NEW EXTENSIONS OF CLINE’S FORMULA FOR GENERALIZED DRAZIN–RIESZ INVERSES

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ABSTRACT. In this note, Cline’s formula for generalized Drazin–Riesz inverses is proved. We prove that if $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ are such that $ACD = DBD$ and $DBA = ACA$, then AC is generalized Drazin–Riesz invertible if and only if BD is generalized Drazin–Riesz invertible, and that, in such a case, if S is a generalized Drazin–Riesz inverse of AC then $T := BS^2D$ is a generalized Drazin–Riesz inverse of BD .

1. INTRODUCTION

Let X, Y be Banach spaces and let $\mathcal{B}(X, Y)$ denote the algebra of all bounded linear operators from X to Y . For $Y = X$ we write $\mathcal{B}(X, X) = \mathcal{B}(X)$. A bounded operator $T \in \mathcal{B}(X)$ is said to be Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that (see [5]):

$$ST = TS, \quad S^2T = S, \quad \text{and} \quad T^{k+1}S = T^k.$$

The concept of Drazin invertible operators has been generalized by Koliha [7] by replacing the third condition in this definition with the condition that $TST - T$ is quasi-nilpotent. Recall that a bounded operator $T \in \mathcal{B}(X)$ is Riesz, if $T - \lambda I$ is Fredholm in the usual sense for every $\lambda \in \mathbb{C} \setminus \{0\}$ (see [1]).

Recently, Živković-Zlatanović and Cvetković [14] introduced and studied a new concept of pseudo-inverse to extend the Koliha concept to “generalized Drazin–Riesz invertible”. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin–Riesz invertible, if there exists $S \in \mathcal{B}(X)$ such that

$$TS = ST, \quad STS = S, \quad \text{and} \quad TST - T \text{ is Riesz.}$$

In this case S is called a generalized Drazin–Riesz inverse of T . Until now, the uniqueness of Drazin–Riesz’s generalized inverse remains an open question. Živković-Zlatanović and Cvetković also showed that T is generalized Drazin–Riesz invertible iff it has a direct sum decomposition $T = T_1 \oplus T_0$ with T_1 invertible and T_0 Riesz. The generalized Drazin–Riesz spectrum of $T \in \mathcal{B}(X)$ is defined by

$$\sigma_{gDR}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not generalized Drazin–Riesz invertible}\}.$$

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Jacobson's Lemma [2] asserts that if $A, B \in \mathcal{B}(X)$, then

$$AB - I \text{ is invertible} \iff BA - I \text{ is invertible.} \quad (1.1)$$

As extensions of Jacobson's lemma, Corach et al. [4] investigated (1.1) under the assumption $ABA = ACA$. They studied common properties of AC and BA from an algebraic viewpoint and also obtained some nice topological analogues ([10][11][12]).

For an associative ring R with unit, Cline [3] showed that if $a, b \in R$ such that ab is Drazin invertible then so is ba and in this case the Drazin inverse of ba is $(ba)^D = b((ab)^D)^2a$. This formula is called *Cline's formula*. Lately, Cline's formula for Drazin and generalized Drazin inverses in a ring under the conditions $acd = dbd$, $dba = aca$, was extended respectively by Miller and Zguitti [8] and by Zenga et al. [13]. Recently, we gave Cline's formula for generalized Drazin–Riesz inverse under the assumption $ABA = ACA$, where $A, B, C \in \mathcal{B}(X)$, see [6].

In this note, we propose to give Cline's formula for the generalized Drazin–Riesz inverse for bounded linear operators satisfying the operator equation set

$$\begin{cases} ACD = DBD, \\ DBA = ACA, \end{cases}$$

where $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$.

2. MAIN RESULTS

Lemma 2.1. *Let $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ such that $\begin{cases} ACD = DBD, \\ DBA = ACA. \end{cases}$ Then AC is Riesz if and only if BD is Riesz.*

Proof.

$$\begin{aligned} AC \text{ is Riesz} &\iff \lambda I - AC \text{ is Fredholm for all } \lambda \in \mathbb{C} \setminus \{0\} \\ &\iff \lambda I - BD \text{ is Fredholm for all } \lambda \in \mathbb{C} \setminus \{0\} \\ &\iff BD \text{ is Riesz.} \end{aligned}$$

See [9, Theorem 3.1]. □

Theorem 2.2. *Let $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ such that $\begin{cases} ACD = DBD, \\ DBA = ACA. \end{cases}$ Then, AC is generalized Drazin–Riesz invertible if and only if BD is generalized Drazin–Riesz invertible. In this case if S is a generalized Drazin–Riesz inverse of AC then $T = BS^2D$ is a generalized Drazin–Riesz inverse of BD .*

Proof. $\Rightarrow$) Suppose that AC is generalized Drazin–Riesz inverse, then there exists $S \in \mathcal{B}(Y)$ such that

$$\begin{cases} S(AC) = (AC)S, \\ S(AC)S = S, \\ (AC)S(AC) - AC \text{ is Riesz.} \end{cases}$$

