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STRUCTURE OF SIMPLE MULTIPLICATIVE HOM-JORDAN ALGEBRAS

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ABSTRACT. We study the structure of simple multiplicative Hom-Jordan algebras. We discuss equivalent conditions for multiplicative Hom-Jordan algebras to be solvable, simple, and semi-simple. Moreover, we give a theorem on the classification of simple multiplicative Hom-Jordan algebras and obtain some propositions about bimodules of multiplicative Hom-Jordan algebras.

1. Introduction

Algebras where the identities defining the structure are twisted by a homomorphism are called Hom-algebras. These algebras have recently been investigated by many authors. The theory of Hom-algebras started from Hom-Lie algebras introduced and discussed in [6, 10, 11, 12]. Hom-associative algebras were introduced in [15], while Hom-Jordan algebras were introduced in [14] as twisted generalization of Jordan algebras.

In recent years, vertex operator algebras are becoming more and more popular because of their importance. In [9], by using the structure of Heisenberg algebras, Lam constructed a vertex operator algebra such that the weight two space $V_2 \cong J$ for a given simple Jordan algebra J of type A, B or C over $\mathbb{C}$. In [2], Ashihara gave a counterexample to the following assertion: If R is a subalgebra of the Griess algebra, then the weight two space of the vertex operator subalgebra $\mathrm{VOA}(R)$ generated by R coincides with R by using a vertex operator algebra associated with the simple Jordan algebra of type D. H. B. Zhao constructed simple quotients $\bar{V}_{\mathscr{J},r}$ for $r \in \mathbb{Z}_{\neq 0}$ using dual-pair type constructions, where $\bar{V}_{\mathscr{J},r}$ is such that $(\bar{V}_{\mathscr{J},r})_0 = \mathbb{C}1$, $(\bar{V}_{\mathscr{J},r})_1 = \{0\}$, and $(\bar{V}_{\mathscr{J},r})_2$ is isomorphic to the type B Jordan algebra $\mathscr{J}$. Moreover, in his paper [19] he reproved that $V_{\mathscr{J},r}$ is simple if $r \notin \mathbb{Z}$.

The structure of Hom-algebras seems to be more complex because of the variety of twisted maps. But the structure of the original algebras is pretty clear. So one of the ways to study the structure of Hom-algebras is to look for relationships between Hom-algebras and their induced algebras. In [15], Makhlouf and Silvestrov

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introduced the structure of Hom-associative algebras and Hom-Leibniz algebras together with their induced algebras. In [13], X. X. Li studied the structure of multiplicative Hom-Lie algebras and gave equivalent conditions for a multiplicative Hom-Lie algebra to be solvable, simple, and semi-simple. By a similar analysis, in this paper we successfully generalize the above results to Hom-Jordan algebras.

It is well known that simple algebras play an important role in structure theory. Similarly, it is necessary to study simple Hom-algebras in the theory of Hom-algebras. In [5], X. Chen and W. Han gave a classification theorem about simple multiplicative Hom-Lie algebras. Using some theorems obtained in Section 3, we generalize the above theorem to Hom-Jordan algebras.

Nowadays, one of the current trends in mathematics has to do with representations and deformations. The two topics are important tools in most parts of mathematics and physics. Representations of Hom-Lie algebras were introduced and studied in [4, 17], but representations of Hom-Jordan algebras came much later. In 2018, Attan gave the definition of bimodules of Hom-Jordan algebras in his paper [3]. In [1], Agrebaoui, Benali, and Makhlouf studied representations of simple Hom-Lie algebras and gave some propositions about them. In this paper, we also give some propositions about bimodules of Hom-Jordan algebras using similar methods.

The paper is organised as follows: In Section 2, we introduce some basic definitions and prove a few lemmas which can be used in what follows. In Section 3, we mainly show three important results, Theorems 3.3, 3.5, and 3.6, which are about solvability, simplicity, and semi-simplicity of multiplicative Hom-Jordan algebras, respectively. In Section 4, we first give a result, Theorem 4.1, on the construction of n-dimensional simple Hom-Jordan algebras. Next we give our main theorem in this section, Theorem 4.3, which is about classification of simple multiplicative Hom-Jordan algebras. In Section 5, we prove a very important result, Theorem 5.5, which is about relationships between bimodules of Hom-Jordan algebras and modules of their induced Jordan algebras. Moreover, some propositions about bimodules of simple Hom-Jordan algebras are also obtained as an application of Theorem 5.5.

2. Preliminaries

Definition 2.1 ([16, pp. 152–157])**.** A *Jordan algebra* J over a field F is an algebra satisfying, for any $x, y \in J$,

(1) $x \circ y = y \circ x$;
(2) $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$.

Definition 2.2 ([14])**.** A *Hom-Jordan algebra* over a field F is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$ which is commutative, and a linear map $\alpha : V \to V$ satisfying, for any $x, y \in V$,

$$\mu(\alpha^2(x), \mu(y, \mu(x, x))) = \mu(\mu(\alpha(x), y), \alpha(\mu(x, x))),$$

where $\alpha^2 = \alpha \circ \alpha$.

Definition 2.3. A Hom-Jordan algebra (V, μ, α) is called *multiplicative* if for any $x, y \in V$, $\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y))$.

Definition 2.4 ([7])**.** A subspace $W \subseteq V$ is a *Hom-subalgebra* of (V, μ, α) if $\alpha(W) \subseteq W$ and

$$\mu(x, y) \in W, \quad \forall x, y \in W.$$

Definition 2.5 ([7])**.** A subspace $W \subseteq V$ is a *Hom-ideal* of (V, μ, α) if $\alpha(W) \subseteq W$ and

$$\mu(x, y) \in W, \quad \forall x \in W, y \in V.$$

Definition 2.6 ([7])**.** Let (V, μ, α) and (V', μ', β) be two Hom-Jordan algebras. A linear map $\phi : V \to V'$ is said to be a *homomorphism* of Hom-Jordan algebras if

(1) $\phi(\mu(x, y)) = \mu'(\phi(x), \phi(y))$;
(2) $\phi \circ \alpha = \beta \circ \phi$.

In particular, ϕ is an isomorphism if ϕ is bijective.

Definition 2.7. A Hom-Jordan algebra (V, μ, α) is called a *Jordan-type Hom-Jordan algebra* if there exists a Jordan algebra (V, μ') such that

$$\mu(x, y) = \alpha(\mu'(x, y)) = \mu'(\alpha(x), \alpha(y)), \quad \forall x, y \in V,$$

and (V, μ') is called the induced Jordan algebra.

Lemma 2.8. *(1) Suppose that (V, μ) is a Jordan algebra and $\alpha : V \to V$ is a homomorphism. Then $(V, \tilde{\mu}, \alpha)$ is a multiplicative Hom-Jordan algebra with $\tilde{\mu}(x, y) = \alpha(\mu(x, y))$, $\forall x, y \in V$.*
(2) Suppose that (V, μ, α) is a multiplicative Hom-Jordan algebra and α is invertible. Then (V, μ, α) is a Jordan-type Hom-Jordan algebra and its induced Jordan algebra is (V, μ') with $\mu'(x, y) = \alpha^{-1}(\mu(x, y))$, $\forall x, y \in V$.

Proof. (1). We have that $\tilde{\mu}$ is commutative, since μ is commutative.

For all $x, y \in V$, we have

$$\begin{aligned}
&\tilde{\mu}(\alpha^2(x), \tilde{\mu}(y, \tilde{\mu}(x, x))) = \alpha(\mu(\alpha^2(x), \alpha(\mu(y, \alpha(\mu(x, x))))))\\
&\quad = \mu(\alpha^3(x), \mu(\alpha^2(y), \mu(\alpha^3(x), \alpha^3(x)))) = \mu(\mu(\alpha^3(x), \alpha^2(y)), \mu(\alpha^3(x), \alpha^3(x)))\\
&\quad = \alpha(\mu(\alpha(\mu(\alpha(x), y)), \alpha^2(\mu(x, x)))) = \tilde{\mu}(\tilde{\mu}(\alpha(x), y), \alpha(\tilde{\mu}(x, x))),
\end{aligned}$$

which implies that $(V, \tilde{\mu}, \alpha)$ is a Hom-Jordan algebra. We see that

$$\alpha(\tilde{\mu}(x, y)) = \alpha^2(\mu(x, y)) = \alpha(\mu(\alpha(x), \alpha(y))) = \tilde{\mu}(\alpha(x), \alpha(y)),$$

which implies that $(V, \tilde{\mu}, \alpha)$ is multiplicative. Hence, $(V, \tilde{\mu}, \alpha)$ is a multiplicative Hom-Jordan algebra.

(2). We have that μ' is commutative, since μ is commutative.

