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REGULAR AUTOMORPHISMS AND CALOGERO-MOSER FAMILIES

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Abstract. We study the subvariety of fixed points of an automorphism of a Calogero-Moser space induced by a regular element of finite order of the normalizer of the associated complex reflection group W. We determine some of (and conjecturally all) the $\mathbb{C}^\times$-fixed points of its unique irreducible component of maximal dimension in terms of the character table of W. This is inspired by the mysterious relations between the geometry of Calogero-Moser spaces and unipotent representations of finite reductive groups, which is the theme of another paper [4].

If $\mathbf{G}$ is a split reductive group over a finite field with q elements $\mathbb{F}_q$ with Weyl group W, Deligne and Lusztig [12] have defined a particular class of irreducible characters of the finite group $G = \mathbf{G}(\mathbb{F}_q)$, called the *unipotent characters* of G. To W, one can also associate a Calogero-Moser space $\mathcal{Z}$ *at equal parameters*, which is a complex irreducible normal affine Poisson variety endowed with a $\mathbb{C}^\times$-action [13]. The main theme of a previous paper of the author [4] is the observation that many aspects of the combinatorics of unipotent characters of G have a conjectural analogue in the geometry of $\mathcal{Z}$, thanks to the Poisson structure and the $\mathbb{C}^\times$-action. Here are two examples:

- Unipotent characters have been partitioned by Lusztig [20] into *families* and it has been conjectured by Gordon-Martino [17] that these families are in bijection with $\mathbb{C}^\times$-fixed points of $\mathcal{Z}$. Note that this conjecture has been proved in all cases except type E_6, E_7 and E_8 (see [17, 1, 7]).
- For d a natural number, Broué-Malle-Michel [9] defined a partition of unipotent characters into *d-Harish-Chandra series* (generalizing the classical partition into *Harish-Chandra series*, which correspond to $d = 1$ case). This partition is conjecturally related to the stratification of $\mathcal{Z}^{\boldsymbol{\mu}_d}$ by symplectic leaves (here, $\boldsymbol{\mu}_d$ denotes the group of complex d-th roots of unity): the reader may find more details in [4, §12.C]. See [4, Part IV] for a list of cases where this conjecture is proved.

In the second point, whenever d is a *regular number* in the sense of Springer [21], it has been observed by Broué-Malle-Michel [10, Rem. 4.21] that the families of unipotent characters which meet the principal d-Harish-Chandra series are characterized by a property involving character values of W (again, more details may be

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found in [4, Ex. 12.9]). If one believes in the analogy between unipotent characters and geometry of $\mathcal{Z}$, this suggests [4, Conj. 7.5] a conjectural characterization of $\mathbb{C}^\times$-fixed points meeting the unique irreducible component of $\mathcal{Z}^{\boldsymbol{\mu}_d}$ of maximal dimension in terms of the character table of W. The proof of one direction of this conjecture is the theme of the present paper.

Note that the conjecture [4, Conj. 7.5] involves only the Calogero-Moser space and can be studied without any reference to unipotent characters. Moreover, since Calogero-Moser spaces are defined for any finite complex reflection group (and not only for Weyl groups) and for a bigger family of parameters, this conjecture is somewhat more general than what has been explained above in this introduction, which can be seen as a motivation for the results obtained here. Therefore, from now on, we will work in this more general context of complex reflection groups. Let V be a finite dimensional complex vector space and let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ generated by reflections (i.e. automorphisms of V whose fixed points space is an hyperplane). To some parameter k, Etingof and Ginzburg [13] have associated a normal irreducible affine complex variety $\mathcal{Z}_k = \mathcal{Z}_k(V,W)$ called a (generalized) *Calogero-Moser space*. If τ is an element of finite order of the normalizer of W in $\mathbf{GL}_{\mathbb{C}}(V)$ stabilizing the parameter k, it induces an automorphism of $\mathcal{Z}_k$.