Let $T = BS^2D$, then we obtain

$$\begin{aligned} T(BD) &= (BS^2D)BD \\ &= BS^2(DBD) \\ &= BS^2(ACD) \\ &= B(S^2AC)D \\ &= BSD. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (BD)T &= (BD)BS^2D \\ &= BDB(S)SD \\ &= BDB(ACS^2)SD \\ &= B(DBA)CS^3D \\ &= B(ACA)CS^3D \\ &= B(ACACS^3)D \\ &= B(S)D \\ &= BSD. \end{aligned}$$

Hence, we deduce that

$$T(BD) = (BD)T.$$

Next, also we have

$$\begin{aligned} T(BD)T &= BS^2D(BD)BS^2D \\ &= BS^2DBDB(S)SD \\ &= BS^2DBDB(ACS^2)SD \\ &= BS^2((DBD)BAC)S^3D \\ &= BS^2((ACD)BAC)S^3D \\ &= BS^2(AC(DBA)C)S^3D \\ &= BS^2(AC(ACA)C)S^3D \\ &= BS^2(ACACAC)S^3D \\ &= BS(ACACACS^4)D \\ &= BS(S)D \\ &= BS^2D \\ &= T. \end{aligned}$$

So

$$T(BD)T = T.$$

Finally, let $Q = I - ACS$. Since we have $QAC = (I - ACS)AC = AC - ACSAC$, QAC is Riesz. Also we have

$$\begin{aligned}
 BD - (BD)^2T &= BD - (BD)^2T \\
 &= BD - (BD)^2BS^2D \\
 &= BD - (BDBD)B(S)SD \\
 &= BD - BDBDB(ACS^2)SD \\
 &= BD - BDBDBACS^3D \\
 &= BD - BDB(DBA)CS^3D \\
 &= BD - BDB(ACA)CS^3D \\
 &= BD - B(DBA)CAC S^3D \\
 &= BD - B(ACA)CAC S^3D \\
 &= BD - BAC(ACACS^3)D \\
 &= BD - BAC(S)D \\
 &= B(I - ACS)D \\
 &= BQD,
 \end{aligned}$$

$$\begin{aligned}
 DBQA &= DB(I - ACS)A \\
 &= DBA - (DBA)CSA \\
 &= ACA - (ACA)CSA \\
 &= AC(I - ACS)A \\
 &= ACQA
 \end{aligned}$$

and

$$\begin{aligned}
 DBQD &= DB(I - ACS)D \\
 &= DBA - (DBA)CSD \\
 &= ACA - (ACA)CSD \\
 &= AC(I - ACS)D \\
 &= ACQD.
 \end{aligned}$$

Then, we deduce that

$$\begin{cases} DBQAC = ACQAC, \\ DBQDB = ACQDB. \end{cases}$$

Therefore, by Lemma 2.1, we deduce that $DBQ = BD - (BD)^2T$ is Riesz.

In summary, BD is generalized Drazin–Riesz invertible.

$\Leftrightarrow$) The other direction is automatic by symmetry of A with D and C with B . $\square$

As a direct consequence of Theorem 2.2, we have the following corollaries.

Corollary 2.3. Let $A, D \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ such that $\begin{cases} ACD = DBD, \\ DBA = ACA. \end{cases}$
Then,

$$\sigma_{gDR}(AC) = \sigma_{gDR}(BD).$$

Corollary 2.4. Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Then,

$$\sigma_{gDR}(AB) = \sigma_{gDR}(BA).$$

If $A = D$ and $B = C$, we have ([6, Theorem 2.1]):

Corollary 2.5. If $A, B, C \in \mathcal{B}(X)$ satisfy $ABA = ACA$, then

AC is generalized Drazin–Riesz invertible

$$\iff BA \text{ is generalized Drazin–Riesz invertible.}$$

In this case if S is a generalized Drazin–Riesz inverse of AC then $T = BS^2A$ is a generalized Drazin–Riesz inverse of BA .

In the following example, we prove that the results obtained in this work are a proper generalization of the corresponding ones in [6]

Example. Let $S_1 \in \mathcal{B}(X)$, $S_2, T_1 \in \mathcal{B}(Y, X)$, and $T_2 \in \mathcal{B}(X, Y)$ satisfying $S_1 = S_2T_2$ and $T_2T_1 \neq I$. Let

$$A = \begin{pmatrix} 0 & S_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = D = \begin{pmatrix} I & T_1 \\ T_2 & I \end{pmatrix}.$$

It is easy to see that $ACB = BCB$ and $BCA = ACA$, but $CAC \neq CBC$. Then Cline's formula for generalized Drazin–Riesz inverses can only be obtained from the results above, but not from the corresponding ones in [6].

We finish this work with the following natural question.

Question: Is Cline's formula for generalized Drazin–Riesz inverses in a ring with identity true?

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