For any $x, y \in V$, we have

$$\begin{aligned}
&\mu'(\mu'(\mu'(x, x), y), x) = \alpha^{-1}(\mu(\alpha^{-1}(\mu(\alpha^{-1}(\mu(x, x)), y)), x))\\
&\quad = \alpha^{-3}(\mu(\mu(\mu(x, x), \alpha(y)), \alpha^2(x))) = \alpha^{-3}(\mu(\alpha(\mu(x, x)), \mu(\alpha(y), \alpha(x))))\\
&\quad = \alpha^{-1}(\mu(\alpha^{-1}(\mu(x, x)), \alpha^{-1}(\mu(y, x)))) = \mu'(\mu'(x, x), \mu'(y, x)),
\end{aligned}$$

which implies that (V, μ') is a Jordan algebra.

It is obvious that $\mu(x, y) = \alpha(\mu'(x, y)) = \mu'(\alpha(x), \alpha(y))$ for any $x, y \in V$. Hence, (V, μ, α) is a Jordan-type Hom-Jordan algebra. □

Definition 2.9. Suppose that (V, μ, α) is a Hom-Jordan algebra. Define its derived sequence as follows:

$$V^{(1)} = \mu(V, V),\ V^{(2)} = \mu(V^{(1)}, V^{(1)}),\ \ldots,\ V^{(k)} = \mu(V^{(k-1)}, V^{(k-1)}),\ \ldots.$$

If there exists $m \in \mathbb{Z}^+$ such that $V^{(m)} = 0$, then (V, μ, α) is called *solvable*.

Definition 2.10. Suppose that (V, μ, α) is a Hom-Jordan algebra and $\alpha \neq 0$. If (V, μ, α) has no nontrivial Hom-ideals and satisfies $\mu(V, V) = V$, then (V, μ, α) is called *simple*. If

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s,$$

where V_i $(1 \leq i \leq s)$ are simple Hom-ideals of (V, μ, α), then (V, μ, α) is called *semi-simple*.

Proposition 2.11. *Suppose that $(V_1, \tilde{\mu_1}, \alpha)$ and $(V_2, \tilde{\mu_2}, \beta)$ are two Jordan-type Hom-Jordan algebras and β is injective. Then ϕ is an isomorphism from $(V_1, \tilde{\mu_1}, \alpha)$ to $(V_2, \tilde{\mu_2}, \beta)$ if and only if ϕ is an isomorphism between their induced Jordan algebras (V_1, μ_1) and (V_2, μ_2) and ϕ satisfies $\beta \circ \phi = \phi \circ \alpha$.*

Proof. $(\Rightarrow)$ For any $x, y \in V_1$, we have

$$\phi(\tilde{\mu_1}(x, y)) = \tilde{\mu_2}(\phi(x), \phi(y)),$$

i.e.,

$$\phi(\alpha(\mu_1(x, y))) = \beta(\mu_2(\phi(x), \phi(y))).$$

Note that $\phi \circ \alpha = \beta \circ \phi$. We have

$$\beta(\phi(\mu_1(x, y))) = \beta(\mu_2(\phi(x), \phi(y))).$$

Since β is injective, we have

$$\phi(\mu_1(x, y)) = \mu_2(\phi(x), \phi(y)),$$

which implies that ϕ is an isomorphism from (V_1, μ_1) to (V_2, μ_2).

$(\Leftarrow)$ For any $x, y \in V_1$, we have

$$\begin{aligned}\phi(\tilde{\mu_1}(x, y)) &= \phi(\alpha(\mu_1(x, y))) = \beta(\phi(\mu_1(x, y))) \\ &= \beta(\mu_2(\phi(x), \phi(y))) = \tilde{\mu_2}(\phi(x), \phi(y)).\end{aligned}$$

Note that $\beta \circ \phi = \phi \circ \alpha$. We have that ϕ is an isomorphism from $(V_1, \tilde{\mu_1}, \alpha)$ to $(V_2, \tilde{\mu_2}, \beta)$. □

Lemma 2.12. *Simple multiplicative Hom-Jordan algebras with $\alpha \neq 0$ are Jordan-type Hom-Jordan algebras.*

Proof. Suppose that (V, μ, α) is a simple multiplicative Hom-Jordan algebra. According to Lemma 2.8 (2), we only need to show that α is invertible. If α is not invertible, then $\mathrm{Ker}(\alpha) \neq 0$. It is obvious that $\alpha(\mathrm{Ker}(\alpha)) \subseteq \mathrm{Ker}(\alpha)$. For any $x \in \mathrm{Ker}(\alpha)$, $y \in V$, we have

$$\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)) = \mu(0, \alpha(y)) = 0,$$

which implies that $\mu(\mathrm{Ker}(\alpha), V) \subseteq \mathrm{Ker}(\alpha)$. Then $\mathrm{Ker}(\alpha)$ is a nontrivial Hom-ideal of (V, μ, α), contradicting the assumption that (V, μ, α) is simple. Therefore,

$\mathrm{Ker}(\alpha) = 0$, i.e., α is invertible. Hence, (V, μ, α) is a Jordan-type Hom-Jordan algebra. □

Now we give a corollary of Proposition 2.11 using Lemma 2.12.

Corollary 2.13. *Two simple multiplicative Hom-Jordan algebras $(V_1, \tilde{\mu_1}, \alpha)$ and $(V_2, \tilde{\mu_2}, \beta)$ are isomorphic if and only if there exists an isomorphism ϕ between their induced Jordan algebras (V_1, μ_1) and (V_2, μ_2) and ϕ satisfies $\beta \circ \phi = \phi \circ \alpha$.*

3. Structure of multiplicative Hom-Jordan algebras

In this section, we discuss sufficient and necessary conditions for multiplicative Hom-Jordan algebras to be solvable, simple, and semi-simple.

Proposition 3.1. *Suppose that (V, μ, α) is a multiplicative Hom-Jordan algebra and I is a Hom-ideal of (V, μ, α). Then $(V/I, \bar{\mu}, \bar{\alpha})$ is a multiplicative Hom-Jordan algebra, where $\bar{\mu}(\bar{x}, \bar{y}) = \overline{\mu(x, y)}$, $\bar{\alpha}(\bar{x}) = \overline{\alpha(x)}$ for all $\bar{x}, \bar{y} \in V/I$.*

Proof. We have that $\bar{\mu}$ is commutative, since μ is commutative.

For any $\bar{x}, \bar{y} \in V/I$, we have

$$\begin{aligned}\bar{\mu}(\bar{\alpha}^2(\bar{x}), \bar{\mu}(\bar{y}, \bar{\mu}(\bar{x}, \bar{x}))) &= \overline{\mu(\alpha^2(x), \mu(y, \mu(x, x)))} \\ &= \overline{\mu(\mu(\alpha(x), y), \alpha(\mu(x, x)))} \\ &= \bar{\mu}(\bar{\mu}(\bar{\alpha}(\bar{x}), \bar{y}), \bar{\alpha}(\bar{\mu}(\bar{x}, \bar{x}))).\end{aligned}$$

Hence, $(V/I, \bar{\mu}, \bar{\alpha})$ is a Hom-Jordan algebra. Moreover, we see that

$$\bar{\alpha}(\bar{\mu}(\bar{x}, \bar{y})) = \overline{\alpha(\mu(x, y))} = \overline{\mu(\alpha(x), \alpha(y))} = \bar{\mu}(\bar{\alpha}(\bar{x}, \bar{y})),$$

which implies that $(V/I, \bar{\mu}, \bar{\alpha})$ is multiplicative. □

Corollary 3.2. *Suppose that (V, μ, α) is a multiplicative Hom-Jordan algebra and satisfies $\alpha^2 = \alpha$. Then $(V/\mathrm{Ker}(\alpha), \bar{\mu}, \bar{\alpha})$ is a Jordan-type Hom-Jordan algebra.*

Proof. If α is invertible, $\mathrm{Ker}(\alpha) = 0$. According to Lemma 2.8 (2), the conclusion is valid. If α isn't invertible, according to the proof of Lemma 2.12, we have that $\mathrm{Ker}(\alpha)$ is a Hom-ideal of (V, μ, α). Then we have that $(V/\mathrm{Ker}(\alpha), \bar{\mu}, \bar{\alpha})$ is a multiplicative Hom-Jordan algebra according to Proposition 3.1. Now we show that $\bar{\alpha}$ is invertible on $V/\mathrm{Ker}(\alpha)$.