We denote by V_{reg} the open subset of V on which W acts freely, and we assume that $V_{\mathrm{reg}}^\tau \neq \varnothing$ (then τ is called *regular*). In this case, there exists a unique irreducible component $(\mathcal{Z}_k^\tau)_{\max}$ of $\mathcal{Z}_k^\tau$ of maximal dimension (as it will be explained in Section 2). Recall that $\mathcal{Z}_k$ is endowed with a $\mathbb{C}^\times$-action and that we have a surjective map $\mathrm{Irr}(W) \to \mathcal{Z}_k^{\mathbb{C}^\times}$ defined by Gordon [16] (induced by the action of the center of a rational Cherednik algebra on *baby Verma modules*) whose fibers are called the *Calogero-Moser k-families* of W. Here, $\mathrm{Irr}(W)$ is the set of irreducible characters of W. If $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_p$ its associated Calogero-Moser k-family. It is a natural question to wonder which $\mathbb{C}^\times$-fixed points of $\mathcal{Z}_k^\tau$ belong to $(\mathcal{Z}_k^\tau)_{\max}$. The aim of this note is to provide a partial answer in terms of the character table of W:

Theorem A. — *Assume that $V_{\mathrm{reg}}^\tau \neq \varnothing$. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. If there exists χ in $\mathfrak{F}_p^\tau$ such that $\tilde{\chi}(\tau) \neq 0$, then $p \in (\mathcal{Z}_k^\tau)_{\max}$.*

In this statement, if χ is a τ-stable irreducible character of W, we denote by $\tilde{\chi}$ an extension of χ to the finite group $W\langle\tau\rangle$ (see [18, Coro. 11.22] for the existence of $\tilde{\chi}$): note that $|\tilde{\chi}(\tau)|^2$ does not depend on the choice of $\tilde{\chi}$ (see [18, Coro. 6.17]). Our proof of Theorem A makes an extensive use of the Gaudin operators introduced in [6, §8.3.B]: whenever $\tilde{\chi}(\tau) \neq 0$, the decomposition of a representation affording χ with respect to generalized eigenspaces of the Gaudin operators allows to construct a τ-fixed point p' in $(\mathcal{Z}_k^\tau)_{\max}$ such that $p = \lim_{\xi \to 0} \xi \cdot p'$.

We conjecture that the converse of Theorem A holds [4, Conj. 7.5]:

> ***Conjecture B.*** — *Assume that $V_{\mathrm{reg}}^\tau \neq \varnothing$. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$ and $p \in (\mathcal{Z}_k^\tau)_{\max}$. Then there exists χ in $\mathfrak{F}_p^\tau$ such that $\tilde{\chi}(\tau) \neq 0$.*

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General notation. Throughout this paper, we will abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$ and all varieties will be algebraic, complex, quasi-projective and reduced. If $\mathcal{X}$ is an affine variety, we denote by $\mathbb{C}[\mathcal{X}]$ its coordinate ring.

If X is a subset of a vector space V (or of its dual V^*), and if Γ is a subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, we denote by Γ_X the pointwise stabilizer of X. If moreover Γ is finite, we will identify $(V^\Gamma)^*$ and $(V^*)^\Gamma$.

1. Calogero-Moser spaces and families

Hypothesis and notation. *We fix in this paper a finite dimensional complex vector space V and a finite subgroup W of $\mathbf{GL}_{\mathbb{C}}(V)$. We set*

$$\operatorname{Ref}(W) = \{s \in W \mid \operatorname{codim}_{\mathbb{C}} V^s = 1\}$$

and we assume throughout this paper that

$$W = \langle \operatorname{Ref}(W) \rangle,$$

i.e. that W is a complex reflection group.

1.A. About W.

We set $\varepsilon : W \to \mathbb{C}^\times$, $w \mapsto \det(w)$. We identify $\mathbb{C}[V]$ (resp. $\mathbb{C}[V^*]$) with the symmetric algebra $\mathrm{S}(V^*)$ (resp. $\mathrm{S}(V)$).

We denote by $\mathcal{A}$ the set of *reflecting hyperplanes* of W, namely

$$\mathcal{A} = \{V^s \mid s \in \operatorname{Ref}(W)\}.$$

If $H \in \mathcal{A}$, we denote by α_H an element of V^* such that $H = \operatorname{Ker}(\alpha_H)$ and by $\alpha_H^\vee$ an element of V such that $V = H \oplus \mathbb{C}\alpha_H^\vee$ and the line $\mathbb{C}\alpha_H^\vee$ is W_H-stable. We set $e_H = |W_H|$. Note that W_H is cyclic of order e_H and that $\operatorname{Irr}(W_H) = \{\operatorname{Res}_{W_H}^W \varepsilon^j \mid 0 \le j \le e-1\}$. We denote by $\varepsilon_{H,j}$ the (central) primitive idempotent of $\mathbb{C}W_H$ associated with the character $\operatorname{Res}_{W_H}^W \varepsilon^{-j}$, namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \varepsilon(w)^j w \in \mathbb{C}W_H.$$

