SECOND COHOMOLOGY SPACE OF $\mathfrak{sl}(2)$ ACTING ON THE SPACE OF BILINEAR BIDIFFERENTIAL OPERATORS

IMED BASDOURI, SARRA HAMMAMI, AND OLFA MESSAOUD

Abstract. We consider the $\mathfrak{sl}(2)$-module structure on the spaces of bilinear bidifferential operators acting on the spaces of weighted densities. We compute the second cohomology group of the Lie algebra $\mathfrak{sl}(2)$ with coefficients in the space of bilinear bidifferential operators that act on tensor densities $D_{\lambda,\nu,\mu}$.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra and $M$ a $\mathfrak{g}$-module. We shall associate a cochain complex known as the Chevalley–Eilenberg differential. The $n$-th space of this complex will be denoted by $C^n(\mathfrak{g}, M)$.

For $n > 0$, it is the space of $n$-linear antisymmetric mappings of $\mathfrak{g}$ into $M$: they will be called $n$-cochains of $\mathfrak{g}$ with coefficients in $M$. The space of 0-cochains $C^0(\mathfrak{g}, M)$ reduces to $M$. The differential $\delta^n$ will be defined by the following formula:

$$\delta^n c(g_1, \ldots, g_{n+1}) = \sum_{1 \leq s < t \leq n+1} (-1)^{s+t-1} c([g_s, g_t], g_1, \ldots, \hat{g}_s, \ldots, \hat{g}_t, \ldots, g_{n+1}) + \sum_{1 \leq s \leq n+1} (-1)^s g_s c(g_1, \ldots, \hat{g}_s, \ldots, g_{n+1});$$

the notation $\hat{g}_i$ indicates that the $i$-th term is omitted.

We check that $\delta^{n+1} \circ \delta^n = 0$, so we have a complex:

$$0 \rightarrow C^0(\mathfrak{g}, M) \rightarrow \cdots \rightarrow C^{n-1}(\mathfrak{g}, M) \xrightarrow{\delta^{n-1}} C^n(\mathfrak{g}, M) \rightarrow \cdots$$

We denote by $H^n(\mathfrak{g}, M) = \ker \delta^n / \operatorname{Im} \delta^{n-1}$ the quotient space. This space is called the space of $n$-cohomology of $\mathfrak{g}$ with coefficients in $M$. We denote by:

$$Z^n(\mathfrak{g}, M) = \ker \delta_n:$$ the space of $n$-cocycles;

$$B^n(\mathfrak{g}, M) = \operatorname{Im} \delta_{n-1}:$$ the space of $n$-coboundaries.

For $M = \mathbb{R}$ (or $\mathbb{C}$) considered as a trivial module, we denote the cohomologies, in this case, by $H^n(\mathfrak{g})$.

We shall now recall classical interpretations of cohomology spaces of low degrees.

2010 Mathematics Subject Classification. 53D55, 14F10, 17B10, 17B68.

Key words and phrases. Invariant operators; cohomology; superalgebras.
• The space \( H^0(\mathfrak{g}, M) \simeq \text{Inv}_g(M) := \{ m \in M : \forall X \in \mathfrak{g}, X.m = 0 \} \).

• The space \( H^1(\mathfrak{g}, M) \) classifies derivations of \( \mathfrak{g} \) with values in \( M \) modulo inner ones (see [1]). This result is particularly useful when \( M = \mathfrak{g} \) with the adjoint representation. In this case, a derivation is a map \( \varrho : \mathfrak{g} \rightarrow \mathfrak{g} \) such that
\[
\varrho([X, Y]) - [\varrho(X), Y] - [X, \varrho(Y)] = 0,
\]
while an inner derivation is given by the adjoint action of some element \( Z \in \mathfrak{g} \).

○ If \( M = \text{Hom}(\mathcal{N}, \mathcal{M}) \), the nontrivial extensions of \( \mathfrak{g} \)-modules are classified by the first cohomology group \( H^1(\mathfrak{g}, \text{Hom}(\mathcal{N}, \mathcal{M})) \) (see e.g. [4, 5]). Any 1-cocycle \( \Upsilon \) generates a new action on \( \mathcal{M} \oplus \mathcal{N} \) as follows: for all \( g \in \mathfrak{g} \) and for all \( (\phi, \varphi) \in \mathcal{M} \oplus \mathcal{N} \), we define
\[
g^*(\phi, \varphi) := (g^*\phi + \Upsilon(\varphi), g^*\varphi).
\]

○ Let \( \rho_0 : \mathfrak{g} \rightarrow \text{End}(V) \) be an action of a Lie algebra \( \mathfrak{g} \) on a vector space \( V \). It is well known that the first cohomology space \( H^1(\mathfrak{g}; \text{End}(V)) \) determines and classifies infinitesimal deformations up to equivalence. Thus, if \( \dim H^1(\mathfrak{g}; \text{End}(V)) = m \), then choose 1-cocycles \( \Upsilon_1, \ldots, \Upsilon_m \) representing a basis of \( H^1(\mathfrak{g}; \text{End}(V)) \) and consider the infinitesimal deformation
\[
\rho = \rho_0 + \sum_{i=1}^m t_i \Upsilon_i,
\]
where \( t_1, \ldots, t_m \) are independent parameters.