Assume that $\bar{x} \in \mathrm{Ker}(\bar{\alpha})$. Then we have $\overline{\alpha(x)} = \bar{\alpha}(\bar{x}) = \bar{0}$, i.e., $\alpha(x) \in \mathrm{Ker}(\alpha)$. Note that $\alpha^2 = \alpha$. We have

$$\alpha(x) = \alpha^2(x) = \alpha(\alpha(x)) = 0,$$

which implies that $x \in \mathrm{Ker}(\alpha)$, i.e., $\bar{x} = \bar{0}$. Hence, $\bar{\alpha}$ is invertible. According to Lemma 2.8 (2), $(V/\mathrm{Ker}(\alpha), \bar{\mu}, \bar{\alpha})$ is a Jordan-type Hom-Jordan algebra. □

Theorem 3.3. *Suppose that (V, μ, α) is a multiplicative Hom-Jordan algebra and α is invertible. Then (V, μ, α) is solvable if and only if its induced Jordan algebra (V, μ') is solvable.*

Proof. Denote the derived series of (V,μ') and (V,μ,α) by $V^{(i)}$, $\tilde{V}^{(i)}$ $(i=1,2,\dots)$, respectively. Suppose that (V,μ') is solvable. Then there exists $m\in\mathbb{Z}^+$ such that $V^{(m)}=0$. Note that

$$\tilde{V}^{(1)}=\mu(V,V)=\alpha(\mu'(V,V))=\alpha(V^{(1)}),$$

$$\tilde{V}^{(2)}=\mu(\tilde{V}^{(1)},\tilde{V}^{(1)})=\mu(\alpha(V^{(1)}),\alpha(V^{(1)}))=\alpha^2(\mu'(V^{(1)},V^{(1)}))=\alpha^2(V^{(2)}).$$

We have $\tilde{V}^{(m)}=\alpha^m(V^{(m)})$ by induction. Hence, $\tilde{V}^{(m)}=0$, i.e., (V,μ,α) is solvable.

On the other hand, assume that (V,μ,α) is solvable. Then there exists $m\in\mathbb{Z}^+$ such that $\tilde{V}^{(m)}=0$. We have $\tilde{V}^{(m)}=\alpha^m(V^{(m)})$ by the above proof. Hence we have $V^{(m)}=0$, since α is invertible. Therefore, (V,μ') is solvable. □

Lemma 3.4. *Suppose that an algebra $\mathcal{A}$ over* F *can be decomposed into the unique direct sum of simple ideals $\mathcal{A}=\bigoplus_{i=1}^{s}\mathcal{A}_i$, where $\mathcal{A}_i$ aren't isomorphic to each other and $\alpha\in\operatorname{Aut}(\mathcal{A})$. Then $\alpha(\mathcal{A}_i)=\mathcal{A}_i$ $(1\le i\le s)$.*

Proof. For any $1\le i\le s$, we have

$$\alpha(\mathcal{A}_i)\mathcal{A}=\alpha(\mathcal{A}_i)\alpha(\mathcal{A})=\alpha(\mathcal{A}_i\mathcal{A})\subseteq\alpha(\mathcal{A}_i),$$

since $\mathcal{A}_i$ are ideals of $\mathcal{A}$. Similarly, we have $\mathcal{A}\alpha(\mathcal{A}_i)\subseteq\alpha(\mathcal{A}_i)$. Hence, $\alpha(\mathcal{A}_i)$ are also ideals of $\mathcal{A}$. Moreover, $\alpha(\mathcal{A}_i)$ are simple since $\mathcal{A}_i$ are simple.

Note that $\mathcal{A}=\bigoplus_{i=1}^{s}\mathcal{A}_i$. We have

$$\mathcal{A}=\alpha(\mathcal{A})=\alpha(\oplus_{i=1}^{s}\mathcal{A}_i)=\oplus_{i=1}^{s}\alpha(\mathcal{A}_i).$$

Note that the decomposition is unique. There exists $1\le j\le s$ such that $\alpha(\mathcal{A}_i)=\mathcal{A}_j$ for any $1\le i\le s$.

If $j\ne i$, then we have

$$\mathcal{A}_i\cong\alpha(\mathcal{A}_i)=\mathcal{A}_j,$$

contradicting the assumption that $\mathcal{A}_i$ aren't isomorphic to each other. Hence, we have $\alpha(\mathcal{A}_i)=\mathcal{A}_i$ $(1\le i\le s)$ for any $s\in\mathbb{N}$. □

Theorem 3.5. *(1) Suppose that (V,μ,α) is a simple multiplicative Hom-Jordan algebra. Then its induced Jordan algebra (V,μ') is semi-simple. Moreover, (V,μ') can be decomposed into a direct sum of isomorphic simple ideals; in addition, α acts simply transitively on simple ideals of the induced Jordan algebra.*

(2) Suppose that (V,μ') is a simple Jordan algebra and $\alpha\in\operatorname{Aut}(V)$. Define $\mu: V\times V\to V$ by

$$\mu(x,y)=\alpha(\mu'(x,y)),\quad \forall x,y\in V.$$

Then (V,μ,α) is a simple multiplicative Hom-Jordan algebra.

Proof. (1) According to the proof of Lemma 2.8 (2) and Lemma 2.12, α is an automorphism both on (V,μ,α) and (V,μ').

Suppose that V_1 is the maximal solvable ideal of (V,μ'). Then there exists $m\in\mathbb{Z}^+$ such that $V_1^{(m)}=0$.

Note that

$$\mu'(\alpha(V_1), V) = \mu'(\alpha(V_1), \alpha(V)) = \alpha(\mu'(V_1, V)) \subseteq \alpha(V_1),$$

$$(\alpha(V_1))^{(m)} = \alpha(V_1^{(m)}) = 0.$$

We have $\alpha(V_1)$ is also a solvable ideal of (V, μ'). Then we have $\alpha(V_1) \subseteq V_1$. Moreover,

$$\mu(V_1, V) = \alpha(\mu'(V_1, V)) \subseteq \alpha(V_1) \subseteq V_1,$$

so V_1 is a Hom-ideal of (V, μ, α). Then we have $V_1 = 0$ or $V_1 = V$ since (V, μ, α) is simple. If $V_1 = V$, according to the proof of Theorem 3.3, we have

$$\tilde{V}^{(m)} = \alpha^m(V^{(m)}) = \alpha^m(V_1^{(m)}) = 0;$$

on the other hand, $V = \mu(V, V)$ since (V, μ, α) is simple. Then we have $\tilde{V}^{(m)} = V$, a contradiction. Hence, $V_1 = 0$. Therefore, (V, μ') is semi-simple.

Since (V, μ') is semi-simple, we have $V = \bigoplus_{i=1}^{s} V_i$, where V_i $(1 \leq i \leq s)$ are simple ideals of (V, μ'). Because there may be isomorphic Jordan algebras in $V_1, V_2, \ldots, V_s$, we rearrange the order as follows:

$$V = V_{11} \oplus V_{12} \oplus \cdots \oplus V_{1m_1} \oplus V_{21} \oplus V_{22} \oplus \cdots \oplus V_{2m_2} \oplus \cdots \oplus V_{t1} \oplus V_{t2} \oplus \cdots \oplus V_{tm_t},$$

where

$$(V_{ij}, \mu') \cong (V_{ik}, \mu'), \quad 1 \leq j, k \leq m_i,\ i = 1, 2, \ldots, t.$$

According to Lemma 3.4, we have

$$\alpha(V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i}) = V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i},$$

$$\begin{aligned}\mu(V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i}, V) &= \alpha(\mu'(V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i}, V))\\ &\subseteq \alpha(V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i}) = V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i},\end{aligned}$$

and so we have that $V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i}$ are Hom-ideals of (V, μ, α). Since (V, μ, α) is simple, we have $V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i} = 0$ or V. So all but one $V_{i1} \oplus V_{i2} \oplus \cdots \oplus V_{im_i}$ must be 0. Without loss of generality, we can assume that

$$V = V_{11} \oplus V_{12} \oplus \cdots \oplus V_{1m_1}.$$

When $m_1 = 1$, (V, μ') is simple. When $m_1 > 1$, if

$$\alpha(V_{1p}) = V_{1p} \quad (1 \leq p \leq m_1)$$

then V_{1p} is a nontrivial ideal of (V, μ, α), which contradicts the fact that (V, μ, α) is simple. Hence,

$$\alpha(V_{1p}) = V_{1l} \quad (1 \leq l \neq p \leq m_1).$$

In addition, it is easy to show that $V_{11} \oplus \alpha(V_{11}) \oplus \cdots \oplus \alpha^{m_1-1}(V_{11})$ is a Hom-ideal of (V, μ, α). Therefore,

$$V = V_{11} \oplus \alpha(V_{11}) \oplus \cdots \oplus \alpha^{m_1-1}(V_{11}).$$

That is, α acts simply transitively on simple ideals of the induced Jordan algebra.

