x_{n+1} \right) \quad (6.1) \]
and it is easy to check that
\[ \langle J\psi, N_1 \rangle = \sqrt{c}x_{n+1} \quad \text{and} \quad \langle J\psi, N_2 \rangle = 0 . \]
Expressing \( J\psi = v + \mathcal{N} \), where \( v \in \mathfrak{X}(S^{2n}(c)) \), we get
\[ v = J\psi - \sqrt{c}x_{n+1} \left( -\sqrt{c-1}x_1, \ldots, -\sqrt{c-1}x_{2n+1}, \frac{1}{\sqrt{c}} \right) \quad (6.2) \]
that is,
\[ v = \left( -x_{n+2}, \ldots, -x_{2n+1}, -\sqrt{1 - \frac{1}{c}}, x_1, \ldots, x_{n+1} \right) \]
\[ + \left( \sqrt{c^2 - cx_1x_{n+1}}, \ldots, \sqrt{c^2 - cx_{n+1}x_{2n+1}}, -x_{n+1} \right) \]
\[ = \left( \sqrt{c^2 - cx_1x_{n+1} - x_{n+2}}, \ldots, \sqrt{c^2 - cx_{n+1}x_{2n+1}}, -x_{n+1} \right) \]
\[ \sqrt{c^2 - cx_{n+1}x_{n+2} + x_1, \ldots, \sqrt{c^2 - cx_{n+1}x_{2n+1} + x_{n+1}}}, 0 \right) \quad (6.3) \]
Now, using expressions of \( N_1 \) and \( N_2 \) it is straightforward to show that
\[ A_{N_1} = \sqrt{c-1}I \quad \text{and} \quad A_{N_2} = -I , \]
and consequently that
\[ A_{\mathcal{N}} = \sqrt{c^2 - cx_{n+1}}I . \]
This proves that the vector field \( v \) given by equation \( (6.3) \) satisfies
\[ \mathcal{L}_v g = 2\sqrt{c^2 - cx_{n+1}}g , \]
that is, \( v \) is a conformal vector field. Note that this vector field is not a Killing vector field on \( S^{2n}(c) \). To verify the last assertion, we see from the last equation that if \( v \) is Killing, \( x_{n+1} = 0 \), and consequently equation \( (6.2) \) gives that \( v = J\psi \).
Moreover, \( S^{2n}(c) \) being an even-dimensional compact and connected manifold of
positive sectional curvature, there would exist a point \( p \) where \( (J\psi)(p) = 0 \); using this in equation (5.1) we get \( c = 1 \), a contradiction.

**(ii)** Consider the unit sphere \( S^{2n-1} \) in \( R^{2n} \) and an immersion \( \psi : S^{2n-1} \to \mathbb{C}^m \), \( m > n \), defined by

\[
\psi(x_1, \ldots, x_n, \ldots, x_{2n}) = (x_1, \ldots, x_{2n}, c_1, \ldots, c_{2m-2n}),
\]

where \( c_i, 1 \leq i \leq 2m - 2n \), are constants and \( \mathbb{C}^m \) is identified with \( R^{2m} \). A local frame of orthonormal normal vector fields for this immersion is given by \( \{N_1, N_2, \ldots, N_{2m-2n+1}\} \), where

\[
N_1 = (x_1, \ldots, x_{2n}, 0, \ldots, 0)
\]

and

\[
N_\alpha = (0, \ldots, 0, 1, 0, \ldots, 0), \quad 1 \text{ at the } (2n + \alpha)^{\text{th}} \text{ place, } 2 \leq \alpha \leq 2m - 2n + 1.
\]

Consider a complex structure \( J \) on \( \mathbb{C}^m \) defined by

\[
JE = (-E(x_2), E(x_1), -E(x_4), E(x_3), \ldots, -E(x_{2m}), E(x_{2m-1})), \quad E \in \mathfrak{X}(\mathbb{C}^m),
\]

which makes \( (\mathbb{C}^m, J, \langle \cdot, \cdot \rangle) \) a Kaehler manifold. Now set \( J\psi = v + \overline{N} \), where \( v \in \mathfrak{X}(S^{2n-1}) \) is the tangential component and \( \overline{N} \) is the normal component of \( J\psi \). We get

\[
J\psi = (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1} - c_2, c_1, \ldots, -c_{2m-2n}, c_{2m-2n-1}),
\]

\[
\langle J\psi, N_1 \rangle = 0, \quad \langle J\psi, N_\alpha \rangle = (-1)^\alpha c_\alpha, \quad 2 \leq \alpha \leq 2m - 2n + 1,
\]

and consequently,

\[
\overline{N} = \sum_{\alpha=1}^{2m-2n+1} \langle N_\alpha, N_\alpha \rangle N_\alpha = (0, \ldots, 0, -c_2, c_1, \ldots, -c_{2m-2n}, c_{2m-2n-1}).
\]

Thus, equations (6.4) and (6.5) imply

\[
v = J\psi - \overline{N} = (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1}, 0, \ldots, 0).
\]

Let \( \nabla \) and \( \overline{\nabla} \) be the Euclidean connection on \( \mathbb{C}^m \) and the Riemannian connection on the real submanifold \( (S^{2n-1}, g) \) with respect to the induced metric \( g \). Then using equation (6.6) we get

\[
\nabla_X v = \overline{\nabla}_X v - h(X, v)
\]

\[
= (-X(x_2), X(x_1), \ldots, -X(x_{2n}), X(x_{2n-1}), 0, \ldots, 0) - h(X, v),
\]

\( X \in \mathfrak{X}(S^{2n-1}) \), where \( h \) is the second fundamental form. Taking the inner product with \( Y \in \mathfrak{X}(S^{2n-1}) \) in the above equation we arrive at

\[
g(\nabla_X v, Y) = -X(x_2)Y(x_1) + \cdots - X(x_{2n})Y(x_{2n-1}) + X(x_{2n-1})Y(x_{2n}),
\]

which leads to

\[
g(\nabla_X v, Y) + g(\nabla_Y v, X) = 0, \quad X, Y \in \mathfrak{X}(S^{2n-1}).
\]

Thus, the vector field \( v \) satisfies

\[
\mathcal{L}_v g = 0,
\]

that is, $v$ is a Killing vector field on $S^{2n-1}$. That the Killing vector field $v$ is not parallel follows from equation (6.7), that is, $v$ is a nontrivial Killing vector field.

References