• The space \( H^2(\mathfrak{g}, M) \) classifies central extensions of \( \mathfrak{g} \) by \( M \) (see [6, 7]), i.e. short exact sequences of Lie algebras
\[
0 \rightarrow M \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0
\]
in which \( M \) is considered as an abelian Lie algebra. We shall mainly consider two particular cases of this situation which will be extensively studied in the sequel:

○ If \( M \) is a trivial \( \mathfrak{g} \)-module (typically \( M = \mathbb{R} \) or \( \mathbb{C} \)), \( H^2(\mathfrak{g}, M) \) classifies central extensions modulo trivial ones. Recall that a central extension of \( \mathfrak{g} \) by \( \mathbb{R} \) produces a new Lie bracket on \( \hat{\mathfrak{g}} = \mathfrak{g} \oplus M \) by setting
\[
[(X, \lambda), (Y, \mu)] = ([X, Y], c(X, Y)).
\]

It is trivial if the cocycle \( c = dl \) is a coboundary of a 1-cochain \( l \), in which case the map \( (X, \lambda) \rightarrow (X, \lambda - l(X)) \) yields a Lie isomorphism between \( \hat{\mathfrak{g}} \) and \( \mathfrak{g} \oplus \mathbb{R} \) considered as a direct sum of Lie algebras.

○ If \( M = \mathfrak{g} \) with the adjoint representation, then \( H^2(\mathfrak{g}, \mathfrak{g}) \) classifies infinitesimal deformations modulo trivial ones. By definition, a (formal) series
\[
(X, Y) \rightarrow \Phi_\lambda(X, Y) := [X, Y] + \lambda f_1(X, Y) + \lambda^2 f_2(X, Y) + \cdots \tag{1.1}
\]
is a deformation of Lie bracket \([,] \) if \( \Phi_\lambda \) is a Lie bracket for every \( \lambda \), i.e. it is an antisymmetric bilinear form in \( X, Y \) and satisfies Jacobi’s identity. If one sets simply
\[
[X, Y]_\lambda = [X, Y] + \lambda c(X, Y), \tag{1.2}
\]
c being a 2-cochain with values in \( \mathfrak{g} \) and \( \lambda \) being a scalar, then this bracket satisfies Jacobi’s identity modulo terms of order \( O(\lambda^2) \) if and only if \( c \) is a 2-cocycle.
Let $\text{Vect}(\mathbb{R})$ be the Lie algebra of all vector fields $X_h = h \frac{d}{dx}$, where $h \in C^\infty(\mathbb{R})$ on $\mathbb{R}$. Consider the 1-parameter deformation of the $\text{Vect}(\mathbb{R})$ action on $C^\infty(\mathbb{R})$: 

$$L^\lambda_{X_h}(f) = h f' + \lambda h f,$$

where $f'$, $h'$ are respectively $\frac{df}{dx}$, $\frac{dh}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\text{Vect}(\mathbb{R})$-module structure on $C^\infty(\mathbb{R})$ defined by $L^\lambda$ for a fixed $\lambda$.

Each bilinear bidifferential operator $A$ on $\mathbb{R}$ gives thus rise to a morphism from $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ to $\mathcal{F}_\mu$, for any $\lambda, \nu, \mu \in \mathbb{R}$, by $f dx^\lambda \otimes g dx^\nu \mapsto A(f \otimes g) dx^\mu$, 

$$A(f dx^\lambda \otimes g dx^\nu) = \sum_{k=0}^{m} \sum_{i+j=k} a_{i,j} f^i g^j dx^\mu,$$

where the coefficients $a_{i,j}$ are constants.

The Lie algebra $\text{Vect}(\mathbb{R})$ acts on the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$ as follows:

$$X_h.A = L^\mu_{X_h} \circ A - A \circ L^{(\lambda,\nu)}_{X_h},$$

(1.3)

where $L^{(\lambda,\nu)}_{X_h}$ is the Lie derivative on $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ defined by the Leibniz rule:

$$L^{(\lambda,\nu)}_{X_h}(f \otimes g) = L^\lambda_{X_h}(f) \otimes g + f \otimes L^\nu_{X_h}(g).$$

If we restrict ourselves to the Lie algebra $\mathfrak{sl}(2)$, which is isomorphic to the Lie subalgebra of $\text{Vect}(\mathbb{R})$ spanned by

$$\{X_1, X_x, X_{x^2}\},$$

we have a family of infinite dimensional $\mathfrak{sl}(2)$-modules still denoted by $\mathcal{D}_{\lambda,\nu,\mu}$. Bouarroudj, in [5], computes the cohomology space $H^1_{\text{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu})$, where $H^1_{\text{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered (see e.g. [6]). In this paper we compute the second cohomology space $H^2_{\text{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu})$ of the Lie algebra $\mathfrak{sl}(2)$ with coefficients in the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$. Moreover, we give explicit formulae for non trivial 2-cocycles which generate these spaces.