(2) Suppose that (V,μ') is a simple Jordan algebra. According to Lemma 2.8 (1), we have that (V,μ,α) is a multiplicative Hom-Jordan algebra. Suppose that V_1 is a nontrivial Hom-ideal of (V,μ,α); then we have

$$\mu'(V_1,V)=\alpha^{-1}(\mu(V_1,V))\subseteq\alpha^{-1}(V_1)=V_1.$$

So V_1 is a nontrivial ideal of (V,μ'), a contradiction. So (V,μ,α) has no nontrivial ideal. If $\mu(V,V)\subsetneqq V$, then

$$\mu'(V,V)=\alpha^{-1}(\mu(V,V))\subsetneqq\alpha^{-1}(V)=V,$$

contradicting the assumption that (V,μ') is a simple Jordan algebra. Hence, (V,μ,α) is simple. □

Theorem 3.6. *(1) Suppose that (V,μ,α) is a semi-simple multiplicative Hom-Jordan algebra. Then (V,μ,α) is a Jordan-type Hom-Jordan algebra and its induced Jordan algebra (V,μ') is also semi-simple.*

(2) Suppose that (V,μ') is a semi-simple Jordan algebra and has the decomposition $V=\bigoplus_{i=1}^{s}V_i$, where V_i $(1\leq i\leq s)$ are simple ideals of (V,μ'), and $\alpha\in\mathrm{Aut}(V)$ satisfies $\alpha(V_i)=V_i$ $(1\leq i\leq s)$. Then (V,μ,α) is a semi-simple multiplicative Hom-Jordan algebra and has a unique decomposition.

Proof. (1) According to the assumption, (V,μ,α) has the decomposition $V=\bigoplus_{i=1}^{s}V_i$, where V_i $(1\leq i\leq s)$ are simple Hom-ideals of (V,μ,α). Then $(V_i,\mu,\alpha|_{V_i})$ $(1\leq i\leq s)$ are simple Hom-Jordan algebras. According to the proof of Lemma 2.12, $\alpha|_{V_i}$ $(1\leq i\leq s)$ are invertible. Therefore, α is invertible on V. According to Lemma 2.8 (2), (V,μ,α) is a Jordan-type Hom-Jordan algebra and its induced Jordan algebra is (V,μ'), where $\mu'(x,y)=\alpha^{-1}(\mu(x,y))$ for all $x,y\in V$.

According to the proof of Theorem 3.5 (2), V_i $(i=1,2,\ldots,s)$ are ideals of (V,μ'). Moreover, (V_i,μ') are induced Jordan algebras of simple Hom-Jordan algebras $(V_i,\mu,\alpha|_{V_i})$, respectively. According to Theorem 3.5 (1), (V_i,μ') are semi-simple Jordan algebras and can be decomposed into a direct sum of isomorphic simple ideals $V_i=V_{i1}\oplus V_{i2}\oplus\cdots\oplus V_{im_i}$. Therefore, (V,μ') is semi-simple and has the decomposition into a direct sum of simple ideals

$$V=V_{11}\oplus V_{12}\oplus\cdots\oplus V_{1m_1}\oplus V_{21}\oplus V_{22}\oplus\cdots\oplus V_{2m_2}\oplus\cdots\oplus V_{s1}\oplus V_{s2}\oplus\cdots\oplus V_{sm_s}.$$

(2) According to Lemma 2.8 (1), (V,μ,α) is a multiplicative Hom-Jordan algebra. For all $1\leq i\leq s$, we have

$$\mu(V_i,V)=\alpha(\mu'(V_i,V))\subseteq\alpha(V_i)=V_i.$$

Note that $\alpha(V_i)=V_i$. We have that V_i are Hom-ideals of (V,μ,α).

If there exists $V_{i0}\subsetneqq V_i$ which is a nontrivial Hom-ideal of $(V_i,\mu,\alpha|_{V_i})$, then we have

$$\mu(V_{i0},V)=\mu(V_{i0},V_1\oplus V_2\oplus\cdots\oplus V_s)=\mu(V_{i0},V_i)\subseteq V_{i0},$$

so V_{i0} is a nontrivial Hom-ideal of (V,μ,α). According to the proof of Theorem 3.5 (2), V_{i0} is also a nontrivial ideal of (V,μ'). Hence, V_{i0} is also a nontrivial ideal of (V_i,μ'), a contradiction. Hence, V_i $(i=1,2,\ldots,s)$ are simple Hom-ideals of (V,μ,α). Therefore, (V,μ,α) is semi-simple and has a unique decomposition. □

Proposition 3.7. *Suppose that* (V, μ, α) *is a multiplicative Hom-Jordan algebra satisfying* $\alpha^2 = \alpha$ *and* $\mu(\mathrm{Im}(\alpha), V) \subseteq \mathrm{Im}(\alpha)$. *Then* (V, μ, α) *is isomorphic to the decomposition into a direct sum of Hom-Jordan algebras*

$$V \cong (V/\operatorname{Ker}(\alpha)) \oplus \operatorname{Ker}(\alpha).$$

Proof. Set $V_1 = (V/\operatorname{Ker}(\alpha)) \oplus \operatorname{Ker}(\alpha)$. According to Corollary 3.2, $(V/\operatorname{Ker}(\alpha), \bar{\mu}, \bar{\alpha})$ is a Hom-Jordan algebra. It is obvious that $(\operatorname{Ker}(\alpha), \mu, \alpha|_{\operatorname{Ker}(\alpha)})$ is a Hom-Jordan algebra. Define $\mu_1 : V_1 \times V_1 \to V_1$ and $\alpha_1 : V_1 \to V_1$ by

$$\mu_1((\bar{x}, k_1), (\bar{y}, k_2)) = (\overline{\mu(x, y)}, \mu(k_1, k_2)),$$
$$\alpha_1((\bar{x}, k_1)) = (\overline{\alpha(x)}, 0).$$

Then (V_1, μ_1, α_1) is a Hom-Jordan algebra and $V_1 = (V/\operatorname{Ker}(\alpha)) \oplus \operatorname{Ker}(\alpha)$ is the direct sum of ideals.

Now we show that $(V, \mu, \alpha) \cong (V_1, \mu_1, \alpha_1)$. According to the assumption, we have that $\mathrm{Im}(\alpha)$ is a Hom-ideal of (V, μ, α). For any $x \in \operatorname{Ker}(\alpha) \cap \mathrm{Im}(\alpha)$, there exists $y \in V$ such that $x = \alpha(y)$. Then we have

$$0 = \alpha(x) = \alpha^2(y) = \alpha(y) = x,$$

so $\operatorname{Ker}(\alpha) \cap \mathrm{Im}(\alpha) = \{0\}$. So for any $x \in V$, we have $x = x - \alpha(x) + \alpha(x)$, where $x - \alpha(x) \in \operatorname{Ker}(\alpha)$ and $\alpha(x) \in \mathrm{Im}(\alpha)$. Therefore, $V = \operatorname{Ker}(\alpha) \oplus \mathrm{Im}(\alpha)$.

Obviously, $(\mathrm{Im}(\alpha), \mu, \alpha|_{\mathrm{Im}(\alpha)})$ is a Hom-Jordan algebra.

Next we will show that $(\mathrm{Im}(\alpha), \mu, \alpha|_{\mathrm{Im}(\alpha)}) \cong (V/\operatorname{Ker}(\alpha), \bar{\mu}, \bar{\alpha})$. Define the map $\varphi : V/\operatorname{Ker}(\alpha) \to \mathrm{Im}(\alpha)$ by $\varphi(\bar{x}) = \alpha(x)$ for all $\bar{x} \in V/\operatorname{Ker}(\alpha)$. Obviously, φ is bijective. For all $\bar{x}, \bar{y} \in V/\operatorname{Ker}(\alpha)$, we have

$$\varphi(\bar{\mu}(\bar{x}, \bar{y})) = \varphi(\overline{\mu(x, y)}) = \alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)) = \mu(\varphi(\bar{x}), \varphi(\bar{y})),$$
$$\varphi(\bar{\alpha}(\bar{x})) = \varphi(\overline{\alpha(x)}) = \alpha^2(x) = \alpha(\varphi(\bar{x})),$$

which implies that $\varphi \circ \bar{\alpha} = \alpha \circ \varphi$. So φ is an isomorphism, i.e.,

$$(\mathrm{Im}(\alpha), \mu, \alpha|_{\mathrm{Im}(\alpha)}) \cong (V/\operatorname{Ker}(\alpha), \bar{\mu}, \bar{\alpha}).$$

Therefore, $V = \operatorname{Ker}(\alpha) \oplus \mathrm{Im}(\alpha) \cong (V/\operatorname{Ker}(\alpha)) \oplus \operatorname{Ker}(\alpha)$. □

4. Classification of simple multiplicative Hom-Jordan algebras

In this section we present a theorem about classification of simple multiplicative Hom-Jordan algebras. First, we give a construction of n-dimensional simple Hom-Jordan algebras.

