HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE OF AN ANNUlus

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Abstract. A bounded operator $S$ on a Hilbert space is hyponormal if $S^*S - SS^*$ is positive. In this work we find necessary conditions for the hyponormality of the Toeplitz operator $T_{f+g}$ on the Bergman space of the annulus $\{1/2 < |z| < 1\}$, where $f$ and $g$ are analytic and $f$ satisfies a smoothness condition.

1. Introduction

A bounded operator $S$ on a Hilbert space is hyponormal if $S^*S - SS^*$ is positive. Hyponormality of Toeplitz operators has been studied by many authors. Hyponormality of these operators on the Hardy space was considered in [3, 4]. Hyponormality of these operators with a symbol of the form $g_1 + g_2$ on the Bergman space of the unit disk was first considered in [8]. Therein a necessary condition was proved, which was later improved in [1]. Some special cases are treated in [7]. A sufficient condition when $g_1$ is a monomial and $g_2$ is a polynomial is proved in [9]. An improvement of the necessary condition in the case when $g_1$ and $g_2$ are binomials is given in [5]. Basic material on Toeplitz operators on the Bergman space of the unit disk can be found in [2]. In this work we consider hyponormality of Toeplitz operators on the Bergman space of an annulus.

We start with definitions and notations. Denote by $A^2_{1/2}$ the space of holomorphic functions on the annulus $C_{1/2} = \{z \in \mathbb{C} : 1/2 < |z| < 1\}$ such that $\int |h|^2 \, dm(z) < \infty$, where $dm(z) = (4/3\pi)d\lambda(z)$ and $\lambda$ is the Lebesgue measure on the annulus. If $h \in A^2_{1/2}$ we write $h = a_0 + \sum_{n=1}^{\infty} a_n z^n + a_{-n} z^{-n}$ and we have $\|h\|^2 = \int |h|^2 \, dm(z) = \sum_{n=0}^{\infty} \frac{4(1-(1/2)^{2n+2})}{3(n+1)} |a_n|^2 + \frac{8}{3} \ln 2 |a_{-1}|^2 + \sum_{n=2}^{\infty} \frac{4(2^{2n-2}-1)}{3(n-1)} |a_{-n}|^2$. We denote by $L^2(C_{1/2})$ the space of measurable and square integrable functions with respect to $dm$ on $C_{1/2}$. Toeplitz operators on $A^2_{1/2}$ are defined by $T_f(h) = P(hf)$, where $f$ is bounded and measurable on $C_{1/2}$, $P$ is the orthogonal projection on

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$A_{1/2}^2$, and $h$ is in $A_{1/2}^2$. The Hankel operators on the space $A_{1/2}^2$ are defined by $H_f(h) = (I - P)(hf)$. The space $A_{1/2}^2$ has an orthonormal basis given by the union of the sets

$$\{ e_n = \frac{\sqrt{3(n + 1)}}{2\sqrt{(1 - (1/2)^{2n+2})^2}}, n \geq 0 \}$$

$$\{ e_{-1} = \frac{\sqrt{3}}{\sqrt{8\ln 2}}, \text{ and} \}$$

$$\{ e_{-n} = \frac{\sqrt{3(n - 1)}}{2\sqrt{(2^{2n-2} - 1)}}, n \geq 2 \}.$$

We consider hyponormality of Toeplitz operators with a symbol of the form $f = g_1 + \overline{g_2}$, where $g_1$ and $g_2$ are bounded analytic functions on $C_{1/2}$. We begin by recalling some known properties of Toeplitz operators.

2. Some basic properties

Lemma 2.1. Let $f$ and $g$ be bounded and measurable on $C_{1/2}$. The following properties hold:

a) $T_{f+g} = T_f + T_g$.

b) $T^* f = T_f$.

c) $T_f T_g = T_{fg}$ if $g$ is analytic on $C_{1/2}$ or $f$ is conjugate analytic.

d) $T_f T_f - T_{f} T_f = H^* f H f$ if $f$ is analytic.

The next proposition is easy to prove and its proof is omitted.