2. $\text{Vect}(\mathbb{R})$-module structures on the space of bilinear bidifferential operators

The Lie algebra $\mathfrak{sl}(2)$ is realized as subalgebra of the Lie algebra $\text{Vect}(\mathbb{R})$,

$$\mathfrak{sl}(2) = \text{Span} \left( X_1 = \frac{d}{dx}, X_x = x \frac{d}{dx}, X_{x^2} = x^2 \frac{d}{dx} \right),$$

(2.1)

corresponding to the fraction-linear transformations

$$x \mapsto \frac{ax + b}{cx + d}, \quad ad - bc = 1.$$

A projective structure on $\mathbb{R}$ (or $S^1$) is given by an atlas with fraction-linear coordinate transformations (in other words, by an atlas such that the $\mathfrak{sl}(2)$-action (2.1) is well-defined).
The commutation relations are

\[[X_1, X_x] = X_1, \quad [X_x, X_x] = 0, \quad [X_1, X_1] = 0,\]
\[[X_1, X_{x^2}] = 2X_x, \quad [X_x, X_{x^2}] = X_{x^2}, \quad [X_{x^2}, X_{x^2}] = 0.\]

2.1. The space of tensor densities on \(\mathbb{R}\). Let \(\text{Vect}(\mathbb{R})\) be the Lie algebra of vector fields on \(\mathbb{R}\). Consider the 1-parameter deformation of the \(\text{Vect}(\mathbb{R})\) action on \(C^\infty(\mathbb{R})\):

\[L^\lambda_{X_h}(f) = hf' + \lambda h' f,\]

where \(f', h'\) are respectively \(\frac{df}{dx}\), \(\frac{dh}{dx}\). Denote by \(\mathcal{F}_\lambda\) the \(\text{Vect}(\mathbb{R})\)-module structure on \(C^\infty(\mathbb{R})\) defined by \(L^\lambda\) for a fixed \(\lambda\). Geometrically, \(\mathcal{F}_\lambda = \{ fdx^\lambda : f \in C^\infty(\mathbb{R}) \}\) is the space of weighted densities of weight \(\lambda \in \mathbb{R}\), so its elements can be represented as \(f(x)dx^\lambda\), where \(f(x)\) is a function and \(dx^\lambda\) is a formal (for the time being) symbol. This space coincides with the space of vector fields, functions, and differential forms for \(\lambda = -1, 0,\) and 1, respectively.

The space \(\mathcal{F}_\lambda\) is a \(\text{Vect}(\mathbb{R})\)-module for the action defined by

\[L^\lambda_{g \frac{d}{dx}}(fdx^\lambda) = (gf' + \lambda g' f)dx^\lambda.\]  

(2.2)

2.2. The space of bilinear bidifferential operators as a \(\text{Vect}(\mathbb{R})\)-module. We are interested in defining a cohomology of the Lie algebra \(\text{Vect}(\mathbb{R})\) with coefficients in the space of bilinear bidifferential operators \(D^\lambda_{\nu,\mu}\). The counterpart \(\text{Vect}(\mathbb{R})\)-modules of the space of linear differential operators is a classical object (see e.g. [9]).

Consider bilinear bidifferential operators that act on tensor densities:

\[A : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \longrightarrow \mathcal{F}_\mu.\]  

(2.3)

The Lie algebra \(\text{Vect}(\mathbb{R})\) acts on the space of bilinear bidifferential operators as follows. For all \(\phi \in \mathcal{F}_\lambda\) and for all \(\psi \in \mathcal{F}_\nu\),

\[L^\lambda_{X}A(\phi, \psi) = L^\mu_X \circ A(\phi, \psi) - A(L^\lambda_X(\phi), \psi) - A(\phi, L^\nu_X(\psi)),\]  

(2.4)

where \(L^\lambda_X\) is the action (2.2). We denote by \(D^\lambda_{\nu,\mu}\) the space of bilinear bidifferential operators (2.3) endowed with the defined \(\text{Vect}(\mathbb{R})\)-module structure (2.4).

3. The second differentiable cohomology space of \(\mathfrak{sl}(2)\) acting on \(D^\lambda_{\nu,\mu}\)

In this section, we investigate the second space differentiable cohomology of the Lie algebra \(\mathfrak{sl}(2)\) with coefficients in the space of bilinear bidifferential operators that act on tensor densities \(D^\lambda_{\nu,\mu}\). Following Sofiane Bouarroudj, we give explicit expressions of the basis cocycles. Namely, we consider only cochains that are given by differentiable maps.
3.1. The main theorem.

**Theorem 3.1.** The second differentiable cohomology space of the $\mathfrak{sl}(2)$-module $D_{\lambda,\nu,\mu}$ has the following structure:

1. If $\mu - \lambda - \nu = 0$, then
   $$H^2(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) = \mathbb{R}.$$

2. If $\mu - \lambda - \nu = k$, where $k$ is a positive integer, then
   $$H^2(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \simeq \begin{cases} \mathbb{R}^4, & \text{if } (\lambda, \mu) = (-\frac{s}{2}, -\frac{t}{2}), \text{where } 0 \leq s, k - s - 2 < t \leq k - 1; \\ \mathbb{R}, & \text{otherwise}. \end{cases}$$

3. If $\mu - \lambda - \nu = k$, where $k$ is not a positive integer, then
   $$H^2(\mathfrak{sl}(2), D_{\lambda,\nu,\mu}) \simeq \mathbb{0}.$$

Before proving the theorem, we are required to prove the following two lemmas.