Theorem 4.1. *There exist* n*-dimensional simple Hom-Jordan algebras for any* $n \in \mathbb{Z}^+$.

Proof. When $n = 1$, let $V = \mathbb{R}^+$ over $\mathbb{R}$, i.e., $\mu : V \times V \to V$, $\mu(a, b) = \frac{1}{2}(ab + ba)$ for all $a, b \in \mathbb{R}$. It is obvious that $\dim(V) = 1$. Take $\alpha = k\,\mathrm{id}_{\mathbb{R}}$ for $k \in \mathbb{R}$. Then (V, μ, α) is a 1-dimensional Hom-Jordan algebra. Obviously, (V, μ, α) is simple, since (V, μ, α) has no nontrivial Hom-ideal and $\mu(V, V) = V$.

When $n = 2$, let $\{e_0, e_1\}$ be a basis of a 2-dimensional vector space V over $\mathbb{C}$. Define a bilinear symmetric binary operation $\mu : V \times V \to V$ as follows:

$$\mu(e_0, e_0) = e_0, \quad \mu(e_1, e_1) = e_1, \quad \mu(e_0, e_1) = \mu(e_1, e_0) = e_0 + e_1.$$

Obviously, $\mu(V, V) = V$. Take $\alpha \in \mathrm{End}(V)$ such that

$$\alpha(e_0) = pe_0, \quad \alpha(e_1) = qe_1, \qquad p, q \in \mathbb{C}.$$

One can verify that (V, μ, α) is a 2-dimensional Hom-Jordan algebra. Next we will show that (V, μ, α) is simple.

Suppose that I is a nontrivial Hom-ideal of (V, μ, α). Then there exists $0 \neq a = t_1e_0 + t_2e_1 \in I$, where $t_1, t_2 \in \mathbb{C}$. Then we have $(t_1 + t_2)e_0 + t_2e_1 = \mu(a, e_0) \in I$, i.e., $\frac{t_1+t_2}{t_1} = \frac{t_2}{t_2}$; $t_1e_0 + (t_1 + t_2)e_1 = \mu(a, e_1) \in I$, i.e., $\frac{t_2}{t_1+t_2} = \frac{t_1}{t_1}$ since $\dim(I) = 1$. So $t_1 + t_2 = t_1$, $t_1 + t_2 = t_2$, which imply that $t_1 = 0$, $t_2 = 0$. Hence, $I = 0$, a contradiction. Therefore, (V, μ, α) is a 2-dimensional simple Hom-Jordan algebra.

When $n \geq 3$, let $\{a_{\bar{i}} \mid i \in \mathbb{Z}_n\}$ be a basis of an n-dimensional vector space V over $\mathbb{C}$. Define a bilinear symmetric binary operation $\mu : V \times V \to V$ as follows:

$$\mu\big(a_{\bar{i}}, a_{\overline{i+1}}\big) = \mu\big(a_{\overline{i+1}}, a_{\bar{i}}\big) = a_{\overline{i+2}};$$

all the others are zero. Then for any linear map $\alpha \in \mathrm{End}(V)$, (V, μ, α) is a Hom-Jordan algebra.

Next, we prove that (V, μ, α) is simple. Clearly, we have $\mu(V, V) = V$. Let W be a nonzero Hom-ideal of (V, μ, α); then there exists a nonzero element $x = \sum_{i=0}^{n-1} x_i a_{\bar{i}} \in W$. Suppose that $x_t \neq 0$. Since $x_t a_{\overline{t+2}} = \mu\big(a_{\bar{t}}, \mu(a_{\overline{t-1}}, x)\big) \in W$, we have $a_{\overline{t+2}} \in W$, so $a_{\overline{n-2}}, a_{\overline{n-1}} \in W$. Then $a_{\bar{0}} = \mu\big(a_{\overline{n-2}}, a_{\overline{n-1}}\big) \in W$, $a_{\bar{1}} = \mu\big(a_{\overline{n-1}}, a_{\bar{0}}\big) \in W$. Hence, we have all $a_{\bar{i}} \in W$ $(i \in \mathbb{Z}_n)$. Therefore $W = V$ and (V, μ, α) is simple. ☐

According to Theorem 3.5 (1) and Corollary 2.13, the dimension of a simple multiplicative Hom-Jordan algebra can only be an integer multiple of dimensions of simple Jordan algebras.

Also by Theorem 3.5 (1) and Corollary 2.13, in order to classify simple multiplicative Hom-Jordan algebras, we just classify automorphisms on their induced Jordan algebras; in particular, automorphisms on semi-simple Jordan algebras which are direct sum of finite isomorphic simple ideals.

Theorem 4.2. *Let J be a semi-simple Jordan algebra such that its n simple ideals are mutually isomorphic; moreover, J can be generated by its automorphism α (or β) and any simple ideal, and α_n (β_n) leaves each simple ideal of J invariant, where $\alpha_n = \alpha^n$ and $\beta_n = \beta^n$. Then there exists an automorphism φ on J satisfying $\varphi \circ \alpha = \beta \circ \varphi$ if and only if there exists an automorphism ψ on the simple ideal of J satisfying $\psi \circ \alpha^n = \beta^n \circ \psi$.*

Proof. Let J_1 be a simple ideal of J. Since α_n (or β_n) leaves each simple ideal of J invariant, we have $\alpha^n(J_1) = J_1$ (or $\beta^n(J_1) = J_1$) and we have

$$J = J_1 \oplus \alpha(J_1) \oplus \cdots \oplus \alpha^{n-1}(J_1)$$
$$\text{or} \quad J = J_1 \oplus \beta(J_1) \oplus \cdots \oplus \beta^{n-1}(J_1),$$

since J can be generated by its automorphism α (or β) and any simple ideal.

Choose a basis $x = (x_1, x_2, \dots, x_m)$ of J_1; then

$$x' = (x, \alpha(x), \alpha^2(x), \dots, \alpha^{n-1}(x)) \quad \text{and} \quad x'' = (x, \beta(x), \beta^2(x), \dots, \beta^{n-1}(x))$$

are both bases of J. Let $\alpha(x') = x'A$, $\beta(x'') = x''B$; then

$$A = \begin{pmatrix} 0 & & & & A_1 \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & & & & B_1 \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{pmatrix},$$

where $\alpha_n(x) = xA_1$, $\beta_n(x) = xB_1$.

If there exists an automorphism ψ on J_1 such that $\psi \circ \alpha_n = \beta_n \circ \psi$, then $MA_1 = B_1 M$, where M is defined by $\psi(x) = xM$. Define $\varphi(x') = x'' \operatorname{diag}(M, \dots, M)$. Then we have $\varphi(x') = (\psi(x), \beta(\psi(x)), \dots, \beta^{n-1}(\psi(x)))$. It is easy to verify that φ is an automorphism, since ψ is an automorphism. Moreover,

$$\begin{aligned} \varphi \circ \alpha(x') &= x'' \begin{pmatrix} M & & & & \\ & M & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & M \end{pmatrix} \begin{pmatrix} 0 & & & & A_1 \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{pmatrix} \\ &= x'' \begin{pmatrix} 0 & \cdots & \cdots & 0 & MA_1 \\ M & 0 & & & \\ & M & 0 & & \\ & & & \ddots & \\ & & & M & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \beta \circ \varphi(x') &= x'' \begin{pmatrix} 0 & & & & B_1 \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{pmatrix} \begin{pmatrix} M & & & & \\ & M & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & M \end{pmatrix} \\ &= x'' \begin{pmatrix} 0 & \cdots & \cdots & 0 & B_1 M \\ M & 0 & & & \\ & M & 0 & & \\ & & & \ddots & \\ & & & M & 0 \end{pmatrix}. \end{aligned}$$

Note that $MA_1 = B_1 M$. We have $\varphi \circ \alpha = \beta \circ \varphi$.