Proposition 2.2. Let $g_1$ and $g_2$ be polynomials. The following are equivalent:

a) $T_{g_1 + \overline{g_2}}$ is hyponormal.

b) $T_{g_2} T_{g_2} - T_{g_1} T_{g_1} \leq T_{g_1} T_{g_1} - T_{g_1} T_{g_1}$.

c) $H^*_{g_1} H_{g_1} \leq H^*_{g_1} H_{g_1}$.

d) $H_{g_2} = K H_{g_1}$, where $K$ is an operator of norm less than one.

The following lemma provides computations that will be needed.

Lemma 2.3. The projection $P$ on $A_{1/2}^2$ satisfies the following relations:

1) $P(z^m \overline{z^n}) = \frac{(m - n + 1)(1 - (1/2)^{2m+2})}{(m + 1)(1 - (1/2)^{2m-2n+2})} z^{m-n}$, if $m \geq n$.

2) $P(z^m \overline{z^n}) = \frac{(n - m - 1)(1 - (1/2)^{2m+2})}{(m + 1)(2^{2n-2m-2} - 1)} \frac{1}{z^{n-m}}$, if $n \geq m + 2$.

3) $P(z^m \overline{z^{m+1}}) = \frac{(1 - (1/2)^{2m+2})}{2\ln 2(m + 1)} \frac{1}{z}$, if $n = m + 1$.

4) $P\left(\frac{1}{z^{m+n}}\right) = \frac{(m + n - 1)(2^{2m-2} - 1)}{2^{2(m+n)-2} - 1} \frac{1}{(m - 1)} z^{m+n}$, if $m \geq 2$. 
5) \[ P \left( \frac{1}{z^n} \right) = \frac{2n \ln 2}{(2^n - 1)^{n+1}} \], if \( n \geq 1 \).

6) \[ P \left( \frac{1}{z^m z^n} \right) = \frac{(m + n + 1)(1 - (1/2)^{2n+2})}{(n + 1)(1 - (1/2)^{2(m+n)+2})} z^{m+n} \].

7) \[ P \left( \frac{1}{z^m z^n} \right) = \frac{((m-n)+1)(2^{2n-2} - 1)}{(n-1)(1 - (1/2)^{2(m-n)+2})} z^{m-n} \], if \( m \geq n, n \neq 1 \).

8) \[ P \left( \frac{1}{z^m z^n} \right) = \frac{2m \ln 2}{(1 - (1/2)^{2m})} z^{m-1} \], if \( m \geq 1 \).

9) \[ P \left( \frac{1}{z^m z^n} \right) = \frac{(n-m-1)(2^{2n-2} - 1)}{(n-1)(2^{(m-n)-2} - 1)} \frac{1}{z^{n-m}} \], if \( m \geq 1, n-m > 1 \).

10) \[ P \left( \frac{1}{z^m z^{n+1}} \right) = \frac{(2^{m-1} - 1)}{2m \ln 2} \frac{1}{z} \], if \( m \geq 1 \).

3. First main result

We begin with a matrix computation.

**Lemma 3.1.** Let \( f = \sum_{k=1}^{\infty} a_k z^k \) be bounded on \( C_{1/2} \). Then for \( i, j \geq 1 \) we have

\[
\langle T_f T_f - T_f \rangle (e_i, e_j) = \sum_{1 \leq k+j-i} a_{k+j-i} a_k \frac{\sqrt{i+1} \sqrt{j+1} (1 - (1/2)^{2(k+j)+2})}{\sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2j+2}} (k+j+1)}
- \sum_{1 \leq k \leq j, 1 \leq k+i-j} a_k a_{k+j-i} \frac{(j-k+1) \sqrt{1 - (1/2)^{2j+2}} \sqrt{1 - (1/2)^{2j+2}}}{(1 - (1/2)^{2(j-k)+2}) \sqrt{i+1} \sqrt{j+1}}
- a_{j+1} a_{i+1} \frac{\sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2j+2}}}{2 \ln 2 \sqrt{i+1} \sqrt{j+1}}
- \sum_{j+2 \leq k, 1 \leq k+i-j} a_k a_{k+i-j} \frac{(k-i-1) \sqrt{1 - (1/2)^{2j+2}} \sqrt{1 - (1/2)^{2j+2}}}{\sqrt{i+1} \sqrt{j+1}}.
\]

