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# GENERALIZED METALLIC STRUCTURES

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Abstract. We study the properties of a generalized metallic, a generalized product and a generalized complex structure induced on the generalized tangent bundle of a smooth manifold $M$ by a metallic Riemannian structure $(J, g)$ on $M$, providing conditions for their integrability with respect to a suitable connection. Moreover, by using methods of generalized geometry, we lift $(J, g)$ to metallic Riemannian structures on the tangent and cotangent bundles of $M$, underlying the relations between them.

## 1. Preliminaries

On a smooth manifold $M$, besides the almost complex, almost tangent, almost product structures, etc., some other polynomial structures can be considered as $C^\infty$-tensor fields $J$ of $(1,1)$-type which are roots of the algebraic equation

$$Q(J) := J^n + a_n J^{n-1} + \cdots + a_2 J + a_1 I = 0,$$

where $I$ is the identity operator on the Lie algebra of vector fields on $M$. In particular, if $Q(J) := J^2 - pJ - qI$, with $p$ and $q$ positive integers, its solution $J$ will be called *metallic structure* [2]. The name is motivated by the fact that the $(p, q)$-*metallic number* introduced by Vera W. de Spinadel [8] is precisely the positive root of the quadratic equation $x^2 - px - q = 0$, namely $\sigma_{p,q} := \frac{p + \sqrt{p^2 + 4q}}{2}$. For example: if $p = q = 1$ we get the *golden number* $\sigma = \frac{1 + \sqrt{5}}{2}$; if $p = 2$ and $q = 1$ we get the *silver number* $\sigma_{2,1} = 1 + \sqrt{2}$; if $p = 3$ and $q = 1$ we get the *bronze number* $\sigma_{3,1} = \frac{3 + \sqrt{13}}{2}$; if $p = 1$ and $q = 2$ we get the *copper number* $\sigma_{1,2} = 2$; if $p = 1$ and $q = 3$ we get the *nickel number* $\sigma_{1,3} = \frac{1 + \sqrt{13}}{2}$, and so on.

We shall briefly recall the basic notions of metallic (Riemannian) geometry.

**Definition 1.1** ([3]). *A metallic structure* $J$ on $M$ is an endomorphism $J : TM \to TM$ satisfying

$$J^2 = pJ + qI, \tag{1.1}$$

for some $p$, $q \in \mathbb{N}^*$. The pair $(M, J)$ is called a *metallic manifold.* Moreover, if a Riemannian metric $g$ on $M$ is compatible with $J$, that is $g(JX, Y) = g(X, JY)$,

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for any $X, Y \in C^\infty(TM)$, we call the pair $(J, g)$ a *metallic Riemannian structure* and $(M, J, g)$ a *metallic Riemannian manifold.*

The concept of integrability for a metallic structure is defined in the classical manner.

**Definition 1.2.** A metallic structure $J$ is called *integrable* if its Nijenhuis tensor field

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]$$

vanishes for all $X, Y \in C^\infty(TM)$.

It is known [3] that an almost product structure $F$ on $M$ induces two metallic structures:

$$J^\pm = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I$$

and, conversely, every metallic structure $J$ on $M$ induces two almost product structures:

$$F^\pm = \pm\left(\frac{2}{2\sigma_{p,q} - p}J - \frac{p}{2\sigma_{p,q} - p}I\right),$$

where $\sigma_{p,q} = \dfrac{p + \sqrt{p^2 + 4q}}{2}$ is the metallic number, for $p, q \in \mathbb{N}^*$.

In particular, if the almost product structure $F$ is compatible with a Riemannian metric $g$, then $(J^+, g)$ and $(J^-, g)$ are metallic Riemannian structures.

The analogue concept of locally product manifold is considered in the context of metallic geometry.

**Definition 1.3** ([1]). A metallic Riemannian manifold $(M, J, g)$ is called *locally metallic* if $J$ is parallel with respect to the Levi-Civita connection $\nabla$ of $g$, that is $\nabla J = 0$.

In the following, we shall extend the definition of a metallic structure for any real numbers $p$ and $q$. In this way, we also include some other well-known structures; for instance, if $(p, q) \in \{(0, -1), (0, 0), (0, 1), (1, 0)\}$, the solution of (1.1) would yield an almost complex, an almost tangent, an almost product and a $J(2, 1)$-structure, respectively.

## 2. Generalized structures induced by metallic structures

Let $TM \oplus T^*M$ be the generalized tangent bundle of a smooth manifold $M$.

**Definition 2.1.** A *generalized metallic structure* $\hat{J}$ on $M$ is an endomorphism $\hat{J} : TM \oplus T^*M \to TM \oplus T^*M$ satisfying

$$\hat{J}^2 = p\hat{J} + qI,$$

for some real numbers $p$ and $q$.

For a linear connection $\nabla$ on $M$, we consider the bracket $[\cdot, \cdot]_\nabla$ on $C^\infty(TM \oplus T^*M)$ [6]:

$$[X + \alpha, Y + \beta]_\nabla := [X, Y] + \nabla_X\beta - \nabla_Y\alpha,$$

for all $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$.

**Definition 2.2.** A generalized metallic structure $\hat{J}$ is called $\nabla$-*integrable* if its Nijenhuis tensor field $N^{\nabla}_{\hat{J}}$ with respect to $\nabla$,

$$N^{\nabla}_{\hat{J}}(\sigma,\tau) := [\hat{J}\sigma,\hat{J}\tau]_{\nabla} - \hat{J}[\hat{J}\sigma,\tau]_{\nabla} - \hat{J}[\sigma,\hat{J}\tau]_{\nabla} + \hat{J}^2[\sigma,\tau]_{\nabla},$$

vanishes for all $\sigma,\tau \in C^{\infty}(TM \oplus T^*M)$.