Now suppose that there exists an automorphism φ on J satisfying $\varphi \circ \alpha = \beta \circ \varphi$. According to the proof of Lemma 3.4, there exists $0 \le i \le n-1$ such that $\varphi(J_1) =$

$\beta^i(J_1)$. Then $\varphi\circ\alpha^j(J_1)=\beta^{i+j}(J_1)$ $(0\le i,j\le n-1)$. Let $\varphi(x)=\beta^i(x)M_1$; then $\varphi(x')=x''M$, where M is

$$\begin{pmatrix} M_1 & & & & \\ & M_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & M_1 \end{pmatrix} \quad (i=0),$$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ i+1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \cdots & n-i & n-i+1 & \cdots & n \end{array} \\ \begin{pmatrix} & & & B_1M_1 & & \\ & & & & B_1M_1 & \\ & & & & & \ddots \\ & & & & & & B_1M_1 \\ M_1 & & & & & \\ & \ddots & & & & \\ & & M_1 & & & \end{pmatrix} \end{array} \quad (1\le i\le n-1).$$

Defining $\psi(x)=xM_1$, we have

$$\varphi(x')=\begin{cases}(\psi(x),\beta(\psi(x)),\dots,\beta^{n-1}(\psi(x))), & (i=0);\\ (\beta^i(\psi(x)),\dots,\beta^{n-1}(\psi(x)),\beta^n(\psi(x)),\dots,\beta^{n+i-1}(\psi(x))), & (1\le i\le n-1).\end{cases}$$

Therefore, ψ is an automorphism on J_1, since φ is an automorphism on J. Moreover, we have $\psi\circ\alpha^n=\beta^n\circ\psi$, since $\varphi\circ\alpha=\beta\circ\varphi$. □

By Theorem 4.2, it is obvious that two simple multiplicative Hom-Jordan algebras (V_1,μ_1,α) and (V_2,μ_2,β) are isomorphic if and only if the automorphisms α^n and β^n on two simple ideals (as simple Jordan algebras) of the corresponding induced Jordan algebras are conjugate.

Combining Corollary 2.13, Theorem 3.5 (1) and Theorem 4.2, we get the following theorem.

Theorem 4.3. *All finite-dimensional simple multiplicative Hom-Jordan algebras can be denoted as (X,n,Γ_α), where X represents the type of the simple ideal (as the simple Jordan algebra) of the corresponding induced Jordan algebras, n represents numbers of simple ideals, and Γ_α represents the set of conjugate classes of the automorphism α^n on the simple Jordan algebra X, i.e., $\Gamma_\alpha=\{\phi\circ\alpha^n\circ\phi^{-1}\mid\phi\in\mathrm{Aut}(X)\}$.*

Example 4.4. Suppose that V is a 2-dimensional vector space with basis $\{e_1,e_2\}$. Define a bilinear map $\mu: V\times V\to V$ by

$$\begin{cases}\mu(e_1,e_1)=e_2,\\ \mu(e_2,e_2)=e_1,\\ \mu(e_1,e_2)=\mu(e_2,e_1)=0,\end{cases}$$

and a linear map $\alpha : V \to V$ by

$$\begin{cases} \alpha(e_1) = e_2, \\ \alpha(e_2) = e_1. \end{cases}$$

Then (V, μ, α) is a simple multiplicative Hom-Jordan algebra. Moreover, its induced Jordan algebra is (V, μ'), where $\mu' : V \times V \to V$ satisfies

$$\begin{cases} \mu'(e_1, e_1) = e_1, \\ \mu'(e_2, e_2) = e_2, \\ \mu'(e_1, e_2) = \mu(e_2, e_1) = 0. \end{cases}$$

(V, μ') is semi-simple and has the decomposition into simple ideals $V = V_1 \oplus V_2$, where V_1 and V_2 are simple ideals generated by e_1 and e_2, respectively. Moreover, we get that V_1 is isomorphic to V_2.

According to Theorem 4.3, (V, μ, α) can be denoted as $(V_1, 2, \alpha^2)$ or $(V_2, 2, \alpha^2)$.

5. Bimodules of simple multiplicative Hom-Jordan algebras

In this section we mainly study bimodules of simple multiplicative Hom-Jordan algebras. We give a theorem on relationships between bimodules of Jordan-type Hom-Jordan algebras and modules of their induced Jordan algebras. Moreover, some propositions about bimoudles of simple multiplicative Hom-Jordan algebras are also obtained.

Definition 5.1 ([3])**.** Let (V, μ, α) be a Hom-Jordan algebra. A V*-bimodule* is a Hom-module (W, α_W) that comes equipped with a left structure map $\rho_l : V \otimes W \to W (\rho_l(a \otimes w) = a \cdot w)$ and a right structure map $\rho_r : W \otimes V \to W (\rho_l(w \otimes a) = w \cdot a)$ such that the following conditions hold for all $a, b, c \in V$, $w \in W$:

(1) $\rho_r \circ \tau_1 = \rho_l$, where $\tau_1 : V \otimes W \to W \otimes V$, $a \otimes w \mapsto w \otimes a$;
(2) $\alpha_W(w \cdot a) \cdot \alpha(\mu(b, c)) + \alpha_W(w \cdot b) \cdot \alpha(\mu(c, a)) + \alpha_W(w \cdot c) \cdot \alpha(\mu(a, b)) = (\alpha_W(w) \cdot \mu(b, c)) \cdot \alpha^2(a) + (\alpha_W(w) \cdot \mu(c, a)) \cdot \alpha^2(b) + (\alpha_W(w) \cdot \mu(a, b)) \cdot \alpha^2(c)$;
(3) $\alpha_W(w \cdot a) \cdot \alpha(\mu(b, c)) + \alpha_W(w \cdot b) \cdot \alpha(\mu(c, a)) + \alpha_W(w \cdot c) \cdot \alpha(\mu(a, b)) = ((w \cdot a) \cdot \alpha(b)) \cdot \alpha^2(c) + ((w \cdot c) \cdot \alpha(b)) \cdot \alpha^2(a) + \mu(\mu(a, c), \alpha(b)) \cdot \alpha_W^2(w)$.

Example 5.2. In [3], suppose that (V, μ, α) is a Hom-Jordan algebra; then (V, α) is a V-bimodule where the structure maps are $\rho_l = \rho_r = \mu$.

Next, we construct an example over a field of prime characteristic.

Example 5.3. Let (V, μ, α) be the Hom-Jordan algebra in Example 4.4 and W a 1-dimensional vector space with basis $\{w_1\}$. Define $\alpha_W : W \to W$ to be a linear map by $\alpha_W(w_1) = w_1$. Define two linear maps: $\rho_l : V \otimes W \to W$ by

$$\begin{cases} \rho_l(e_1, w_1) = w_1, \\ \rho_l(e_2, w_1) = w_1 \end{cases}$$

and $\rho_r : W \otimes V \to W$ by

$$\begin{cases} \rho_l(w_1, e_1) = w_1, \\ \rho_l(w_1, e_2) = w_1. \end{cases}$$

Then (W, α_W) is a V-bimodule with the structure maps defined as above when the characteristic of the ground field is two.

Definition 5.4 ([8])**.** A *Jordan module* is a system consisting of a vector space V, a Jordan algebra J, and two compositions $x \cdot a$, $a \cdot x$ for x in V, a in J which are bilinear and satisfy

(i) $a \cdot x = x \cdot a$;
(ii) $(x{\cdot}a){\cdot}(b{\circ}c)+(x{\cdot}b){\cdot}(c{\circ}a)+(x{\cdot}c){\cdot}(a{\circ}b) = (x{\cdot}(b{\circ}c)){\cdot}a+(x{\cdot}(c{\circ}a)){\cdot}b+(x{\cdot}(a{\circ}b)){\cdot}c$;
(iii) $(((x{\cdot}a){\cdot}b){\cdot}c)+(((x{\cdot}c){\cdot}b){\cdot}a)+x{\cdot}(a{\circ}c{\circ}b) = (x{\cdot}a){\cdot}(b{\circ}c)+(x{\cdot}b){\cdot}(c{\circ}a)+(x{\cdot}c){\cdot}(a{\circ}b)$,

where $x \in V$, $a, b, c \in J$, and $a_1 \circ a_2 \circ a_3$ stands for $((a_1 \circ a_2) \circ a_3)$.

Theorem 5.5. *Let (V, μ, α) be a Jordan-type Hom-Jordan algebra with (V, μ') the induced Jordan algebra.*

(1) Let (W, α_W) be a V-bimodule of (V, μ, α) with $\rho_l(\rho_r)$ the left structure map (respectively, the right structure map). Suppose that α_W is invertible and satisfies $\alpha_W(w \cdot a) = \alpha_W(w) \cdot \alpha(a)$ for all $a \in V$, $w \in W$. Then W is a module of the induced Jordan algebra (V, μ') with two compositions $w \cdot' a = \alpha_W^{-1}(w \cdot a)$ and $a \cdot' w = \alpha_W^{-1}(a \cdot w)$ for all $a \in V$, $w \in W$.

(2) Let W be a module of the induced Jordan algebra (V, μ') with two compositions $w \cdot' a$ and $a \cdot' w$ for all $a \in V$, $w \in W$. If there exists $\alpha_W \in \mathrm{End}(W)$ such that $\alpha_W(w \cdot' a) = \alpha_W(w) \cdot' \alpha(a)$ for all $a \in V$, $w \in W$, then (W, α_W) is a V-bimodule of (V, μ, α) with the left structure map $\rho_l : V \otimes W \to W$ $(\rho_l(a \otimes w) = \alpha_W(a \cdot' w))$ and the right structure map $\rho_r : W \otimes V \to W$ $(\rho_r(w \otimes a) = \alpha_W(w \cdot' a))$.