**Proof.** We have

\[
\langle T_f T_f (e_i, e_j) = \sum_{k,l=1}^{\infty} a_{k+l-i} a_k \frac{\sqrt{3(i+1)}}{2 \sqrt{1 - (1/2)^{2i+2}}} \frac{\sqrt{3(j+1)}}{2 \sqrt{1 - (1/2)^{2j+2}}} \langle z^{k+j}, z^{i+l} \rangle
= \sum_{1 \leq k \leq j+i} a_k a_{i+j-k} \frac{(1 - (1/2)^{2(k+j)+2}) \sqrt{(i+1)(j+1)}}{(k+j+1) \sqrt{(1 - (1/2)^{2i+2}) (1 - (1/2)^{2j+2})}}.
\]
Similarly, we get
\[
\langle T_j T_f (e_i), e_i \rangle = \sum_{1 \leq k+i-j \leq j} \sum_{1 \leq k \leq j} \overline{a_k} a_{k+i-j} (j - k + 1) \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2j+2}} \sqrt{i + 1} \sqrt{j + 1} / (1 - (1/2)^{2(j-k)+2}) \]  
\[+ a_{j+1} a_{i+1} \sqrt{(1 - (1/2)^{2i+2})} \sqrt{(1 - (1/2)^{2j+2})} / \sqrt{j + 1} \]  
\[+ \sum_{j+2 \leq k \leq 1 \leq k+i-j} \overline{a_k} a_{k+i-j} (k - j + 1) \sqrt{1 - (1/2)^{2i+2}} (1 - (1/2)^{2j+2}) / \sqrt{(i + 1) (j + 1)} \].
\]
□

Set \( \beta_{i,j} = \langle T_j T_f - T_j T_f (e_i), e_i \rangle, \ i, j \geq 1 \). By rewriting the expression for \( \beta_{i,j} \) we obtain
\[
\beta_{i+p,i} = \sum_{1 \leq k \leq i} \sum_{1 \leq k+p} \overline{a_k} a_{k+p} \sqrt{i+1} \sqrt{i+p+1} (1 - (1/2)^{2(k+p+i)+2}) \quad 1 - (1/2)^{2i+2} \sqrt{1 - (1/2)^{2(i+p)+2}} (k + p + i + 1) \]  
\[= \sum_{i+2 \leq k} \overline{a_k} a_{k+p} \sqrt{i+1} \sqrt{i+p+1} (1 - (1/2)^{2(k+p+i)+2}) \]  
\[= \sum_{i+2 \leq k} \overline{a_k} a_{k+p} Q_{i,k,p} + \sum_{i+2 \leq k} \overline{a_k} a_{k+p} S_{i,k,p} + \sum_{i+2 \leq k} \overline{a_k} a_{k+p} S_{i,k,p} + \sum_{i+2 \leq k} \overline{a_k} a_{k+p} S_{i,k,p} \hspace{1cm} \text{Lemma 3.2. We have } \lim_{i \to \infty} i^2 \beta_{i+p,i} = \gamma_{i+p,i}, \text{ where } (\gamma_{i,j}) \text{ is the matrix of the Hardy space Topelitz operator } T_f |f|^2. \]
\[
\text{Proof. An elementary computation shows that } \lim_{i \to \infty} i^2 Q_{i,k,p} = k(k+p). \text{ Set } h_i(k) = i^2 \chi_{\{1, \ldots, i\}}(k) \overline{a_k} a_{k+p} Q_{i,k,p}. \text{ The first sum in the above expression of } \beta_{i+p,i} \text{ can be written as } \int h_i(k) \, d\mu(k), \text{ where } d\mu \text{ is the counting measure. It is easy to see that for } i \text{ sufficiently large, } |h_i(k)| \leq 2|a_k a_{k+p}| \leq k^2 |a_k|^2 + (k+p)^2 |a_{k+p}|^2 = M(k). \text{ Since } f' \in H^2, \text{ the function } M(k) \text{ is integrable with respect to the counting measure.} \]