2.1. **Generalized metallic structure induced by** $(J,g)$**.** Let $(J,g)$ be a metallic Riemannian structure on $M$ such that $J^2 = pJ + qI$, $p,q \in \mathbb{R}$. If we denote by $\sharp_g : T^*M \to TM$ the inverse of the isomorphism $\flat_g : TM \to T^*M$, $\flat_g(X) := i_X g$, from the $g$-symmetry of $J$ we have $\sharp_g \circ J^* = J \circ \sharp_g$ and $\flat_g \circ J = J^* \circ \flat_g$, where $(J^*\alpha)(X) := \alpha(JX)$. Also notice that $J^*$ is a metallic structure too, namely, $(J^*)^2 = pJ^* + qI$, and we easily get that $\sharp_g \circ (J^*)^k = J^k \circ \sharp_g$ and $\flat_g \circ J^k = (J^*)^k \circ \flat_g$, for any $k \in \mathbb{N}$.

On $TM \oplus T^*M$ we consider the Riemannian metric

$$\hat{g}(X+\alpha, Y+\beta) := g(X,Y) + g(\sharp_g\alpha, \sharp_g\beta), \tag{2.1}$$

for any $X,Y \in C^{\infty}(TM)$ and $\alpha,\beta \in C^{\infty}(T^*M)$.

**Definition 2.3.** A pair $(\hat{J},\hat{g})$ of a generalized metallic structure $\hat{J}$ and a Riemannian metric $\hat{g}$ such that $\hat{J}$ is $\hat{g}$-symmetric is called *generalized metallic Riemannian structure.*

Remark that the generalized metallic structure $\hat{J}_m := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}$ induced by the metallic Riemannian structure $(J,g)$ is $\hat{g}$-symmetric; hence, $(\hat{J}_m,\hat{g})$ is a generalized metallic Riemannian structure.

**Proposition 2.4.** *The generalized metallic structure* $\hat{J}_m$ *induced by the metallic Riemannian structure* $(J,g)$ *on* $M$ *is* $\nabla$*-integrable if and only if* $J$ *is integrable and* $(\nabla_{JX}J) = (\nabla_X J)J$*, for any* $X \in C^{\infty}(TM)$*.*

*Proof.* We have:

$$\begin{aligned}
N^{\nabla}_{\hat{J}_m}(X,Y) &= [JX,JY] - J[JX,Y] - J[X,JY] + J^2[X,Y] = N_J(X,Y) \\
N^{\nabla}_{\hat{J}_m}(X,\beta) &= [JX,J^*\beta]_{\nabla} - J^*[JX,\beta]_{\nabla} - J^*[X,J^*\beta]_{\nabla} + (J^*)^2[X,\beta]_{\nabla} \\
&= \nabla_{JX}J^*\beta - J^*\nabla_{JX}\beta - J^*\nabla_X J^*\beta + (J^*)^2\nabla_X\beta \\
&= ((\nabla_{JX}J^*) - J^*(\nabla_X J^*))(\beta) \\
&= \beta((\nabla_{JX}J) - (\nabla_X J)J) \\
N^{\nabla}_{\hat{J}_m}(\alpha,\beta) &= 0,
\end{aligned}$$

for all $X,Y \in C^{\infty}(TM)$ and $\alpha,\beta \in C^{\infty}(T^*M)$. The proof is thus complete. □

Remark that if $\nabla$ is a $J$-*connection*, that is $\nabla J = 0$, then $\hat{J}_m$ is $\nabla$-integrable if and only if $J$ is integrable. Moreover, if $T^{\nabla}$ is the torsion of $\nabla$, $T^{\nabla}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$, then a direct computation gives:

$$N_J(X,Y) = (\nabla_{JX}J)Y - (\nabla_{JY}J)X + J(\nabla_Y J)X - J(\nabla_X J)Y + \Phi(T^{\nabla})(X,Y),$$

where

$$\Phi(T^\nabla)(X,Y) := -T^\nabla(JX,JY) + JT^\nabla(JX,Y) + JT^\nabla(X,JY) - J^2T^\nabla(X,Y).$$

In particular, if $\nabla$ is a torsion free $J$-connection, then $\hat{J}_m$ is $\nabla$-integrable.

Let $\nabla^g$ be the Levi-Civita connection of $g$ and define a linear connection $D$ on $M$ by $D := \nabla^g + F$, where $F$ is a $(1,2)$-type tensor field such that

$$\begin{cases} DJ = 0 \\ Dg = 0. \end{cases}$$

This is equivalent to

$$\begin{cases} (\nabla^g_X J)Y = J(F(X,Y)) - F(X,JY) \\ g(F(X,Y),Z) + g(Y,F(X,Z)) = 0 \end{cases}$$

for any $X,Y,Z \in C^\infty(TM)$.

Consider the bracket $[\cdot,\cdot]_D$ on $C^\infty(TM \oplus T^*M)$ [6]:

$$[X+\alpha, Y+\beta]_D := [X,Y] + D_X\beta - D_Y\alpha,$$

for any $X,Y \in C^\infty(TM)$ and $\alpha,\beta \in C^\infty(T^*M)$.