Proof. (1) For any $x \in W$, $a, b, c \in V$, we have

$$a \cdot' x = \alpha_W^{-1}(a \cdot x) = \alpha_W^{-1}(\rho_l(a \otimes x)) = \alpha_W^{-1}(\rho_r \circ \tau_1(a \otimes x)) = \alpha_W^{-1}(x \cdot a) = x \cdot' a;$$

$$\begin{aligned}
&(x \cdot' a) \cdot' \mu'(b, c) + (x \cdot' b) \cdot' \mu'(c, a) + (x \cdot' c) \cdot' \mu'(a, b) \\
&\quad = \alpha_W^{-1}(\alpha_W^{-1}(x \cdot a) \cdot \alpha^{-1}(\mu(b, c))) + \alpha_W^{-1}(\alpha_W^{-1}(x \cdot b) \cdot \alpha^{-1}(\mu(c, a))) \\
&\qquad + \alpha_W^{-1}(\alpha_W^{-1}(x \cdot c) \cdot \alpha^{-1}(\mu(a, b))) \\
&\quad = \alpha_W^{-3}(\alpha_W^2(\alpha_W^{-1}(x \cdot a) \cdot \alpha^{-1}(\mu(b, c))) + \alpha_W^2(\alpha_W^{-1}(x \cdot b) \cdot \alpha^{-1}(\mu(c, a))) \\
&\qquad + \alpha_W^2(\alpha_W^{-1}(x \cdot c) \cdot \alpha^{-1}(\mu(a, b)))) \\
&\quad = \alpha_W^{-3}(\alpha_W((x \cdot a) \cdot \mu(b, c)) + \alpha_W((x \cdot b) \cdot \mu(c, a)) + \alpha_W((x \cdot c) \cdot \mu(a, b))) \\
&\quad = \alpha_W^{-3}(\alpha_W(x \cdot a) \cdot \alpha(\mu(b, c)) + \alpha_W(x \cdot b) \cdot \alpha(\mu(c, a)) + \alpha_W(x \cdot c) \cdot \alpha(\mu(a, b))) \\
&\quad = \alpha_W^{-3}((\alpha_W(x) \cdot \mu(b, c)) \cdot \alpha^2(a) + (\alpha_W(x) \cdot \mu(c, a)) \cdot \alpha^2(b) \\
&\qquad + (\alpha_W(x) \cdot \mu(a, b)) \cdot \alpha^2(c)) \\
&\quad = \alpha_W^{-3}(\alpha_W(x \cdot \alpha^{-1}(\mu(b, c))) \cdot \alpha^2(a) + \alpha_W(x \cdot \alpha^{-1}(\mu(c, a))) \cdot \alpha^2(b) \\
&\qquad + \alpha_W(x \cdot \alpha^{-1}(\mu(a, b))) \cdot \alpha^2(c)) \\
&\quad = \alpha_W^{-3}(\alpha_W^2(x \cdot' \mu'(b, c)) \cdot \alpha^2(a) + \alpha_W^2(x \cdot' \mu'(c, a)) \cdot \alpha^2(b) \\
&\qquad + \alpha_W^2(x \cdot' \mu'(a, b)) \cdot \alpha^2(c))
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_W^{-3}(\alpha_W^2((x \cdot' \mu'(b,c)) \cdot a) + \alpha_W^2((x \cdot' \mu'(c,a)) \cdot b) + \alpha_W^2((x \cdot' \mu'(a,b)) \cdot c)) \\
&= \alpha_W^{-1}((x \cdot' \mu'(b,c)) \cdot a) + \alpha_W^{-1}((x \cdot' \mu'(c,a)) \cdot b) + \alpha_W^{-1}((x \cdot' \mu'(a,b)) \cdot c) \\
&= (x \cdot' \mu'(b,c)) \cdot' a + (x \cdot' \mu'(c,a)) \cdot' b + (x \cdot' \mu'(a,b)) \cdot' c;
\end{aligned}
$$

$$
\begin{aligned}
&(x \cdot' a) \cdot' \mu'(b,c) + (x \cdot' b) \cdot' \mu'(c,a) + (x \cdot' c) \cdot' \mu'(a,b) \\
&\quad = \alpha_W^{-3}(\alpha_W(x \cdot a) \cdot \alpha(\mu(b,c)) + \alpha_W(x \cdot b) \cdot \alpha(\mu(c,a)) + \alpha_W(x \cdot c) \cdot \alpha(\mu(a,b))) \\
&\quad = \alpha_W^{-3}(((x \cdot a) \cdot \alpha(b)) \cdot \alpha^2(c) + ((x \cdot c) \cdot \alpha(b)) \cdot \alpha^2(a) + \mu(\mu(a,c), \alpha(b)) \cdot \alpha_W^2(x)) \\
&\quad = \alpha_W^{-3}((\alpha_W(x \cdot' a) \cdot \alpha(b)) \cdot \alpha^2(c) + (\alpha_W(x \cdot' c) \cdot \alpha(b)) \cdot \alpha^2(a) \\
&\qquad + \alpha(\mu'(\alpha(\mu'(a,c)), \alpha(b))) \cdot \alpha_W^2(x)) \\
&\quad = \alpha_W^{-3}(\alpha_W((x \cdot' a) \cdot b) \cdot \alpha^2(c) + \alpha_W((x \cdot' c) \cdot b) \cdot \alpha^2(a) \\
&\qquad + \alpha^2(\mu'(\mu'(a,c), b)) \cdot \alpha_W^2(x)) \\
&\quad = \alpha_W^{-3}(\alpha_W^2((x \cdot' a) \cdot' b) \cdot \alpha^2(c) + \alpha_W^2((x \cdot' c) \cdot' b) \cdot \alpha^2(a) \\
&\qquad + \alpha^2(\mu'(\mu'(a,c), b)) \cdot \alpha_W^2(x)) \\
&\quad = \alpha_W^{-3}(\alpha_W^2(((x \cdot' a) \cdot' b) \cdot c) + \alpha_W^2(((x \cdot' c) \cdot' b) \cdot a) + \alpha_W^2(\mu'(\mu'(a,c), b) \cdot x)) \\
&\quad = \alpha_W^{-1}(((x \cdot' a) \cdot' b) \cdot c) + \alpha_W^{-1}(((x \cdot' c) \cdot' b) \cdot a) + \alpha_W^{-1}(\mu'(\mu'(a,c), b) \cdot x) \\
&\quad = ((x \cdot' a) \cdot' b) \cdot' c + ((x \cdot' c) \cdot' b) \cdot' a + \mu'(\mu'(a,c), b) \cdot' x.
\end{aligned}
$$

Therefore, W is a module of the induced Jordan algebra (V, μ').

(2) For any $w \in W$, $a, b, c \in V$, we have

$$\rho_r \circ \tau_1(a \otimes w) = \alpha_W(w \cdot' a) = \alpha_W(a \cdot' w) = \rho_l(a \otimes w),$$

which implies that $\rho_r \circ \tau_1 = \rho_l$. Then we see that

$$
\begin{aligned}
&\alpha_W(w \cdot a) \cdot \alpha(\mu(b,c)) + \alpha_W(w \cdot b) \cdot \alpha(\mu(c,a)) + \alpha_W(w \cdot c) \cdot \alpha(\mu(a,b)) \\
&\quad = \alpha_W(\alpha_W^2(w \cdot' a) \cdot' \alpha^2(\mu'(b,c))) + \alpha_W(\alpha_W^2(w \cdot' b) \cdot' \alpha^2(\mu'(c,a))) \\
&\qquad + \alpha_W(\alpha_W^2(w \cdot' c) \cdot' \alpha^2(\mu'(a,b))) \\
&\quad = \alpha_W(\alpha_W^2((w \cdot' a) \cdot' \mu'(b,c)) + \alpha_W^2((w \cdot' b) \cdot' \mu'(c,a)) + \alpha_W^2((w \cdot' c) \cdot' \mu'(a,b))) \\
&\quad = \alpha_W^3((w \cdot' a) \cdot' \mu'(b,c) + (w \cdot' b) \cdot' \mu'(c,a) + (w \cdot' c) \cdot' \mu'(a,b)) \\
&\quad = \alpha_W^3((w \cdot' \mu'(b,c)) \cdot' a + (w \cdot' \mu'(c,a)) \cdot' b + (w \cdot' \mu'(a,b)) \cdot' c) \\
&\quad = \alpha_W^2(\alpha_W(w \cdot' \mu'(b,c)) \cdot' \alpha(a)) + \alpha_W^2(\alpha_W(w \cdot' \mu'(c,a)) \cdot' \alpha(b)) \\
&\qquad + \alpha_W^2(\alpha_W(w \cdot' \mu'(a,b)) \cdot' \alpha(c)) \\
&\quad = \alpha_W(\alpha_W^2(w \cdot' \mu'(b,c)) \cdot' \alpha^2(a)) + \alpha_W(\alpha_W^2(w \cdot' \mu'(c,a)) \cdot' \alpha^2(b)) \\
&\qquad + \alpha_W(\alpha_W^2(w \cdot' \mu'(a,b)) \cdot' \alpha^2(c)) \\
&\quad = \alpha_W^2(w \cdot' \mu'(b,c)) \cdot \alpha^2(a) + \alpha_W^2(w \cdot' \mu'(c,a)) \cdot \alpha^2(b) + \alpha_W^2(w \cdot' \mu'(a,b)) \cdot \alpha^2(c) \\
&\quad = \alpha_W(\alpha_W(w) \cdot' \alpha(\mu'(b,c))) \cdot \alpha^2(a) + \alpha_W(\alpha_W(w) \cdot' \alpha(\mu'(c,a))) \cdot \alpha^2(b) \\
&\qquad + \alpha_W(\alpha_W(w) \cdot' \alpha(\mu'(a,b))) \cdot \alpha^2(c)
\end{aligned}
$$