By the dominated convergence theorem we obtain:

$$
\lim_{i \to \infty} i^2 \sum_{1 \leq k \leq i} \sum_{1 \leq k + p} \overline{a_k} a_{k+p} Q_{i,k,p} = \sum k(k + p) \overline{a_k} a_{k+p}.
$$

Also, for $i$ large, there exists a constant $C$ such that

$$
|\overline{a_i+1} a_{i+p+1} R_{i,p}| \leq C \left((i + 1)^2 |a_{i+1}|^2 + (i + p + 1)^2 |a_{i+p+1}|^2\right).
$$

Thus $\lim_{i \to \infty} i^2 \overline{a_i+1} a_{i+p+1} R_{i,p} = 0$. Finally, it is not difficult to see that $i^2 |S_{i,k,p}| \leq k(k + p)$. Using the dominated convergence theorem we obtain

$$
\lim_{i \to \infty} i^2 \sum_{i+2 \leq k} \overline{a_k} a_{k+p} S_{i,k,p} = 0.
$$

We deduce that $\lim_{i \to \infty} i^2 \beta_{i+p,i} = \sum k(k + p) \overline{a_k} a_{k+p}$ and recognize this last limit as being equal to $\gamma_{i+p,i}$, where $(\gamma_{i,j})$ is the matrix of the Hardy space Toeplitz operator $T_{|f'|^2}$.

We are led to the following necessary condition for hyponormality.

**Theorem 3.3.** Let $f = \sum a_k z^k$ and $g = \sum b_k z^k$ be bounded on $C_{1/2}$. Assume that $f' \in H^2$. If $T_{f+g}$ is hyponormal then $g' \in H^2$ and $|g'| \leq |f'|$ a.e. on the unit circle.

**Proof.** If $(\theta_{i,j})$ denotes the matrix of $T_{f}T_j - T_jT_f - T_{g}T_g - T_gT_f$ and $(\sigma_{i,j})$ denotes the matrix of $T_{f}T_{g} - T_{g}T_{f}$, then the inequality $\sigma_{i,j} \leq \beta_{i,i}$ leads to

$$
\sum_{1 \leq k \leq i} |b_k|^2 Q_{i,k,0} + |b_{i+1}|^2 R_{i,0} + \sum_{i+2 \leq k} |b_k|^2 S_{i,k,0} \leq \sum_{1 \leq k \leq i} |a_k|^2 Q_{i,k,0} + |a_{i+1}|^2 R_{i,0} + \sum_{i+2 \leq k} |a_k|^2 S_{i,k,0}.
$$

We deduce that $\sum_{1 \leq k \leq i} i^2 |b_k|^2 Q_{i,k,0} \leq i^2 \beta_{i,i}$. Since $\lim_{i \to \infty} i^2 |b_k|^2 Q_{i,k,0} = k^2$, writing the left hand side of this last inequality as an integral with respect to the counting measure and using Fatou’s lemma we get $\sum k^2 |b_k|^2 \leq \sum k^2 |a_k|$ and $g' \in H^2$. From the previous lemma, $\lim_{i \to \infty} i^2 \theta_{i+p,i} = \lambda_{i+p,i}$, where $(\lambda_{i,j})$ denotes the matrix of the Hardy space Toeplitz operator $T_{|f'|^2 - |g'|^2}$. Hyponormality and a property of Toeplitz matrices lead to $|g'| \leq |f'|$ a.e. on the unit circle. □

**Corollary 3.4.** Let $f = \sum a_k z^k$ and $g = \sum b_k z^k$ be analytic and univalent in an open set containing $C_{1/2}$. Then $T_{f+g}$ is normal if and only if $g = cf$, where $c$ is a constant with $|c| = 1$.