Define the connection $\hat{D}$ on $TM \oplus T^*M$ by [7]:

$$\hat{D}_X(Y+\beta) := D_XY + D_X\beta,$$

for any $X,Y \in C^\infty(TM)$ and $\beta \in C^\infty(T^*M)$. It follows that

$$\hat{D}_X(Y+\beta) = \nabla^g_XY + F(X,Y) + \nabla_X\beta - \beta \circ F(X,\cdot).$$

Let $n$ be the dimension of $M$ and assume that $q \neq 0$. Denote by $\{x^1,\dots,x^n\}$ the local coordinates on $M$ and let $\{X_1,\dots,X_n\}$ be the corresponding local frame for $TM$. Following [4] we define:

$$F(X_i,X_j) := \omega(X_j)X_i - \omega(X_l)g^{lk}g_{ij}X_k + \frac{1}{q}\omega(JX_j)JX_i - \frac{1}{q}\omega(JX_l)g^{lk}J_j^s g_{is}X_k,$$

where $\omega$ is a 1-form on $M$ and we use Einstein's convention of summation.

We immediately have that $g(F(X_i,X_j),X_r)+g(X_j,F(X_i,X_r)) = 0$, for all $i,j,r$; therefore, $Dg = 0$, for any 1-form $\omega$. Moreover, the torsion of $D$ is given by

$$T^D(X,Y) = \omega(Y)X - \omega(X)Y + \frac{1}{q}\omega(JY)JX - \frac{1}{q}\omega(JX)JY,$$

for any $X,Y \in C^\infty(TM)$.

**Lemma 2.5.** *$T^D$ satisfies the following properties:*

$$T^D(JX,Y) = JT^D(X,Y) = T^D(X,JY)$$

$$\Phi(T^D)(X,Y) = 0,$$

*for any $X,Y \in C^\infty(TM)$.*

*Proof.* From a direct computation we get

$$JT^D(X,Y) = \omega(Y)JX - \omega(X)JY + \frac{p}{q}\omega(JY)JX - \frac{p}{q}\omega(JX)JY + \omega(JY)X - \omega(JX)Y,$$

which is equal to $T^D(JX,Y)$ and $T^D(X,JY)$.

Consequently, we have $\Phi(T^\nabla)(X,Y) = 0$. □

Recently, C. Karaman [4] constructed metallic semi-symmetric metric $J$-connections $D$ on locally decomposable metallic Riemannian manifolds $(M,J,g)$. These connections satisfy:

$$DJ = 0,\ Dg = 0,\ T^D(X,Y) = \omega(Y)X - \omega(X)Y + \frac{1}{q}\omega(JY)JX - \frac{1}{q}\omega(JX)JY,$$

for any $X,Y \in C^\infty(TM)$. In particular, we can state the following:

**Proposition 2.6.** *Let $(M,J,g)$ be a locally decomposable metallic Riemannian manifold and let $D$ be a metallic semi-symmetric metric $J$-connection. Then $\hat{J}_m$ is $D$-integrable.*

**Proposition 2.7.** *Let $\left(\hat{J}_m := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \hat{g}\right)$ be the generalized metallic Riemannian structure induced by the metallic Riemannian structure $(J,g)$ on $M$ with $\hat{g}$ the Riemannian metric defined by* (2.1)*. Then:*

(1) $\hat{D}\hat{J}_m = 0$ *if and only if* $DJ = 0$*;*
(2) $\hat{D}\hat{g} = 0$ *if and only if* $Dg$*.*

*Proof.* Remark that $\hat{D}\hat{J}_m = 0$ is equivalent to $(D_XJ)Y + \beta \circ D_XJ = 0$, for any $X,Y \in C^\infty(TM)$ and $\beta \in C^\infty(T^*M)$, and $\hat{D}\hat{g} = 0$ is equivalent to $(D_Xg)(Y,Z) - (D_Xg)(\sharp_g\beta, \sharp_g\gamma) = 0$, for any $X,Y,Z \in C^\infty(TM)$ and $\beta,\gamma \in C^\infty(T^*M)$. □

**Definition 2.8.** A smooth map $f$ between two metallic manifolds $(M_1,J_1)$ and $(M_2,J_2)$ is called *metallic* if $f_* \circ J_1 = J_2 \circ f_*$.

*Remark* 2.9. A metallic diffeomorphism $f$ between two metallic manifolds $(M_1,J_1)$ and $(M_2,J_2)$ naturally induces an isomorphism $\hat{f}$ between their generalized tangent bundles defined by

$$\hat{f} : TM_1 \oplus T^*M_1 \to TM_2 \oplus T^*M_2, \quad \hat{f}(X+\alpha) := f_*X + ((f_*)^*)^{-1}\alpha,$$

where $f_* : TM_1 \to TM_2$ is the tangent map of $f$ and $(f_*)^* : T^*M_2 \to T^*M_1$ is the dual map of $f_*$, that is, $((f_*)^*\alpha)(X) := \alpha(f_*X)$, for all $\alpha \in C^\infty(T^*M_2)$ and $X \in C^\infty(TM_1)$, which preserves the generalized metallic structures $\hat{J}_{i,m} := \begin{pmatrix} J_i & 0 \\ 0 & J_i^* \end{pmatrix}$, $i = 1,2$. Indeed, from $f_* \circ J_1 = J_2 \circ f_*$ follows that $(f_*)^* \circ J_2^* = J_1^* \circ (f_*)^*$, hence $\hat{f} \circ \hat{J}_{1,m} = \hat{J}_{2,m} \circ \hat{f}$.

In particular, if $f : M \to M$ is a diffeomorphism which preserves the metallic structure $J$, then $\hat{f}$ can be defined by

$$\hat{f}(X+\alpha) := f_*X + (f_*)^*\alpha,$$

which coincides with the generalized metallic structure $\hat{J}_m$ when $J = f_*$. In this case, $J$ is invertible and $J^{-1} = \frac{1}{q}J - \frac{p}{q}I$, for $q \neq 0$.

2.2. **Generalized product structure induced by** $(J, g)$**.** Let $(J, g)$ be a metallic Riemannian structure on $M$ such that $J^2 = pJ + qI$, $p, q \in \mathbb{R}$. Then $\hat{J}_p := \begin{pmatrix} J & (I - J^2)\sharp_g \\ \flat_g & -J^* \end{pmatrix}$ is a generalized product structure on $M$, that is $\hat{J}_p^2 = I$.