**Lemma 3.2.** Let $C: \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \rightarrow \mathcal{F}_\mu$ be a bilinear bidifferential operator defined as follows: for all $\phi \in \mathcal{F}_\lambda$ and for all $\psi \in \mathcal{F}_\nu$,

$$C(\phi \otimes \psi) = \sum_{i+j=k} a_{i,j} (XY' - X'Y) \phi^{(i)} \psi^{(j)} + \sum_{i+j = k-1} b_{i,j} (X'Y'' - X''Y) \phi^{(i)} \psi^{(j)} + \sum_{i+j = k-2} c_{i,j} (X''Y' - X'Y'') \phi^{(i)} \psi^{(j)},$$

where the superscript $'$ stands for $\frac{d}{dx}$ and $a_{i,j}$, $b_{i,j}$, and $c_{i,j}$ are constants, and let the 2-cocycle condition read as follows: for all vector fields $X \frac{d}{dx}$, $Y \frac{d}{dx}$, and $Z \frac{d}{dx}$ in $\mathfrak{sl}(2)$,

$$\delta C(\phi \otimes \psi) = \left( L_{X}^{\lambda,\nu,\mu} C \left( X \frac{d}{dx}, Y \frac{d}{dx} \right) - L_{Y}^{\lambda,\nu,\mu} C \left( X \frac{d}{dx}, Z \frac{d}{dx} \right) \right) \left( \frac{d}{dx} \phi \otimes \psi \right)$$

$$- \left( C \left( \left[ X \frac{d}{dx}, Y \frac{d}{dx} \right], Z \frac{d}{dx} \right) + C \left( \left[ X \frac{d}{dx}, Z \frac{d}{dx} \right], Y \frac{d}{dx} \right) \right) \phi \otimes \psi = 0.$$

Then we have

$$\delta C(\phi \otimes \psi) = \frac{1}{2} \sum_{i+j = k-1} \left( (i + 1)(i + 2\lambda) a_{i+1,j} + (j + 1)(j + 2\nu) a_{i,j+1} \right)$$

$$+ (\mu - \lambda - \nu - i - j) b_{i,j} \phi^{(i)} \psi^{(j)}.$$  

(3.1)
Proof. Straightforward computation using the definition (2.2). □

Lemma 3.3. Let \( b : \mathcal{F}_\lambda \otimes \mathcal{F}_\upsilon \rightarrow \mathcal{F}_\mu \) be a bilinear bidifferential operator defined as follows. For all \( \phi \in \mathcal{F}_\lambda \) and for all \( \psi \in \mathcal{F}_\upsilon \):

\[
b \left( \frac{d}{dx} \right) (\phi \otimes \psi) = \sum_{i+j=k} \alpha_{i,j} X^{(i)} \psi^{(j)} + \sum_{i+j=k-1} \beta_{i,j} X' \phi^{(i)} \psi^{(j)},
\]

(3.2)

where \( \alpha_{i,j}, \beta_{i,j} \) are constants. For all \( X \frac{d}{dx}, Y \frac{d}{dx} \in \mathfrak{sl}(2) \), we have

\[
\delta b(\phi \otimes \psi) = \frac{1}{2} \sum_{i+j=k-1} (XY'' - X''Y) \\
\times ((i+1)(i+2\lambda)\alpha_{i+1,j} + (j+1)(j+2\upsilon)\alpha_{i,j+1}) \phi^{(i)} \psi^{(j)} \\
+ \frac{1}{2} \sum_{i+j=k-2} (X'Y'' - X''Y') \\
\times ((i+1)(i+2\lambda)\beta_{i+1,j} + (j+1)(j+2\upsilon)\beta_{i,j+1}) \phi^{(i)} \psi^{(j)}.
\]

(3.3)

Proof. Straightforward computation using the definition (2.2). □

3.2. Proof of Theorem 3.1. Using Lemma 3.3 for all \( X \frac{d}{dx}, Y \frac{d}{dx} \in \mathfrak{sl}(2) \), \( \phi \in \mathcal{F}_\lambda \), and \( \psi \in \mathcal{F}_\upsilon \), we prove that the coefficient of the component \( \phi^{(i)} \psi^{(j)} \) in the 2-cocycle condition above is equal to

\[
\frac{1}{2} ((i+1)(i+2\lambda)a_{i+1,j} + (j+1)(j+2\upsilon)a_{i,j+1} + (\mu - \lambda - \upsilon - i - j)b_{i,j}) \phi^{(i)} \psi^{(j)}. \tag{3.4}
\]

The annihilation of the 2-cocycle condition requires the annihilation of the formula (3.4). So we have

\[
(i+1)(i+2\lambda)a_{i+1,j} + (j+1)(j+2\upsilon)a_{i,j+1} + (\mu - \lambda - \upsilon - i - j)b_{i,j} = 0. \tag{3.5}
\]

We distinguish many cases:

- For \( \mu - \lambda - \upsilon = 0 \), the 2-cocycle on \( \mathfrak{sl}(2) \) has the following form:

\[
C \left( \frac{d}{dx} \right) (\phi \otimes \psi) = a(XY' - X'Y) \phi \psi,
\]

where \( X \frac{d}{dx} \in \mathfrak{sl}(2) \), \( \phi \in \mathcal{F}_\lambda \), \( \psi \in \mathcal{F}_\upsilon \), and \( a \) is a constant. The 2-cocycle condition is proved by a direct computation:

\[
\delta C \left( \frac{d}{dx} \right) (\phi \otimes \psi) = 0.
\]