**Proof.** Only the necessary condition needs to be shown. Normality implies that $|g'| = |f'|$ on the unit circle. Thus $f'$ and $g'$ have the same finite number of zeros (if any) with the same multiplicity. We thus have $\left|\frac{f'}{g'}\right| = \left|\frac{g'}{f'}\right| = 1$ on the unit circle. By the maximum principle, $g' = cf'$ with $|c| = 1$. We get $g = cf$. □
Lemma 3.5. Let \( f = \sum_{k=1}^{\infty} a_k z^k \) be bounded on \( C_{1/2} \). Then for \( i \geq 3, j \geq 3 \) we have

\[
\langle T_f T_f - T_f T_f^*(e_{-j}), e_{-i} \rangle
\]

\[
= \sum_{1 \leq k < j} \frac{a_{k+i-j} a_k}{1 \leq k+i-j} \frac{\sqrt{(i-1)}}{\sqrt{(2^{i-2} - 1)}} \frac{\sqrt{(j-1)}}{\sqrt{(2^{j-2} - 1)}} \frac{(2^{(j-k)-2} - 1)}{(j-k-1)}
\]

\[
+ 2 \ln 2 \frac{a_{i-1} a_{j-1}}{\sqrt{2^{-i-2} - 1}} \frac{\sqrt{(j-1)}}{\sqrt{2^{j-2} - 1}}
\]

\[
+ \sum_{j \leq k} \frac{a_{k+i-j} a_k}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{(j-1)}}{\sqrt{(2^{j-2} - 1)}} \frac{(1 - (1/2)^{(k-j)+2})}{k-j+1}
\]

\[
- \sum_{1 \leq k \leq k+j-i} \frac{a_k a_{k+j-i}}{(2^{(j+k)-2} - 1) \sqrt{(i-1) \sqrt{j-1}}}
\]

Proof. We have

\[
\langle T_f T_f (e_{-j}), e_{-i} \rangle = \sum_{1 \leq k < j} \frac{a_{k+i-j} a_k}{1 \leq k+i-j} \frac{\sqrt{(i-1)}}{\sqrt{(2^{i-2} - 1)}} \frac{\sqrt{(j-1)}}{\sqrt{(2^{j-2} - 1)}} \frac{(2^{(j-k)-2} - 1)}{(j-k-1)}
\]

\[
+ 2 \ln 2 \frac{a_{i-1} a_{j-1}}{\sqrt{2^{-i-2} - 1}} \frac{\sqrt{(j-1)}}{\sqrt{2^{j-2} - 1}}
\]

\[
+ \sum_{j \leq k} \frac{a_{k+i-j} a_k}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{(j-1)}}{\sqrt{(2^{j-2} - 1)}} \frac{(1 - (1/2)^{(k-j)+2})}{k-j+1}
\]

Similarly,

\[
\langle T_f T_f^*(e_{-j}), e_{-i} \rangle = \sum_{k,l=1}^{\infty} \frac{a_k a_l}{2 \sqrt{(2^{(i-2) - 1)} 2 \sqrt{(2^{j-2} - 1)}}} \left\langle P \left( \frac{z^k}{z^l} \right), P \left( \frac{z^{1/2}}{z^{1/2}} \right) \right\rangle
\]

\[
= \sum_{1 \leq k \leq k+j-i} \frac{a_k a_{k+j-i}}{(2^{(j+k)-2} - 1) \sqrt{(i-1) \sqrt{j-1}}}
\]

\[ \square \]

Let \( \beta_{-i,-j} = \langle (T_f T_f - T_f T_f^*)(e_{-j}), e_{-i} \rangle \) and denote by \( (\zeta_{i,j}) \) the matrix of the Toeplitz operator \( T_{f_{1/2}}^2 \) on the Hardy space of the unit disk, where \( f_{1/2}(z) = \sum \frac{a_k}{z^k} \).

We can show the following lemma.
Lemma 3.6. We have \( \lim_{i \to \infty} i^2 \beta_{-i-p,-i} = \zeta_{i+p,i} \).