A direct computation gives the following.

**Proposition 2.10.** *The generalized product structure $\hat{J}_p$ induced by the metallic Riemannian structure $(J, g)$ on $M$ is $\nabla$-integrable if and only if the following conditions are satisfied:*

$$\begin{aligned}
&N_J - (I - J^2)\sharp_g(d^\nabla g) = 0\\
&(\nabla_{JX}g)Y - (\nabla_{JY}g)X + J^*((\nabla_X g)Y - (\nabla_Y g)X) + g((\nabla_Y J)X - (\nabla_X J)Y)\\
&\quad + g(T^\nabla(X, JY) + T^\nabla(JX, Y)) = 0\\
&(d^\nabla g)((I - J^2)Y, X) - (\nabla_X J^*)g(JY) + (\nabla_{JX}J^*)g(Y) = 0\\
&(\nabla_{(I-J^2)X}J^*)g(Y) - (\nabla_{(I-J^2)Y}J^*)g(X) = 0\\
&(\nabla_{(I-J^2)X}J^2)Y - (\nabla_{(I-J^2)Y}J^2)X + T^\nabla((I - J^2)X, (I - J^2)Y)\\
&\quad - (I - J^2)\sharp_g((\nabla_{(I-J^2)X}g)Y - (\nabla_{(I-J^2)Y}g)X) = 0\\
&-(\nabla_{JX}J^2)Y - (\nabla_{(I-J^2)Y}J)X + (\nabla_X J)Y + J(\nabla_X J^2)Y - J^2(\nabla_X J)Y\\
&\quad - (I - J^2)\sharp_g((\nabla_{JX}g)Y - (\nabla_X g)JY)\\
&\quad - T^\nabla(JX, (I - J^2)Y) + JT^\nabla(X, (I - J^2)Y) = 0,
\end{aligned}$$

*for all $X, Y \in C^\infty(TM)$, where we denoted $\flat_g$ by $g$ and the exterior differential associated to $\nabla$ acting on $g$ by $(d^\nabla g)(X, Y) := (\nabla_X g)(Y) - (\nabla_Y g)(X) + g(T^\nabla(X, Y))$.*

**Proposition 2.11.** *Let $(M, J, g)$ be a locally metallic Riemannian manifold. Then $\hat{J}_p$ is $\nabla$-integrable, for $\nabla$ the Levi-Civita connection of $g$.*

*Proof.* From the previous proposition, we have that the generalized product structure $\hat{J}_p$ is $\nabla$-integrable if and only if the following conditions are satisfied:

$$\begin{aligned}
&N_J = 0\\
&(\nabla_Y J)X - (\nabla_X J)Y = 0\\
&(\nabla_X J^*)J^* - (\nabla_{JX}J^*) = 0\\
&(\nabla_{(I-J^2)X}J^*)g(Y) - (\nabla_{(I-J^2)Y}J^*)g(X) = 0\\
&(\nabla_{(I-J^2)X}J^2)Y - (\nabla_{(I-J^2)Y}J^2)X = 0\\
&-(\nabla_{JX}J^2)Y - (\nabla_{(I-J^2)Y}J)X + (\nabla_X J)Y + J(\nabla_X J^2)Y - J^2(\nabla_X J)Y = 0\\
&-(\nabla_{JX}J^2)Y + (\nabla_{(I+J^2)Y}J)X - (\nabla_X J)Y + J(\nabla_X J^2)Y - J^2(\nabla_X J)Y = 0,
\end{aligned}$$

for all $X, Y \in C^\infty(TM)$. In particular, if $\nabla J = 0$, then $\hat{J}_p$ is $\nabla$-integrable. □

**Definition 2.12.** A generalized product structure $\hat{J}$ on $M$ is called *anti-pseudo-calibrated* if it is $(\cdot,\cdot)$-anti-invariant and the bilinear symmetric form defined by $(\cdot, \hat{J}\cdot)$ on $TM$ is non-degenerate, where

$$(X+\alpha, Y+\beta) := -\frac{1}{2}(\alpha(Y)-\beta(X))$$

is the natural symplectic structure on $TM \oplus T^*M$.

*Remark* 2.13. The generalized product structure $\hat{J}_p$ is anti-pseudo-calibrated with respect to $(\cdot,\cdot)$.

**Proposition 2.14.** *Let $\hat{J}_p$ be the generalized product structure defined by the metallic Riemannian structure $(J,g)$ on $M$. Then*

$$G(\sigma,\tau) := (\sigma, \hat{J}_p(\tau)),$$

*with $\sigma,\tau \in C^\infty(TM\oplus T^*M)$, is a neutral metric.*

*Proof.* Locally we can write $2G$ in block matrix form as:

$$\begin{pmatrix} g & -J \\ -J & -(I-J^2)\sharp_g \end{pmatrix}.$$

As $J$ is $g$-symmetric, pointwise, we can take $g=I$ and $J=\Lambda$ the diagonal matrix with eigenvalues $\lambda_1,\dots,\lambda_n$ which are solutions of the metallic equation $\lambda^2-p\lambda-q=0$. Then we get:

$$\begin{pmatrix} I & -\Lambda \\ -\Lambda & p\Lambda+(q-1)I \end{pmatrix}.$$

In order to compute the indices of $2G$, we can use the Gauss–Lagrange algorithm and by elementary operations on rows and columns of the matrix we get the form

$$\begin{pmatrix} I & 0 \\ 0 & -I+(\Lambda^2-p\Lambda-qI) \end{pmatrix},$$

therefore

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};$$

hence $2G$ has $n$ positive and $n$ negative eigenvalues and the proof is complete. □