Thus the space \( Z^2(\mathfrak{sl}(2), D_{\lambda,\upsilon,\mu}) \) is one-dimensional. Now we are going to study the triviality of the general cocycle (3.2). Every trivial 2-cocycle of \( \mathfrak{sl}(2) \) in \( D_{\lambda,\upsilon,\lambda+\upsilon} \) must be of the form \( \delta Q \), where \( Q \) is an element of \( D_{\lambda,\upsilon,\lambda+\upsilon} \) defined as follows:

\[
Q \left( \frac{d}{dx} \right) (\phi \otimes \psi) = X\alpha \phi \psi + X'\beta \phi \psi.
\]
where $\alpha$ and $\beta$ are constants. We have

$$\delta Q\left(\frac{d}{dx}Y, \frac{d}{dx}\right)(\phi, \psi)$$

$$= L^\lambda_{Y,\nu,\lambda}Q\left(\frac{d}{dx}\right)(\phi, \psi) - L^\lambda_{X,\nu,\lambda}Q\left(\frac{d}{dx}\right)(\phi, \psi)$$

$$- Q\left(\left[\frac{d}{dx}, \frac{d}{dx}\right]\right)(\phi, \psi),$$

After a direct computation, the result will be $\delta Q\left(\frac{d}{dx}X, \frac{d}{dx}\right)(\phi, \psi) = 0$; then $\delta Q\left(\frac{d}{dx}X, \frac{d}{dx}\right)(\phi, \psi) \neq C\left(\frac{d}{dx}X, \frac{d}{dx}\right)(\phi, \psi)$ shows that the general cocycle (3.2) cannot be ultimately trivial. Therefore the coboundary space $B^2(sl(2), D_{\lambda,\nu,\mu})$ vanishes. As a consequence,

$$H^2(sl(2), D_{\lambda,\nu,\lambda+\nu}) = Z^2(sl(2), D_{\lambda,\nu,\lambda+\nu}).$$

- For $\mu - \lambda - \nu = k$, where $k$ is a positive integer:
  1. If $\lambda \neq \frac{s}{2}$ and $\nu \neq \frac{t}{2}$, where $s, t \in \{0, \ldots, k - 1\}$, then the space of solutions of the system (3.5) is one-dimensional, generated by $a_{0,k}$. Indeed, in that case $(i+1)(i+2\lambda) \neq 0$ and $(j+1)(j+2\nu) \neq 0$; therefore the system (3.4) is equivalent to

$$a_{i+1,j} = -\frac{(j+1)(j+2\nu)}{(i+1)(i+2\lambda)}a_{i,j+1},$$

where $i + j = k - 1$. By iterations, we get

$$a_{1,k-1} = -\frac{k(k-1+2\nu)}{2\lambda}a_{0,k} = -C^1_k(k-1+2\nu)a_{0,k},$$

$$a_{2,k-2} = -\frac{(k-1)(k-2+2\nu)}{1+2\lambda}a_{1,k-1} = C^2_k(k-1+2\nu)(k-2+2\nu)\frac{2\lambda}{2\lambda(1+2\lambda)}a_{0,k},$$

$$\vdots$$

$$a_{i,k-i} = (-1)^{i+1}C^i_k\frac{(k-i+1+2\nu)(k-i+2+2\nu)\cdots(k-1+2\nu)}{(i-1+2\lambda)(i-2+2\lambda)\cdots2\lambda}a_{0,k}.$$

Now, we show how the constants $b_{i,j}$ and $c_{i,j}$ can be eliminated from our initial 2-cocycle (3.2). We add the coboundary $\delta b$ of the equation (3.3) of our 2-cocycle (3.1). The constants $\alpha_{i,j}$ and $\beta_{i,j}$ are chosen such that

$$\begin{cases} b_{i,j} = -\frac{1}{2}((i+1)(i+2\lambda)\alpha_{i+1,j} + (j+1)(j+2\nu)\alpha_{i,j+1}), \\ c_{i,j} = -\frac{1}{2}((i+1)(i+2\lambda)\beta_{i+1,j} + (j+1)(j+2\nu)\beta_{i,j+1}). \end{cases}$$
Thus, the cohomology group in question is one-dimensional, generated by the 2-cocycle

\[ C \left( X \frac{d}{dx}, Y \frac{d}{dx} \right) \phi \psi(k) \]

\[ = (XY' - X'Y) \phi \psi(k) \]

\[ + \sum_{i+j=k-1} (-1)^{(i+1)} C_k^{(i+1)} (XY' - X'Y) \]

\[ \times \frac{(k-i+2v)(k-i+1+2v)(k-i+2+2v) \cdots (k-1+2v)}{(i-1+2\lambda)(i-2+2\lambda) \cdots 2\lambda} \]

\[ \times \phi^{(i+1)} \psi(j) \].

(2) If \( \lambda \neq \frac{-s}{2} \) and \( v = \frac{-t}{2} \), where \( s, t \in \{0, \ldots, k-1\} \), then the constants \( a_{k-t,k}, a_{k-t+1,t-1}, \ldots, a_{k,0} \) are zero, and the space of solutions of the system (3.5) is one-dimensional, generated by \( a_{0,k} \). Two cases should be studied:

(a) If \( j \leq t \):

- For \( j = t \), we have \((j + 1)(j + 2v) = 0\). So,

\[ (k-t)(k-t-1+2v)a_{k-t,t} = 0. \]

We have \( \lambda \neq \frac{-s}{2} \), for all \( s \in \{0, \ldots, k-1\} \), then \((i+2\lambda) \neq 0\).