**Proof.**

\[ \beta_{-i-p,-i} = \sum_{1 \leq k \leq i-p} \frac{a_{k+p}a_k}{(2^{2i-2} - 1)} \frac{(2^{2(i-k)-2} - 1)}{(i-k-1)} \]

\[ = \sum_{1 \leq k \leq i-p} \frac{a_{k+p}a_k}{(2^{2i-2} - 1)} \frac{(2^{2(i-k)-2} - 1)}{(i-k-1)} \]

\[ + 2 \ln 2a_{i+p-1}a_{i-1} - \sum_{i \leq k} \frac{a_{k+p}a_k}{(2^{2i-2} - 1)} \frac{(2^{2(i-k)-2} - 1)}{(i-k-1)} \]

\[ = \sum_{1 \leq k \leq i-p} \frac{a_{k+p}a_k}{(2^{2i-2} - 1)} \frac{(2^{2(i-k)-2} - 1)}{(i-k-1)} \]

\[ = \lim_{i \to \infty} i^2 Q'_{i,p,k} + \sum_{i \leq k} \frac{a_{k+p}a_k S'_{i,k,p}}{2k} \]

A computation shows that \( \lim_{i \to \infty} i^2 Q'_{i,p,k} = \frac{1}{2^{2i+p}} \). As in the proof of the previous theorem we can show that

\[ \lim_{i \to \infty} i^2 \sum_{1 \leq k \leq i-p} \frac{a_{k+p}a_k Q'_{i,p,k}}{2k} = \sum_{1 \leq k \leq i-p} \frac{k(k+p)}{2k} \frac{a_{k+p}}{2k+p} \]

We see that this last limit is equal to \( \zeta_{i,i+p} \). We also show that

\[ \lim_{i \to \infty} i^2 a_{i+p-1}a_{i-1} R'_{i,p} = 0 \]
and

\[ \lim_{i \to \infty} i^2 \sum_{i \leq k} a_{k+i-p} a_k S'_{i,k,p} = 0. \]

We deduce that

\[ \lim_{i \to \infty} i^2 \beta_{-i-p,-i} = \zeta_{i+p,i}. \]

If \( f = \sum_{1}^{\infty} a_k z^k \) is bounded analytic on \( C_{1/2} \), then clearly \( \sum \frac{k^2}{2\pi} |a_k|^2 < \infty \). We can also see that \( |g'_{1/2}| \leq |f'_{1/2}| \) a.e. on the unit circle is equivalent to \( |g'| \leq |f'| \) a.e. on \( \{ z : |z| = 1/2 \} \).

**Theorem 3.7.** Let \( f = \sum_{1}^{\infty} a_k z^k \) and \( g = \sum_{1}^{\infty} b_k z^k \) be bounded on \( C_{1/2} \). If \( T_{f+g} \) is hyponormal then \( |g'| \leq |f'| \) a.e. on \( \{ z : |z| = 1/2 \} \).

The proof is similar to the proof of the previous theorem and is omitted. Combining the previous two theorems we get our first main result.

**Theorem 3.8.** Let \( f = \sum_{1}^{\infty} a_k z^k \) and \( g = \sum_{1}^{\infty} b_k z^k \) be bounded on \( C_{1/2} \) and assume that \( f' \in H^2 \). If \( T_{f+g} \) is hyponormal then \( g' \in H^2 \) and \( |g'| \leq |f'| \) a.e. on \( \{ z : |z| = 1 \} \cup \{ z : |z| = 1/2 \} \).

4. **Second main result**

We now put \( f = \sum_{1}^{\infty} a_k \frac{1}{z^k} \) and \( g = \sum_{1}^{\infty} b_k \frac{1}{z^k} \) and assume that \( f \) and \( g \) are bounded on \( C_{1/2} \). We need the following computation.

**Lemma 4.1.** For \( i \geq 1, j \geq 1 \) we have

\[ \langle T_f T_f - T_f T_f(e), e_i \rangle \]

\[ = \sum_{1 \leq k, k+1-i-j} a_{k+i-j} a_k \frac{\sqrt{(i+1)\sqrt{(j+1)}}(1-(1/2)^{2(j-k)+2})}{\sqrt{(1-(1/2)^{2i+2})}\sqrt{(1-(1/2)^{2j+2})}(j-k+1)} \]

\[ - \sum_{1 \leq k, k+j-i} \frac{a_k a_{k+j-i}}{\sqrt{(1-(1/2)^{2i+2})}\sqrt{(1-(1/2)^{2j+2})}(j+k+1)} \sqrt{i+j+1(1-(1/2)^{2(j+k)+2})}. \]