$$= (\alpha_W(w) \cdot \mu(b,c)) \cdot \alpha^2(a) + (\alpha_W(w) \cdot \mu(c,a)) \cdot \alpha^2(b) + (\alpha_W(w) \cdot \mu(a,b)) \cdot \alpha^2(c);$$

$$\begin{aligned}
&\alpha_W(w \cdot a) \cdot \alpha(\mu(b,c)) + \alpha_W(w \cdot b) \cdot \alpha(\mu(c,a)) + \alpha_W(w \cdot c) \cdot \alpha(\mu(a,b)) \\
&\quad = \alpha_W^3((w \cdot' a) \cdot' \mu'(b,c) + (w \cdot' b) \cdot' \mu'(c,a) + (w \cdot' c) \cdot' \mu'(a,b)) \\
&\quad = \alpha_W^3(((w \cdot' a) \cdot' b) \cdot' c + ((w \cdot' c) \cdot' b) \cdot' a + \mu'(\mu'(a,c),b) \cdot' w) \\
&\quad = \alpha_W^2(\alpha_W((w \cdot' a) \cdot' b) \cdot' \alpha(c)) + \alpha_W^2(\alpha_W((w \cdot' c) \cdot' b) \cdot' \alpha(a)) \\
&\qquad + \alpha_W^2(\alpha(\mu'(\mu'(a,c),b)) \cdot' \alpha_W(w)) \\
&\quad = \alpha_W(\alpha_W^2((w \cdot' a) \cdot' b) \cdot' \alpha^2(c)) + \alpha_W(\alpha_W^2((w \cdot' c) \cdot' b) \cdot' \alpha^2(a)) \\
&\qquad + \alpha_W(\alpha^2(\mu'(\mu'(a,c),b)) \cdot' \alpha_W^2(w)) \\
&\quad = \alpha_W^2((w \cdot' a) \cdot' b) \cdot \alpha^2(c) + \alpha_W^2((w \cdot' c) \cdot' b) \cdot \alpha^2(a) + \alpha^2(\mu'(\mu'(a,c),b)) \cdot \alpha_W^2(w) \\
&\quad = \alpha_W(\alpha_W(w \cdot' a) \cdot' \alpha(b)) \cdot \alpha^2(c) + \alpha_W(\alpha_W(w \cdot' c) \cdot' \alpha(b)) \cdot \alpha^2(a) \\
&\qquad + \alpha(\mu'(\alpha(\mu'(a,c)), \alpha(b))) \cdot \alpha_W^2(w) \\
&\quad = ((w \cdot a) \cdot \alpha(b)) \cdot \alpha^2(c) + ((w \cdot c) \cdot \alpha(b)) \cdot \alpha^2(a) + \mu(\mu(a,c), \alpha(b)) \cdot \alpha_W^2(w).
\end{aligned}$$

Therefore, (W, α_W) is a V-bimodule of (V, μ, α). ☐

Definition 5.6. For a bimodule (W, α_W) of a Hom-Jordan algebra (V, μ, α), if a subspace $W_0 \subseteq W$ satisfies the conditions $\rho_l(a \otimes w) \in W_0$ for any $a \in V$, $w \in W_0$ and $\alpha_W(W_0) \subseteq W_0$, then $(W_0, \alpha_W|_{W_0})$ is called a *V-submodule* of (W, α_W). A bimodule (W, α_W) of a Hom-Jordan algebra (V, μ, α) is called *irreducible* if it has precisely two V-submodules (itself and 0) and is called *completely reducible* if $W = W_1 \oplus W_2 \oplus \cdots \oplus W_s$, where $(W_i, \alpha_W|_{W_i})$ are irreducible V-submodules.

Proposition 5.7. *Suppose that (W, α_W) is a bimodule of the simple multiplicative Hom-Jordan algebra (V, μ, α) with $\alpha_W(a \cdot w) = \alpha(a) \cdot \alpha_W(w)$ for all $a \in V$, $w \in W$. Then,* $\operatorname{Ker}(\alpha_W)$ *and* $\operatorname{Im}(\alpha_W)$ *are submodules of W for (V, μ, α). Moreover, we have an isomorphism of (V, μ, α)-modules $\overline{\alpha_W} : W/\operatorname{Ker}(\alpha_W) \to \operatorname{Im}(\alpha_W)$.*

Proof. For any $w \in \operatorname{Ker}(\alpha_W)$, we have

$$\alpha_W(a \cdot w) = \alpha(a) \cdot \alpha_W(w) = 0, \quad \forall a \in V,$$

which implies that $a \cdot w \in \operatorname{Ker}(\alpha_W)$. Obviously, $\alpha_W(\operatorname{Ker}(\alpha_W)) \subseteq \operatorname{Ker}(\alpha_W)$. Therefore, $\operatorname{Ker}(\alpha_W)$ is a submodule of W for (V, μ, α).

For any $w \in \operatorname{Im}(\alpha_W)$, $a \in V$, there exist $u \in W$ and $\tilde{a} \in V$ such that $w = \alpha_W(u)$ and $a = \alpha(\tilde{a})$. Then

$$a \cdot w = \alpha(\tilde{a}) \cdot \alpha_W(u) = \alpha_W(\tilde{a} \cdot u) \in \operatorname{Im}(\alpha_W).$$

Note that $\alpha_W(\operatorname{Im}(\alpha_W)) \subseteq \operatorname{Im}(\alpha_W)$. We obtain that $\operatorname{Im}(\alpha_W)$ is a submodule of W for (V, μ, α).

Define $\overline{\alpha_W} : W/\operatorname{Ker}(\alpha_W) \to \operatorname{Im}(\alpha_W)$ by $\overline{\alpha_W}(\bar{w}) = \alpha_W(w)$. It is easy to verify that $\overline{\alpha_W}$ is an isomorphism. ☐

Corollary 5.8. *If* (W, α_W) *is a irreducible bimodule of the simple multiplicative Hom-Jordan algebra* (V, μ, α) *with* $\alpha_W(a \cdot w) = \alpha(a) \cdot \alpha_W(w)$ *for all* $a \in V$, $w \in W$, *then* α_W *is invertible.*

Proposition 5.9. *Suppose that* (V, μ, α) *is a simple multiplicative Hom-Jordan algebra and* (W, α_W) *is a bimodule with* $\alpha_W(a \cdot w) = \alpha(a) \cdot \alpha_W(w)$ *for all* $a \in V$, $w \in W$, *and suppose that* α_W *is invertible. If* W *is an irreducible module of the induced Jordan algebra* (V, μ') *with two compositions* $w \cdot' a = \alpha_W^{-1}(w \cdot a)$, $a \cdot' w = \alpha_W^{-1}(a \cdot w)$ *for all* $a \in V$, $w \in W$, *then* (W, α_W) *is an irreducible bimodule of* (V, μ, α).

Proof. Assume that (W, α_W) is reducible. Then there exists $W_0 \neq \{0_W\}$ a subspace of W such that $(W_0, \alpha_W|_{W0})$ is a submodule of (W, α_W). That is, $\alpha_W(W_0) \subseteq W_0$ and $a \cdot w \in W_0$, for any $a \in V$, $w \in W_0$. Hence, $a \cdot' w = \alpha_W^{-1}(a \cdot w) \in \alpha_W^{-1}(W_0) = W_0$. So W_0 is a nontrivial submodule of W for (V, μ'), a contradiction. Hence, (W, α_W) is an irreducible bimodule of (V, μ, α). □

Remark 5.10. In [18], the author introduced another definition of Hom-Jordan algebras, using which one could verify that all the above results are also valid.

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