If Ω is a W-orbit of reflecting hyperplanes, we write e_Ω for the common value of all the e_H, where $H \in \Omega$. We denote by $\aleph$ the set of pairs (Ω, j) where $\Omega \in \mathcal{A}$ and $0 \le j \le e_\Omega - 1$. The vector space of families of complex numbers indexed by $\aleph$ will be denoted by $\mathbb{C}^\aleph$: elements of $\mathbb{C}^\aleph$ will be called *parameters*. If $k = (k_{\Omega,j})_{(\Omega,j)\in\aleph} \in \mathbb{C}^\aleph$, we define $k_{H,j}$ for all $H \in \Omega$ and $j \in \mathbb{Z}$ by $k_{H,j} = k_{\Omega,j_0}$ where Ω is the W-orbit of H and j_0 is the unique element of $\{0, 1, \ldots, e_H - 1\}$ such that $j \equiv j_0 \mod e_H$.

We denote by V_{reg} the set of elements v of V such that $W_v = 1$. It is an open subset of V and recall from Steinberg-Serre Theorem [8, Theo. 4.7] that

$$V_{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathcal{A}} H. \tag{1.1}$$

In particular, V_{reg} is a principal open affine subset of V and $\mathbb{C}[V_{\mathrm{reg}}] = \mathbb{C}[V][1/\prod_{H\in\mathcal{A}} \alpha_H]$.

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1.B. **Rational Cherednik algebra at $t = 0$.** Let $k \in \mathbb{C}^{\aleph}$. We define the *rational Cherednik algebra* $\mathbf{H}_k$ (*at* $t = 0$) to be the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\mathrm{T}(V \oplus V^*)$ with the group W) by the relations

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \displaystyle\sum_{H \in \mathcal{A}} \sum_{j=0}^{e_H - 1} e_H(k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,j}, \end{cases} \tag{1.2}$$

for all $x, x' \in V^*$, $y, y' \in V$. Here $\langle \ , \ \rangle : V \times V^* \to \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathbf{H}_k$ and $\mathbb{C}[V^*] \to \mathbf{H}_k$. Recall [13, Theo. 1.3] that we have an isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k \tag{1.3}$$

induced by multiplication (this is the so-called *PBW-decomposition*).

Remark 1.4. Let $(l_\Omega)_{\Omega \in \mathcal{A}/W}$ be a family of complex numbers and let $k' \in \mathbb{C}^{\aleph}$ be defined by $k'_{\Omega,j} = k_{\Omega,j} + l_\Omega$. Then $\mathbf{H}_k = \mathbf{H}_{k'}$. This means that there is no loss of generality if we consider for instance only parameters k such that $k_{\Omega,0} = 0$ for all Ω, or only parameters k such that $k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,e_\Omega - 1} = 0$ for all Ω (as in [6]). ■

1.C. **Calogero-Moser space.** We denote by $\mathbf{Z}_k$ the center of the algebra $\mathbf{H}_k$: it is well-known [13, Theo 3.3 and Lem. 3.5] that $\mathbf{Z}_k$ is an integral domain, which is integrally closed. Moreover, it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras [16, Prop. 3.6]. So, by the PBW-decomposition, $\mathbf{Z}_k$ contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$, and it is a free $\mathbf{P}$-module of rank $|W|$ (see [13, Prop. 4.15]). We denote by $\mathcal{Z}_k$ the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{Z}_k]$ is $\mathbf{Z}_k$: this is the *Calogero-Moser space* associated with the datum (V, W, k). It is irreducible and normal.

We set $\mathcal{P} = V/W \times V^*/W$, so that $\mathbb{C}[\mathcal{P}] = \mathbf{P}$ and the inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k$ induces a morphism of varieties

$$\Upsilon_k : \mathcal{Z}_k \longrightarrow \mathcal{P}$$

which is finite and flat.

1.D. **Calogero-Moser families.** Using the PBW-decomposition, we define a $\mathbb{C}$-linear map $\mathbf{\Omega}^{\mathbf{H}_k} : \mathbf{H}_k \longrightarrow \mathbb{C}W$ by

$$\mathbf{\Omega}^{\mathbf{H}_k}(fwg) = f(0)g(0)w$$

for all $f \in \mathbb{C}[V]$, $g \in \mathbb{C}[V^*]$ and $w \in \mathbb{C}W$. This map is W-equivariant for the action on both sides by conjugation, so it induces a well-defined $\mathbb{C}$-linear map

$$\mathbf{\Omega}^k : \mathbf{Z}_k \longrightarrow \mathrm{Z}(\mathbb{C}W).$$

Recall from [6, Cor. 4.2.11] that $\mathbf{\Omega}^k$ is a morphism of algebras.