**Proof.** We have

\[ \langle T_f T_f(e), e_i \rangle = \sum_{k,l=1}^{\infty} a_l a_k \frac{\sqrt{3(i+1)}}{2\sqrt{(1-(1/2)^{2i+2})}} \frac{\sqrt{3(j+1)}}{2\sqrt{(1-(1/2)^{2j+2})}} (z^{j-k}, z^{i-l}) \]

\[ = \sum_{1 \leq k, k+i-j} a_{k+i-j} a_k \frac{\sqrt{(i+1)}}{\sqrt{(1-(1/2)^{2i+2})}} \frac{\sqrt{(j+1)}}{\sqrt{(1-(1/2)^{2j+2})}} (1-(1/2)^{2(j-k)+2}) \frac{1}{j-k+1}. \]
and
\[
\langle T_f T_f^*(e_j), e_i \rangle = \sum_{k,l=1}^{\infty} a_k a_l \frac{\sqrt{3(i + 1)}}{2\sqrt{(1 - (1/2)^{2i+2})}} \frac{\sqrt{3(j + 1)}}{2\sqrt{(1 - (1/2)^{2j+2})}} \\
\times \left\langle P \left( \frac{1}{z^k} z^j \right), P \left( \frac{1}{z^l} z^i \right) \right\rangle
\]
\[
= \sum_{1 \leq k, k+j-i} \infty \alpha_k a_{k+j-i} \frac{\sqrt{(1 - (1/2)^{2i+2})}}{\sqrt{i + 1}} \frac{\sqrt{(1 - (1/2)^{2j+2})}}{\sqrt{j + 1}} \frac{\sqrt{(1 - (1/2)^{2k+2})}}{\sqrt{k + 1}}.
\]

We get, using the same notations as before,
\[
\beta_{i+p,i} = \sum_{1 \leq k-p \leq k} \alpha_k a_{k-p} \frac{\sqrt{(i + 1)}}{\sqrt{1 - (1/2)^{2(i+p)+2}}} \frac{(1 - (1/2)^{2i+p+2})}{(1 - (1/2)^{2i+k+2} + 1) - k + p + 1}
\]
\[
- \sum_{1 \leq k-p \leq k} \alpha_k a_{k-p} \frac{\sqrt{(1 - (1/2)^{2i+p+2})}}{\sqrt{i + 1}} \frac{(1 - (1/2)^{2i+k+2} + 1)}{\sqrt{i + 1}} \frac{1}{\sqrt{j + 1}} \frac{1}{\sqrt{k + 1}}
\]
\[
= \sum_{1 \leq k-p \leq k} \alpha_k a_{k-p} U_{i,k,p}.
\]

A computation shows that
\[
\lim_{i \to \infty} \frac{1}{r^2} \beta_{i+p,i} = \sum_{1 \leq k, k-p} \alpha_k a_{k-p} k(k-p).
\]

We recognize the general element \(\xi_{m+p,m}\) of the matrix of the Toeplitz operator \(T_{f^j}\) on the Hardy space of the unit disk with \(\tilde{f}\) defined by \(\tilde{f}(z) = \sum_{i}^{\infty} a_k z^k\). Obviously the condition \(|\tilde{g}(e^{i\theta})| \leq |\tilde{f}(e^{i\theta})|\) a.e. on the unit circle is the same as \(|g'| \leq |f'|\) a.e. on the unit circle. The condition \(f' \in H^2\) is equivalent to \(\sum k^2 |a_k|^2 < \infty\) and this is satisfied if \(f = \sum_{i}^{\infty} \frac{a_i}{\sqrt{i+n}}\) is bounded on \(C_{1/2}\). Using similar methods we obtain the following theorem.

**Theorem 4.2.** Let \(f = \sum_{i}^{\infty} a_k \frac{1}{\sqrt{i+n}}\) and \(g = \sum_{i}^{\infty} b_k \frac{1}{\sqrt{i+n}}\) be analytic and bounded on \(C_{1/2}\). If \(T_{f'g'}\) is hyponormal then \(|g'| \leq |f'|\) a.e. on the unit circle.