**Proposition 2.15.** *Let* $\left(\hat{J}_p := \begin{pmatrix} J & (I-J^2)\sharp_g \\ \flat_g & -J^* \end{pmatrix}, \hat{g}\right)$ *be the generalized product structure induced by the metallic Riemannian structure $(J,g)$ on $M$, with $\hat{g}$ the Riemannian metric defined by* (2.1)*. Then*

$$\hat{D}\hat{J}_p = 0 \text{ if and only if } DJ=0 \text{ and } Dg=0.$$

*Proof.* Remark that $(\hat{D}_Y\hat{J}_p)X = (D_YJ)X+(D_Yg)X$, for any $X,Y\in C^\infty(TM)$ and $(\hat{D}_Y\hat{J}_p)\alpha = -p(D_Y(J\sharp_g))\alpha-(q-1)(D_Y\sharp_g)\alpha-(D_YJ^*)\alpha$, for any $Y\in C^\infty(TM)$ and $\alpha\in C^\infty(T^*M)$, therefore the statement. □

*Remark* 2.16. Starting with a metallic structure on a manifold, with minimal restrictions on $p$ and $q$, some other generalized metallic structures on its generalized tangent bundle can be constructed as follows.

The metallic structure $J$ on $M$ induces two almost product structures on $M$:

$$F^{\pm} := \pm\left(\frac{2}{2\sigma_{p,q}-p}J - \frac{p}{2\sigma_{p,q}-p}I\right);$$

the almost product structures $F^{\pm}$ induce two generalized product structures on $TM \oplus T^*M$:

$$\hat{F}^{\pm} := \begin{pmatrix} F^{\pm} & 0 \\ 0 & (F^{\pm})^* \end{pmatrix};$$

and the generalized product structures $\hat{F}^{\pm}$ induce two generalized metallic structures on $TM \oplus T^*M$:

$$\hat{J}^{\pm}_{+,m} := \pm\frac{2\sigma_{p,q}-p}{2}\hat{F}^{+} + \frac{p}{2}I, \quad \hat{J}^{\pm}_{-,m} := \pm\frac{2\sigma_{p,q}-p}{2}\hat{F}^{-} + \frac{p}{2}I,$$

where

$$\hat{J}^{+}_{+,m} = \hat{J}^{-}_{-,m} = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}$$

and

$$\hat{J}^{-}_{+,m} = \hat{J}^{+}_{-,m} = \begin{pmatrix} -J+pI & 0 \\ 0 & -J^*+pI \end{pmatrix}.$$

The metallic structure $J$ on $M$ induces a generalized product structure on $TM \oplus T^*M$:

$$\hat{J}_p := \begin{pmatrix} J & (I-J^2)\sharp_g \\ \flat_g & -J^* \end{pmatrix},$$

and the generalized product structure $\hat{J}_p$ induces two generalized metallic structures on $TM \oplus T^*M$:

$$\hat{J}^{\pm}_{m} := \pm\frac{2\sigma_{p,q}-p}{2}\hat{J}_p + \frac{p}{2}I,$$

namely,

$$\hat{J}^{+}_{m} = \begin{pmatrix} \frac{2\sigma_{p,q}-p}{2}J + \frac{p}{2}I & -(pJ+(q-1)I)\sharp_g \\ \flat_g & -\frac{2\sigma_{p,q}-p}{2}J^* + \frac{p}{2}I \end{pmatrix}$$

and

$$\hat{J}^{-}_{m} = \begin{pmatrix} -\frac{2\sigma_{p,q}-p}{2}J + \frac{p}{2}I & -(pJ+(q-1)I)\sharp_g \\ \flat_g & \frac{2\sigma_{p,q}-p}{2}J^* + \frac{p}{2}I \end{pmatrix}.$$

2.3. **Generalized complex structure induced by** $(J, g)$**.** Let $(J, g)$ be a metallic Riemannian structure on $M$ such that $J^2 = pJ + qI$, $p, q \in \mathbb{R}$. Then $\hat{J}_c := \begin{pmatrix} J & -(I+J^2)\sharp_g \\ \flat_g & -J^* \end{pmatrix}$ is a generalized complex structure on $M$, that is, $\hat{J}_c^2 = -I$ [6].

A direct computation gives the following.

**Proposition 2.17.** *The generalized complex structure $\hat{J}_c$ induced by the metallic Riemannian structure $(J,g)$ on $M$ is $\nabla$-integrable if and only if the following conditions are satisfied:*

$$
\begin{aligned}
&N_J + (I+J^2)\sharp_g(d^\nabla g) = 0\\
&(\nabla_{JX}g)Y - (\nabla_{JY}g)X + J^*((\nabla_X g)Y - (\nabla_Y g)X) + g((\nabla_Y J)X - (\nabla_X J)Y)\\
&\quad + g(T^\nabla(X,JY) + T^\nabla(JX,Y)) = 0\\
&(d^\nabla g)((I+J^2)Y,X) + (\nabla_X J^*)g(JY) - (\nabla_{JX}J^*)g(Y) = 0\\
&(\nabla_{(I+J^2)X}J^*)g(Y) - (\nabla_{(I+J^2)Y}J^*)g(X) = 0\\
&(\nabla_{(I+J^2)X}J^2)Y - (\nabla_{(I+J^2)Y}J^2)X - T^\nabla((I+J^2)X,(I+J^2)Y)\\
&\quad - (I+J^2)\sharp_g((\nabla_{(I+J^2)X}g)Y - (\nabla_{(I+J^2)Y}g)X) = 0\\
&-(\nabla_{JX}J^2)Y + (\nabla_{(I+J^2)Y}J)X - (\nabla_X J)Y + J(\nabla_X J^2)Y - J^2(\nabla_X J)Y\\
&\quad + (I+J^2)\sharp_g((\nabla_{JX}g)Y - (\nabla_X g)JY)\\
&\quad + T^\nabla(JX,(I+J^2)Y) - JT^\nabla(X,(I+J^2)Y) = 0,
\end{aligned}
$$