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Calogero-Moser families were defined by Gordon using his theory of *baby Verma modules* [16, §4.2 and §5.4]. We explain here an equivalent definition given in [6, §7.2]. If $\chi \in \mathrm{Irr}(W)$, we denote by $\o_\chi : \mathrm{Z}(\mathbb{C}W) \to \mathbb{C}$ its central character (i.e., $\o_\chi(z) = \chi(z)/\chi(1)$ is the scalar by which z acts on an irreducible representation affording the character χ). We say that two characters χ and χ' belong to the same *Calogero-Moser k-family* if $\o_\chi \circ \boldsymbol{\Omega}^k = \o_{\chi'} \circ \boldsymbol{\Omega}^k$.

In other words, the map $\o_\chi \circ \boldsymbol{\Omega}^k : \mathbf{Z}_k \to \mathbb{C}$ is a morphism of algebras, so it might be viewed as a point $\varphi_k(\chi)$ of $\mathcal{Z}_k$, which is easily checked to be $\mathbb{C}^\times$-fixed. This defines a surjective map [16, §5.4]

$$\varphi_k : \mathrm{Irr}(W) \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$$

whose fibers are the Calogero-Moser k-families. If $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_p$ the corresponding Calogero-Moser k-family.

1.E. **Alternative parameters.** Let $\mathcal{C}$ denote the space of maps $\mathrm{Ref}(W) \to \mathbb{C}$ which are constant on conjugacy classes of reflections. The element

$$\sum_{(\Omega,j)\in\aleph} \sum_{H\in\Omega} (k_{H,j} - k_{H,j+1}) e_H \varepsilon_{H,j}$$

of $\mathrm{Z}(\mathbb{C}W)$ is supported only by reflections, so there exists a unique map $c_k \in \mathcal{C}$ such that

$$\sum_{(\Omega,j)\in\aleph} \sum_{H\in\Omega} (k_{H,j} - k_{H,j+1}) e_H \varepsilon_{H,j} = \sum_{s\in\mathrm{Ref}(W)} (\varepsilon(s) - 1) c_k(s) s.$$

Then the map $\mathbb{C}^\aleph \to \mathcal{C}$, $k \mapsto c_k$ is linear and surjective. With this notation, we have

$$[y, x] = \sum_{s\in\mathrm{Ref}(W)} (\varepsilon(s) - 1)\, c_k(s)\, \frac{\langle y, \alpha_s\rangle \cdot \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle}\, s, \tag{1.4}$$

for all $y \in V$ and $x \in V^*$. Here, $\alpha_s = \alpha_{V^s}$ and $\alpha_s^\vee = \alpha_{V^s}^\vee$.

1.F. **Actions on the Calogero-Moser space.** The Calogero-Moser space $\mathcal{Z}_k$ is endowed with a $\mathbb{C}^\times$-action and an action of the stabilizer of k in $\mathrm{N}_{\mathbf{GL}_\mathbb{C}(V)}(W)$, which are described below.

1.F.1. *Grading, $\mathbb{C}^\times$-action.* The algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ can be $\mathbb{Z}$-graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a $\mathbb{Z}$-grading on $\mathbf{H}_k$, because the defining relations (1.2) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra $\mathbf{Z}_k$ is also $\mathbb{Z}$-graded. So the Calogero-Moser space $\mathcal{Z}_k$ inherits a regular $\mathbb{C}^\times$-action.

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Note also that by definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of $\mathbf{Z}_k$.

1.F.2. *Action of the normalizer.* The group $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ acts on the set $\aleph$ and so on the space of parameters $\mathbb{C}^\aleph$. If $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$, then τ induces an isomorphism of algebras $\mathbf{H}_k \longrightarrow \mathbf{H}_{\tau(k)}$. So, if $\tau(k) = k$, then it induces an action on the algebra $\mathbf{H}_k$ (and so on its center $\mathbf{Z}_k$ and on the Calogero-Moser space $\mathcal{Z}_k$).