Thus \( a_{k-t,t} = 0 \).

- For \( j \in \{0, \ldots, t-1\} \), we have \((j + 1)(j + 2v) \neq 0\).

So, \( a_{k-t+1,t-1} = - \frac{(t-1)(t+1+2v)}{(k-t+1)(k-1+2\lambda)} a_{k-t,t} = 0 \).

Thus, \( a_{k-t+2,t-2} = - \frac{(t-1)(t-2+2v)}{(k-t+2)(k-2+2\lambda)} a_{k-t+1,t-1} = 0 \).

\[ \vdots \]

Finally, \( a_{k,0} = 0 \).

(b) If \( j > t \), then

\[ a_{i+1,j} = - \frac{(j + 1)(j + 2v)}{(i+1)(i+2\lambda)} a_{i,j+1}, \]

where \( i + j = k - 1 \). By iterations, we get

\[ a_{1,k-1} = -C_k^{(1)} \frac{(k-1+2v)}{2\lambda} a_{0,k}, \]

\[ a_{2,k-2} = C_k^{(2)} \frac{(k-1+2v)(k-2+2v)}{2\lambda(1+2\lambda)} a_{0,k}, \]

\[ \vdots \]

\[ a_{,k-1} = (-1)^{i+1} C_k^{(i+1)} \frac{(k-i+2v)(k-i+1+2v)(k-i+2+2v) \cdots (k-1+2v)}{(i-1+2\lambda)(i-2+2\lambda) \cdots 2\lambda} a_{0,k}. \]

The constants \( b_{i,j} \) and \( c_{i,j} \) can be eliminated by the same method as in Part (1). We have just proved that the cohomology group in question...
SECOND COHOMOLOGY SPACE OF $\mathfrak{sl}(2)$ is generated by the 2-cocycle

$$C \left( \frac{X}{dx}, \frac{Y}{dx} \right) (\phi, \psi) = (XY' - X'Y) \left( \phi \psi^{(k)} + \sum_{i+j=k-1} a_{i+1,j} \phi^{i+1} \psi^{(j)} \right),$$

where

$$a_{i+1,j} \simeq \begin{cases} 0, & \text{if } j \leq t; \\ (-1)^{i+1} C_i+1 \left( \frac{1}{2(i-1)} \right) \left( \frac{1}{2(i-2)} \right) \cdots \left( \frac{1}{2(i-2\lambda)} \right), & \text{otherwise}. \end{cases}$$

(3) If $\lambda = \frac{s}{2}$ and $u \neq \frac{k-s-1}{2}$, where $s, t \in \{0, \ldots, k - 1\}$, then we follow the same steps as in (2)(b). Thus, the cohomology group in question is one-dimensional, generated by the 2-cocycle

$$C \left( \frac{X}{dx}, \frac{Y}{dx} \right) (\phi, \psi) = (XY' - X'Y) \left( \phi \psi^{(k)} + \sum_{i+j=k-1} a_{i,j} \phi^{i} \psi^{(j+1)} \right),$$

where

$$a_{i,j+1} \simeq \begin{cases} 0, & \text{if } j \leq t; \\ (-1)^{k-i} C_i+1 \left( \frac{1}{2(i-1)} \right) \left( \frac{1}{2(i-2)} \right) \cdots \left( \frac{1}{2(i-2\lambda)} \right) \cdots \left( \frac{1}{2(i-2\lambda)} \right), & \text{otherwise}. \end{cases}$$

(4) If $\lambda = \frac{s}{2}$ and $u \neq \frac{k-s-1}{2}$, where $s \in \{0, \ldots, k - 1\}$, then the space of solutions of the system (3.5) is two dimensional, generated by $a_{s+1,k-s-1}$ and $a_{s,k-s}$.

(a) If $i = s$, $j = k - s - 1$, we have

$$\begin{cases} (i+1)(i+2\lambda) = 0, \\ (j+1)(j+2\lambda) = 0. \end{cases}$$

(b) If $i \neq s$, we have $(i+1)(i+2\lambda) \neq 0$.

The system (3.5) is equivalent to the system

$$a_{i+1,j} = -\frac{(j+1)(j+2\lambda)}{(i+1)(i+2\lambda)} a_{i,j+1}.$$

(i) If $i + j = k - 1$ for all $i \in \{1, \ldots, s - 1\}$: by iterations, we get

$$a_{1,k-1} = -C_1 \left( \frac{1}{2\lambda} \right) a_{0,k},$$

$$a_{2,k-2} = C_2 \left( \frac{1}{2\lambda(1+2\lambda)} \right) a_{0,k},$$

$$\vdots$$

$$a_{i,k-i} = (-1)^s C_i \left( \frac{1}{2\lambda(s-1+2\lambda)(s-2+2\lambda) \cdots 2\lambda} \right) a_{0,k}.$$
(ii) If \( i + j = k - 1 \) for all \( i \geq s + 1 \): by iterations, we get