*for all $X,Y \in C^\infty(TM)$, where we denoted $\flat_g$ by $g$ and the exterior differential associated to $\nabla$ acting on $g$ by $(d^\nabla g)(X,Y) := (\nabla_X g)(Y) - (\nabla_Y g)(X) + g(T^\nabla(X,Y))$.*

**Proposition 2.18.** *Let $(M,J,g)$ be a locally metallic Riemannian manifold. Then $\hat{J}_c$ is $\nabla$-integrable, for $\nabla$ the Levi-Civita connection of $g$.*

*Proof.* From the previous proposition, we have that the generalized complex structure $\hat{J}_c$ is $\nabla$-integrable if and only if the following conditions are satisfied:

$$
\begin{aligned}
&N_J = 0\\
&(\nabla_Y J)X - (\nabla_X J)Y = 0\\
&(\nabla_X J^*)J^* - (\nabla_{JX}J^*) = 0\\
&(\nabla_{(I+J^2)X}J^*)g(Y) - (\nabla_{(I+J^2)Y}J^*)g(X) = 0\\
&(\nabla_{(I+J^2)X}J^2)Y - (\nabla_{(I+J^2)Y}J^2)X = 0\\
&-(\nabla_{JX}J^2)Y + (\nabla_{(I+J^2)Y}J)X - (\nabla_X J)Y + J(\nabla_X J^2)Y - J^2(\nabla_X J)Y = 0,
\end{aligned}
$$

for all $X,Y \in C^\infty(TM)$. In particular, if $\nabla J = 0$, then $\hat{J}_c$ is $\nabla$-integrable. □

**Definition 2.19.** A generalized complex structure $\hat{J}$ on $M$ is called *calibrated* if it is $(\cdot,\cdot)$-invariant and the bilinear symmetric form defined by $(\cdot,\hat{J}\cdot)$ on $TM$ is non-degenerate and positive definite, where

$$(X+\alpha, Y+\beta) := -\frac{1}{2}(\alpha(Y) - \beta(X))$$

is the natural symplectic structure on $TM \oplus T^*M$.

*Remark* 2.20. The generalized complex structure $\hat{J}_c$ is calibrated with respect to $(\cdot,\cdot)$.

**Proposition 2.21.** *Let* $(\hat{J}_c := \begin{pmatrix} J & -(I+J^2)\sharp_g \\ \flat_g & -J^* \end{pmatrix}, \hat{g})$ *be the generalized complex structure induced by the metallic Riemannian structure* $(J, g)$ *on* $M$ *with* $\hat{g}$ *the Riemannian metric defined by* (2.1)*. Then:*

$$\hat{D}\hat{J}_c = 0 \text{ if and only if } DJ = 0 \text{ and } Dg = 0.$$

*Proof.* Remark that $(\hat{D}_Y\hat{J}_c)X = (D_YJ)X + (D_Yg)X$, for any $X, Y \in C^\infty(TM)$ and $(\hat{D}_Y\hat{J}_c)\alpha = -p(D_Y(J\sharp_g))\alpha - (q+1)(D_Y\sharp_g)\alpha - (D_YJ^*)\alpha$, for any $Y \in C^\infty(TM)$ and $\alpha \in C^\infty(T^*M)$, therefore the statement. □

**Definition 2.22.** A pair $(\hat{J}_c, \hat{J}_p)$ of a generalized complex structure and a generalized product structure is called *generalized complex product structure* if $\hat{J}_c\hat{J}_p = -\hat{J}_p\hat{J}_c$.

*Remark* 2.23. If $(J, g)$ is a metallic Riemannian structure on $M$, then $(\hat{J}_c, \hat{J}_p)$, for $\hat{J}_c := \begin{pmatrix} J & -(I+J^2)\sharp_g \\ \flat_g & -J^* \end{pmatrix}$ and $\hat{J}_p := \begin{pmatrix} J & (I-J^2)\sharp_g \\ \flat_g & -J^* \end{pmatrix}$, is a generalized complex product structure.

## 3. Metallic structures on tangent and cotangent bundles

3.1. **Metallic structure on the tangent bundle.** Let $(M, J, g)$ be a metallic Riemannian manifold and let $\nabla$ be a linear connection on $M$. $\nabla$ defines the decomposition into the horizontal and vertical subbundles of $T(TM)$:

$$T(TM) = T^H(TM) \oplus T^V(TM).$$

Let $\pi : TM \to M$ be the canonical projection and $\pi_* : T(TM) \to TM$ be the tangent map of $\pi$. If $a \in TM$ and $A \in T_a(TM)$, then $\pi_*(A) \in T_{\pi(a)}M$ and we denote by $\chi_a$ the standard identification between $T_{\pi(a)}M$ and its tangent space $T_a(T_{\pi(a)}M)$.

Let $\Psi^\nabla : TM \oplus T^*M \to T(TM)$ be the bundle morphism defined by

$$\Psi^\nabla(X+\alpha) := X_a^H + \chi_a(\sharp_g\alpha),$$

where $a \in TM$ and $X_a^H$ is the horizontal lifting of $X \in T_{\pi(a)}M$.