Notation. *From now on, and until the end of this paper, we fix a parameter $k \in \mathbb{C}^\aleph$ and a* ***regular*** *element τ of* ***finite*** *order of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ such that $\tau(k) = k$.*

We denote by $\mathcal{Z}_k^\tau$ the variety of fixed points of τ in $\mathcal{Z}_k$, endowed with its reduced structure. All the above constructions are τ-equivariant: for instance, the map $\varphi_k : \mathrm{Irr}(W) \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$ is τ-equivariant.

Let us recall the following consequence [21, Prop. 3.5 and Theo. 4.2] of the above hypothesis:

Theorem 1.6 (Springer)**.** *The group W^τ acts as a reflection group on V^τ and the natural map $V^\tau/W^\tau \to (V/W)^\tau$ is an isomorphism of varieties.*

Corollary 1.7. *The natural map $(V_{\mathrm{reg}}^\tau \times V^{*\tau})/W^\tau \to ((V_{\mathrm{reg}} \times V^*)/W)^\tau$ is an isomorphism of varieties.*

Proof. Since W acts freely on the variety $V_{\mathrm{reg}} \times V^*$, the quotient variety $(V_{\mathrm{reg}} \times V^*)/W$ is smooth. Consequently, the variety of fixed points $((V_{\mathrm{reg}} \times V^*)/W)^\tau$ is also smooth. Similarly, $(V_{\mathrm{reg}}^\tau \times V^{*\tau})/W^\tau$ is smooth. Since a bijective morphism between smooth complex varieties is an isomorphism (by Zariski's Main Theorem), we only need to show that the above natural map is bijective.

First, if (v_1, v_1^*) and (v_2, v_2^*) are two elements of $V_{\mathrm{reg}}^\tau \times V^{*\tau}$ belonging to the same W-orbit, there exists $w \in W$ such that $v_2 = w(v_1)$. Since v_1 and v_2 are τ-stable, we also have $\tau(w)(v_1) = v_2$, and so $v_1 = w^{-1}\tau(w)(v_1)$. Since $v_1 \in V_{\mathrm{reg}}$, this forces $\tau(w) = w$ and the injectivity follows.

Now, if $(v, v^*) \in V_{\mathrm{reg}} \times V^*$ is such that its W-orbit is τ-stable, then the W-orbit of v is τ-stable. So Theorem 1.6 shows that we may assume that $\tau(v) = v$. The hypothesis implies that there exists $w \in W$ such that $\tau(v) = w(v)$ and $\tau(v^*) = w(v^*)$. But $\tau(v) = v \in V_{\mathrm{reg}}$, so $w = 1$. In particular, $\tau(v^*) = v^*$, and the surjectivity follows. ◻

2. Irreducible component of maximal dimension

Let $(\mathcal{Z}_k)_{\mathrm{reg}}$ denote the open subset $\Upsilon_k^{-1}(V_{\mathrm{reg}}/W \times V^*/W)$. By [13, Prop. 4.11], we have a $\mathbb{C}^\times$-equivariant and τ-equivariant isomorphism

$$(\mathcal{Z}_k)_{\mathrm{reg}} \simeq (V_{\mathrm{reg}} \times V^*)/W. \tag{2.1}$$

This shows that $(\mathcal{Z}_k)_{\mathrm{reg}}$ is smooth and so $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$ is also smooth. By Corollary 1.7, this implies that

$$(\mathcal{Z}_k)_{\mathrm{reg}}^\tau \simeq (V_{\mathrm{reg}}^\tau \times V^{*\tau})/W^\tau. \tag{2.2}$$

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In particular it is irreducible. We denote by $(\mathcal{Z}_k^\tau)_{\max}$ its closure: it is an irreducible closed subvariety of $\mathcal{Z}_k^\tau$.

Moreover, $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$ has dimension $2\dim V^\tau$ by Corollary 1.7. So $\dim \mathcal{Z}_k^\tau \geq 2\dim V^\tau = \dim(\mathcal{Z}_k^\tau)_{\max}$. But, on the other hand, $\Upsilon_k(\mathcal{Z}_k^\tau) \subset (V/W)^\tau \times (V^*/W)^\tau$. Since Υ_k is a finite morphism, we get from Theorem 1.6 that $\dim \mathcal{Z}_k^\tau \leq 2\dim V^\tau$. Hence

$$\dim \mathcal{Z}_k^\tau = \dim(\mathcal{Z}_k^\tau)_{\max} = 2\dim V^\tau. \tag{2.3}$$