\[
a_{s+2,k-s-2} = \frac{(k-s-1)(k-s-2+2\nu)}{(s+2)(s+1+2\lambda)} a_{s+1,k-s-1},
\]

\[
a_{s+3,k-s-3} = \frac{(k-s-2)(k-s-1)(k-s-3+2\nu)(k-s-2+2\nu)}{(s+3)(s+2)(s+1+2\lambda)} a_{s+1,k-s-1},
\]

\[
a_{2,k-2} = C_2 \frac{(k-1+2\nu)(k-2+2\nu)}{2\lambda(1+2\lambda)} a_{0,k},
\]

\[
\vdots
\]

\[
a_{i,k-i} = (-1)^{i-s+1} 
\times \frac{(k-i+1)(k-i+2) \cdots (k-s-1)(k-i+2\nu)(k-i+1+2\nu) \cdots (k-s+2\nu)}{i(i-1) \cdots (s+2)(i-1+2\lambda)(i-2+2\lambda) \cdots (s+1+2\lambda)}
\times a_{s+1,k-s-1}.
\]

Now we will explain how the constants \( b_{i,j} \) and \( c_{i,j} \) can be eliminated except constants \( b_{s,k-s-1} \) and \( c_{s,k-s-1} \) because the component in (3.3) is zero.

The \( H^2(\mathfrak{sl}(2), D_{\lambda,\nu,\lambda+\nu}) \) is generated by a family of cocycles depending on four free parameters: \( a_{0,k}, a_{s+1,k-s-1}, b_{s,k-s-1}, \) and \( c_{s,k-s-1} \). Thus, the cohomology group in question is four-dimensional, generated by the 2-cocycle

\[
C \left( \frac{d}{dx}, Y \frac{d}{dx} \right) (\phi, \psi)
\]

\[
= b_{s,k-s-1} (XY'' - X''Y) \phi^{(s)} \psi^{(k-s-1)}
\]

\[
+ c_{s,k-s-1} (X'Y'' - X''Y') \phi^{(s)} \psi^{(k-s-1)}
\]

\[
+ \left( a_{0,k} \phi \psi^{(k)} + a_{s+1,k-s-1} \phi^{(s+1)} \psi^{(k-s-1)} \right)
\]

\[
+ \sum_{\substack{i+j=k \atop i \neq (0,s+1)}} a_{i,j} \phi^{(i)} \psi^{(j)} \right) (XY'' - X''Y),
\]

where \( a_{i,j} \) equals

\[
(-1)^{s} C_k \frac{(k-s+2\nu)(k-s+1+2\nu)(k-s+2+2\nu) \cdots (k-1+2\nu)}{(s-1+2\lambda)(s-2+2\lambda) \cdots 2\lambda} a_{0,k},
\]

if \( i \leq s \), and equals

\[
(-1)^{i-s+1} 
\times \frac{(k-i+1)(k-i+2) \cdots (k-s-1)(k-i+2\nu)(k-i+1+2\nu) \cdots (k-s+2\nu)}{i(i-1) \cdots (s+2)(i-1+2\lambda)(i-2+2\lambda) \cdots (s+1+2\lambda)}
\times a_{s+1,k-s-1},
\]

if \( i \geq s + 1 \).

(5) If \( \lambda = \frac{s}{2} \) and \( \nu = \frac{t}{2} \), where \( s, t \in \{0, \ldots, k-1\} \) and \( i + j = k - 1 \), we distinguish many cases:
(a) For $t \leq k - s - 2$, the space of solutions of the system $\text{(3.5)}$ is one-dimensional, generated by $a_{s+1,k-s-1}$. In fact, there are six cases:

(i) If $i = s$, we have $(k - s)(k - s + 2\nu)a_{s,k-s} = 0$; then, $a_{s,k-s} = 0$.

(ii) If $i < s$, we have

$$a_{i,j} = (-1)^i C_k^i (j + 2\nu)(j + 1 + 2\nu) \cdots (k - 1 + 2\nu) \frac{a_{0,k}}{(i - 1 + 2\lambda)(i - 2 + 2\lambda) \cdots 2\lambda}$$

since $a_{s,k-s} = 0$.

(iii) If $i = k - t - 1$ and $j = t$, then we have

$$(k - t)(k - 1 - t + 2\lambda)a_{k-t,t} = 0,$$

and and as the condition $t \leq k - s - 2$ involves $s < k - t - 1$ and $(i + 2\lambda)$ does not vanish only if $i = s$, so $\text{(3.5)}$ implies $(k - t)(k - t - 1 + 2\nu)a_{k-t,t} = 0$; so $a_{k-t,t} = 0$.