If we set \(f_2(z) = \sum \frac{2^k a_k z^k}{\sqrt{k+1}}\), then \(f_2' \in H^2\) is equivalent to \(\sum k^2 2^{2k} |a_k|^2 < \infty\). In this case, \(|g_2'| \leq |f_2'|\) a.e. on the unit circle is equivalent to \(|g'| \leq |f'|\) a.e. on \(\{z : |z| = 1/2\}\). Let \((\rho_{i,j})\) denote the matrix of the Hardy space Toeplitz operator \(T_{f_2^j}|z|^{2j}\). Using the same notations we can show the following lemma, the proof of which is omitted.

**Lemma 4.3.** \(\lim_{i \to \infty} \frac{1}{r^2} \beta_{-i-p,-i} = \rho_{i+p,i}\).

We obtain our second main result.
Theorem 4.4. Let \( f = \sum_{k=0}^{\infty} a_k e^{i\theta} \) and \( g = \sum_{k=0}^{\infty} b_k e^{i\theta} \) be bounded on \( C_{1/2} \), with \( \sum k^2 2^k |a_k|^2 < \infty \). If \( T_f + \overline{g} \) is hyponormal then \( \sum k^2 2^k |b_k|^2 < \infty \) and \( |g'| \leq |f'| \) a.e. on \( \{ z : |z| = 1 \} \cup \{ z : |z| = 1/2 \} \).

An application of the maximum modulus principle allows us to describe the normality of \( T_f + \overline{g} \) under the condition of univalence.

Corollary 4.5. Let \( f = \sum_{k=0}^{\infty} a_k e^{i\theta} \) and \( g = \sum_{k=0}^{\infty} b_k e^{i\theta} \) be analytic and univalent in an open set containing \( C_{1/2} \). Then \( T_f + \overline{g} \) is normal if and only if \( g = cf \), where \( c \) is a constant with \( |c| = 1 \).

We list two more results which are shown using methods similar to the ones used for the previous theorems.

Theorem 4.6. Let \( f = \sum_{k=0}^{\infty} a_k z^k \) and \( g = \sum_{k=0}^{\infty} b_k z^k \) be bounded on \( C_{1/2} \). Assume that \( \sum k^2 |a_k|^2 < \infty \). If \( T_f + \overline{g} \) is hyponormal then \( \sum k^2 |b_k|^2 < \infty \) and \( |g'(e^{i\theta})| \leq |f'(e^{i\theta})| \) a.e. on the unit circle.

Corollary 4.7. Let \( f = \sum_{k=0}^{\infty} a_k z^k \) and \( g = \sum_{k=0}^{\infty} b_k z^k \) be bounded on \( C_{1/2} \). Assume that \( f \) and \( \overline{g} \) are univalent in an open set containing \( C_{1/2} \). Then \( T_f + \overline{g} \) is normal if and only if \( \overline{g} = cf \) for some constant \( c \) with \( |c| = 1 \).

Theorem 4.8. Let \( f = \sum_{k=0}^{\infty} a_k z^k \) and \( g = \sum_{k=0}^{\infty} b_k z^k \) be bounded on \( C_{1/2} \). If \( T_f + \overline{g} \) is hyponormal then \( \sum k^2 2^k |b_k|^2 < \infty \) and \( |g'(1/2 e^{i\theta})| \leq |f'(1/2 e^{i\theta})| \) for almost all \( \theta \).

Corollary 4.9. Let \( f = \sum_{k=0}^{\infty} a_k z^k \) and \( g = \sum_{k=0}^{\infty} b_k z^k \) be bounded on \( C_{1/2} \) and assume that \( T_f + \overline{g} \) is hyponormal. The following holds:

i) \( \sum k^2 2^k |b_k|^2 < \infty \) and \( |g'(1/2 e^{i\theta})| \leq |f'(1/2 e^{i\theta})| \) for almost all \( \theta \).

ii) If \( f' \in H^2 \) then \( |g'(e^{i\theta})| \leq |f'(e^{i\theta})| \) a.e. on the unit circle.

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