Let $\{x^1, \ldots, x^n\}$ be local coordinates on $M$, let $\{\tilde{x}^1, \ldots, \tilde{x}^n, y^1, \ldots, y^n\}$ be respectively the corresponding local coordinates on $TM$, and let $\left\{X_1, \ldots, X_n, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\}$ be a local frame on $T(TM)$, where $X_i = \frac{\partial}{\partial \tilde{x}^i}$. We have:

$$X_i^H = X_i - y^k\Gamma_{ik}^l \frac{\partial}{\partial y^l}$$

$$X_i^V = y^k\Gamma_{ik}^l \frac{\partial}{\partial y^l}$$

$$\left(\frac{\partial}{\partial y^i}\right)^H = 0$$

$$\left(\frac{\partial}{\partial y^i}\right)^V = \frac{\partial}{\partial y^i},$$

where $i, k, l$ run from 1 to $n$ and $\Gamma^k_{il}$ are the Christoffel symbols of $\nabla$.

Let $\Psi^\nabla : TM \oplus T^*M \to T(TM)$ be the bundle morphism defined before (which is an isomorphism on the fibres). In local coordinates, we have the following expressions:

$$\Psi^\nabla\left(\frac{\partial}{\partial x^i}\right) = X_i^H$$

$$\Psi^\nabla\left(dx^j\right) = g^{jk}\frac{\partial}{\partial y^k}.$$

Let $(\hat{J}_m, \hat{g})$ be the generalized metallic structure defined in the previous section. The isomorphism $\Psi^\nabla$ allows us to construct a natural metallic structure $\bar{J}_m$ and a natural Riemannian metric $\bar{g}$ on $TM$ in the following way.

We define $\bar{J}_m : T(TM) \to T(TM)$ by

$$\bar{J}_m := (\Psi^\nabla) \circ \hat{J}_m \circ (\Psi^\nabla)^{-1}$$

and the Riemannian metric $\bar{g}$ on $TM$ by

$$\bar{g} := ((\Psi^\nabla)^{-1})^*(\hat{g}).$$

**Proposition 3.1.** *$(TM, \bar{J}_m, \bar{g})$ is a metallic Riemannian manifold.*

*Proof.* From the definition it follows that $\bar{J}_m^2 = p\bar{J}_m + qI$ and $\bar{g}(\bar{J}_m X, Y) = \bar{g}(X, \bar{J}_m Y)$, for any $X, Y \in C^\infty(T(TM))$. □

In local coordinates, we have the following expressions for $\bar{J}_m$ and $\bar{g}$:

$$\begin{cases} \bar{J}_m\left(X_i^H\right) = J_i^k X_k^H \\ \bar{J}_m\left(\dfrac{\partial}{\partial y^j}\right) = g_{ji}J_k^i g^{kl}\dfrac{\partial}{\partial y^l} = J_j^k\dfrac{\partial}{\partial y^k} \end{cases}$$

$$\begin{cases} \bar{g}\left(X_i^H, X_j^H\right) = g_{ij} \\ \bar{g}\left(X_i^H, \dfrac{\partial}{\partial y^j}\right) = 0 \\ \bar{g}\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right) = g_{ij}. \end{cases}$$

Moreover,

$$\bar{J}_m\left(X_i\right) = J_i^k X_k - y^l(J_i^k\Gamma^s_{kl} - J_r^s\Gamma^r_{il})\frac{\partial}{\partial y^s}$$

$$\begin{cases} \bar{g}\left(X_i, X_j\right) = g_{ij} + y^k y^h \Gamma^l_{ik}\Gamma^s_{jh}g_{hk} \\ \bar{g}\left(X_i, \dfrac{\partial}{\partial y^j}\right) = y^k\Gamma^l_{ik}g_{lj}. \end{cases}$$

Computing the Nijenhuis tensor of $\bar{J}_m$, we get the following:

$$N_{\bar{J}_m}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0$$

$$N_{\bar{J}_m}\left(X_i^H, \frac{\partial}{\partial y^j}\right) = \left(\left(\nabla_{JX_i} J\right) X_j - J\left(\nabla_{X_i} J\right) X_j\right)^k \frac{\partial}{\partial y^k}$$

$$N_{\bar{J}_m}\left(X_i^H, X_j^H\right) = \left(N_J\left(X_i, X_j\right)\right)^k X_k^H$$
$$- y^s\left(J_i^k J_j^h R_{khs}^r - J_l^r J_i^k R_{kjs}^l - J_j^h J_l^r R_{ihs}^l + pJ_l^r R_{ijs}^l + qR_{ijs}^r\right) \frac{\partial}{\partial y^r}.$$

Therefore we can state the following.

**Proposition 3.2.** *Let $(M, J, g)$ be a flat locally metallic Riemannian manifold. If $\nabla$ is the Levi-Civita connection of $g$, then $(\bar{J}_m, \bar{g})$ is an integrable metallic Riemannian structure on $TM$.*

3.2. **Metallic structure on the cotangent bundle.** Let $(M, J, g)$ be a metallic Riemannian manifold and let $\nabla$ be a linear connection on $M$. $\nabla$ defines the decomposition into the horizontal and vertical subbundles of $T(T^*M)$:

$$T(T^*M) = T^H(T^*M) \oplus T^V(T^*M).$$

Let $\pi : T^*M \to M$ be the canonical projection and $\pi_* : T(T^*M) \to TM$ be the tangent map of $\pi$. If $a \in T^*M$ and $A \in T_a(T^*M)$, then $\pi_*(A) \in T_{\pi(a)}M$ and we denote by $\chi_a$ the standard identification between $T^*_{\pi(a)}M$ and its tangent space $T_a(T^*_{\pi(a)}M)$.

Let $\Phi^\nabla : TM \oplus T^*M \to T(T^*M)$ be the bundle morphism defined by [5]:

$$\Phi^\nabla(X + \alpha) := X_a^H + \chi_a(\alpha),$$

where $a \in T^*M$ and $X_a^H$ is the horizontal lifting of $X \in T_{\pi(a)}M$.