(iv) If $i \neq s$ and $j \neq t$, the system $\text{(3.5)}$ implies

$$a_{i+1,j} = -\frac{(j + 1)(j + 2\nu)}{(i + 1)(i + 2\lambda)}a_{i,j+1},$$

and this last equality allows us to obtain

$$a_{i,j} = (-1)^{i-s+1}\frac{(j + 1)(j + 2) \cdots (k - s - 1)}{i(i - 1) \cdots (i + 2)} \times \frac{(j + 2\nu)(j + 1 + 2\nu) \cdots (k - s - 2 + 2\nu)}{(i - 1 + 2\lambda)(i - 2 + 2\lambda) \cdots (s + 1 + 2\lambda)} a_{s+1,k-s-1}.$$

Since $a_{s,k-s} = 0$, we obtain

$$a_{0,k} = a_{1,k-1} = \cdots = a_{s,k-s} = 0.$$

(v) If $s + 1 \leq i < k - t - 1$, we obtain

$$a_{i,j} = (-1)^{i-s+1}\frac{(j + 1)(j + 2) \cdots (k - s - 1)}{i(i - 1) \cdots (i + 2)} \times \frac{(j + 2\nu)(j + 1 + 2\nu) \cdots (k - s - 2 + 2\nu)}{(i - 1 + 2\lambda)(i - 2 + 2\lambda) \cdots (s + 1 + 2\lambda)} a_{s+1,k-s-1}.$$

(vi) If $i > k - t - 1$, we have $a_{i,j} = 0$, since $a_{k-t,t} = 0$.

We conclude that

$$a_{i,j} \approx \begin{cases} 0, & \text{if } i \leq s; \\ 0, & \text{if } j \leq t; \\ (-1)^{i-s+1}\frac{(j + 1)(j + 2) \cdots (k - s - 1)}{i(i - 1) \cdots (i + 2)} \times \frac{(j + 2\nu)(j + 1 + 2\nu) \cdots (k - s - 2 + 2\nu)}{(i - 1 + 2\lambda)(i - 2 + 2\lambda) \cdots (s + 1 + 2\lambda)} a_{s+1,k-s-1}, & \text{otherwise}. \end{cases}$$

The constants $b_{i,j}$ and $c_{i,j}$ are eliminated as explained in the other cases.

Thus, the cohomology group in question is one-dimensional, generated by the 2-cocycles

\[ C \left( X \frac{d}{dx}, Y \frac{d}{dx} \right) (\phi, \psi) \]

\[ = (XY' - X'Y) \left( \phi^{(s+1)} \psi^{(k-s-1)} + \sum_{i+j=k \atop i \neq s+1} a_{i,j} \phi^{(i)} \psi^{(j)} \right). \]

(b) If \( t > k - s - 2 \), then the space of solutions of the system \((3.5)\) is two-dimensional, generated by \( a_{s+1,k-s-1} \) and \( a_{k-t-1,t+1} \).

Secondly, the constants \( b_{i,j} \) and \( c_{i,j} \) are eliminated as explained in the other cases, except \( b_{k-t-1,t} \) and \( c_{k-t-1,t} \).

The \( H_2(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\lambda+\nu}) \) is generated by a family of cocycles depending on four free parameters \( a_{0,k}, a_{s+1,k-s-1}, b_{s,k-s-1}, \) and \( c_{s,k-s-1} \). Thus, the cohomology group in question is four-dimensional, generated by the 2-cocycle

\[ C \left( X \frac{d}{dx}, Y \frac{d}{dx} \right) (\phi, \psi) \]

\[ = b_{k-t-1,t}(XY'' - X''Y)\phi^{(k-t-1)} \psi^t + c_{k-t-1,t}(X'Y'' - X''Y')\phi^{(k-t-1)} \psi^t \]

\[ + \left( a_{0,k} \phi^{(k)} + a_{k,0} \phi^{(s+1)} \psi^{(k-s-1)} \right) + \sum_{i+j=k \atop i,j \neq 0} a_{i,j} \phi^{(i)} \psi^{(j)} \]

\[ (XY' - X'Y), \]

where

\[ a_{i,j} \simeq \begin{cases} 
(-1)^{k-j} C_k^{(j+2\nu)} a_{0,k}, & \text{if } j \geq t + 1; \\
(-1)^{k-i} C_k^{(i+2\nu)(j+1+2\nu)(j+2+2\nu)\ldots(k-1+2\nu)(k-2+2\nu)\ldots(2\nu) a_{k,0}, & \text{if } i \geq s + 1.
\end{cases} \]

• For \( \mu - \lambda - \nu = k \), where \( k \) is not a positive integer, every 2-cocycle on \( \mathfrak{sl}(2) \) retains the following general from:

\[ C \left( X \frac{d}{dx}, Y \frac{d}{dx} \right) (\phi \otimes \psi) = \sum_{0 \leq n, m \leq 2} \sum_{i,j} a_{i,j,n,m} X^{(n)} Y^{(m)} \phi^{(i)} \psi^{(j)}. \]

The 2-cocycle condition is equivalent to \( a_{i,j,n,m} = 0, \forall i, j, n, m \in \mathbb{N} \). So the operator \( C \left( X \frac{d}{dx}, Y \frac{d}{dx} \right) \) is identically the zero map.

Thus,

\[ H^2(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\lambda+\nu}) \simeq 0. \]
SECOND COHOMOLOGY SPACE OF $\mathfrak{sl}(2)$

REFERENCES


Imed Basdouri
Département de Mathématiques, Faculté des Sciences de Gafsa, Zarroug, 2112 Gafsa, Tunisie

Sarra Hammami
Département de Mathématiques, Faculté des Sciences de Sfax, BP 802, 3038 Sfax, Tunisie

Olfa Messaoud
Département de Mathématiques, Faculté des Sciences de Gafsa, Zarroug, 2112 Gafsa, Tunisie

Received: May 25, 2018
Accepted: April 30, 2019