Let $\left\{x^1, \ldots, x^n\right\}$ be local coordinates on $M$, let $\left\{\tilde{x}^1, \ldots, \tilde{x}^n, y_1, \ldots, y_n\right\}$ be respectively the corresponding local coordinates on $T^*M$ and let $\{X_1, \ldots, X_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$ be a local frame on $T(T^*M)$, where $X_i = \frac{\partial}{\partial \tilde{x}^i}$. We have:

$$X_i^H = X_i + y_k \Gamma_{il}^k \frac{\partial}{\partial y_l}$$

$$X_i^V = -y_k \Gamma_{il}^k \frac{\partial}{\partial y_l}$$

$$\left(\frac{\partial}{\partial y_i}\right)^H = 0$$

$$\left(\frac{\partial}{\partial y_i}\right)^V = \frac{\partial}{\partial y_i},$$

where $i, k, l$ run from 1 to $n$ and $\Gamma_{il}^k$ are the Christoffel symbols of $\nabla$.

Let $\Phi^{\nabla} : TM \oplus T^*M \to T(T^*M)$ be the bundle morphism defined before (which is an isomorphism on the fibres). In local coordinates, we have the following expressions:

$$\Phi^{\nabla}\left(\frac{\partial}{\partial x^i}\right) = X_i^H$$
$$\Phi^{\nabla}\left(dx^j\right) = \frac{\partial}{\partial y_j}.$$

Let $(\hat{J}_m, \hat{g})$ be the generalized metallic structure defined in the previous section. The isomorphism $\Phi^{\nabla}$ allows us to construct a natural metallic structure $\tilde{J}_m$ and a natural Riemannian metric $\tilde{g}$ on $T^*M$ in the following way.

We define $\tilde{J}_m : T(T^*M) \to T(T^*M)$ by

$$\tilde{J}_m := (\Phi^{\nabla}) \circ \hat{J}_m \circ (\Phi^{\nabla})^{-1}$$

and the Riemannian metric $\tilde{g}$ on $T^*M$ by

$$\tilde{g} := ((\Phi^{\nabla})^{-1})^*(\hat{g}).$$

**Proposition 3.3.** *$(T^*M, \tilde{J}_m, \tilde{g})$ is a metallic Riemannian manifold.*

*Proof.* From the definition it follows that $\tilde{J}_m^2 = p\tilde{J}_m + qI$ and $\tilde{g}(\tilde{J}_m X, Y) = \tilde{g}(X, \tilde{J}_m Y)$, for any $X, Y \in C^\infty(T(T^*M))$. □

In local coordinates, we have the following expressions for $\tilde{J}_m$ and $\tilde{g}$:

$$\begin{cases} \tilde{J}_m\left(X_i^H\right) = J_i^k X_k^H \\ \tilde{J}_m\left(\dfrac{\partial}{\partial y_j}\right) = J_k^j \dfrac{\partial}{\partial y_k} \end{cases}$$

$$\begin{cases} \tilde{g}\left(X_i^H, X_j^H\right) = g_{ij} \\ \tilde{g}\left(X_i^H, \dfrac{\partial}{\partial y_j}\right) = 0 \\ \tilde{g}\left(\dfrac{\partial}{\partial y_i}, \dfrac{\partial}{\partial y_j}\right) = g^{ij}. \end{cases}$$

Moreover,

$$\tilde{J}_m\left(X_i\right) = J_i^k X_k + y_l(J_i^k \Gamma_{kr}^l - J_r^s \Gamma_{is}^l)\frac{\partial}{\partial y_r}$$

$$\begin{cases} \tilde{g}\left(X_i, X_j\right) = g_{ij} + y_k y_h \Gamma_{il}^k \Gamma_{jr}^h g^{lr} \\ \tilde{g}\left(X_i, \dfrac{\partial}{\partial y_j}\right) = -y_k \Gamma_{il}^k g^{lj}. \end{cases}$$

Computing the Nijenhuis tensor of $\tilde{J}_m$, we get the following:

$$N_{\tilde{J}_m}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = 0$$

$$N_{\tilde{J}_m}\left(X_i^H, \frac{\partial}{\partial y_j}\right) = \left(\left(\nabla_{JX_i} J\right) X_k - J\left(\nabla_{X_i} J\right) X_k\right)^j \frac{\partial}{\partial y_k}$$

$$\begin{aligned} N_{\tilde{J}_m}\left(X_i^H, X_j^H\right) &= \left(N_J\left(X_i, X_j\right)\right)^k X_k^H \\ &\quad + y_l\left(J_i^k J_j^h R_{khs}^l - J_s^r J_i^k R_{kjr}^l - J_s^r J_j^k R_{ikr}^l + pJ_s^k R_{ijk}^l + qR_{ijs}^l\right) \frac{\partial}{\partial y_s}. \end{aligned}$$

Therefore we can state the following.

**Proposition 3.4.** *Let $(M, J, g)$ be a flat locally metallic Riemannian manifold. If $\nabla$ is the Levi-Civita connection of $g$, then $(\tilde{J}_m, \tilde{g})$ is an integrable metallic Riemannian structure on $T^*M$.*

*Remark* 3.5. The metallic structures $\bar{J}_m$ and $\tilde{J}_m$ on the tangent and cotangent bundles respectively, satisfy:

$$\bar{J}_m \circ \left(\Psi^\nabla \circ \left(\Phi^\nabla\right)^{-1}\right) = \left(\Psi^\nabla \circ \left(\Phi^\nabla\right)^{-1}\right) \circ \tilde{J}_m.$$

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