This shows that $(\mathcal{Z}_k^\tau)_{\max}$ is an irreducible component of maximal dimension of $\mathcal{Z}_k^\tau$ and that

$$\Upsilon_k((\mathcal{Z}_k^\tau)_{\max}) = (V/W)^\tau \times (V^*/W)^\tau. \tag{2.4}$$

Proposition 2.5. *The closed subvariety $(\mathcal{Z}_k^\tau)_{\max}$ of $\mathcal{Z}_k^\tau$ is the unique irreducible component of maximal dimension.*

Proof. Let $\mathcal{X}$ be an irreducible component of $\mathcal{Z}_k^\tau$ of dimension $2\dim V^\tau$. Since Υ_k is finite, the image $\Upsilon_k(\mathcal{X})$ is closed in $V/W \times V^*/W$, irreducible of dimension $2\dim(V^\tau)$ and contained in $(V/W)^\tau \times (V^*/W)^\tau$. By Theorem 1.6, we get that $\Upsilon_k(\mathcal{X}) = (V/W)^\tau \times (V^*/W)^\tau$.

Let $\mathcal{U} = \Upsilon_k^{-1}(V_{\mathrm{reg}}/W \times V^*/W) \cap \mathcal{X}$. Then $\mathcal{U}$ is a non-empty open subset of $\mathcal{X}$: since $\mathcal{X}$ is irreducible, this forces $\mathcal{U}$ to have dimension $2\dim(V^\tau)$. But $\mathcal{U}$ is contained in $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$ which is irreducible of the same dimension, so the closure of $\mathcal{U}$ contains $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$. This proves that $\mathcal{X} = (\mathcal{Z}_k^\tau)_{\max}$. □

It is natural to ask which $\mathbb{C}^\times$-fixed points of $\mathcal{Z}_k$ belong to $(\mathcal{Z}_k^\tau)_{\max}$. Inspired by the representation theory of finite reductive groups (see [11] and [10, Rem. 4.21]), we propose an answer to this question in terms of the character table of the finite group $W\langle\tau\rangle$ (see [4, Ex. 12.9] for some explanations). We first need some notation.

If $\chi \in \mathrm{Irr}(W)$, we denote by E_χ a $\mathbb{C}W$-module affording the character χ. If moreover χ is τ-stable, we fix a structure of $\mathbb{C}W\langle\tau\rangle$-module on E_χ extending the structure of $\mathbb{C}W$-module, and we denote by $\tilde{\chi}$ its associated irreducible character of $W\langle\tau\rangle$. Note that the real number $|\tilde{\chi}(\tau)|^2$ does not depend on the choice of $\tilde{\chi}$.

Conjecture 2.6. *Recall that τ is regular. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. Then p belongs to $(\mathcal{Z}_k^\tau)_{\max}$ if and only if $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$.*

Remark 2.7. Let $\mathfrak{F}$ be a τ-stable Calogero-Moser family. Then $\mathfrak{F}$ contains a unique irreducible character $\chi_{\mathfrak{F}}$ with minimal b-invariant [6, Theo. 7.4.1], where the b-invariant of an irreducible character χ is the minimal natural number j such that χ occurs in the j-th symmetric power of the natural representation V of W. From this characterization, we see that $\chi_{\mathfrak{F}}$ is τ-stable. In particular, any τ-stable Calogero-Moser family contains at least one τ-stable character. ■

In general, we are only able to prove the “if” part of Conjecture 2.6.

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Theorem 2.8. *Recall that τ is regular. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. If $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$, then p belongs to $(\mathcal{Z}_k^\tau)_{\max}$.*

The next two sections are devoted to the proof of Theorem 2.8.

3. Verma modules

3.A. **Definition.** Recall that $\mathbb{C}[V] \rtimes W$ is a subalgebra of $\mathbf{H}_k$ (it is the image of $1 \otimes \mathbb{C}W \otimes \mathbb{C}[V]$ by the PBW-decomposition 1.3). If E is a $\mathbb{C}W$-module, we denote by $E^\#$ the $(\mathbb{C}[V^*] \rtimes W)$-module extending E by letting any element $f \in \mathbb{C}[V^*]$ act by multiplication by $f(0)$. If $\chi \in \mathrm{Irr}(W)$, we define an $\mathbf{H}_k$-module $\Delta(\chi)$ as follows:

$$\Delta(\chi) = \mathbf{H}_k \otimes_{\mathbb{C}[V^*] \rtimes W} E_\chi^\#.$$

Then $\Delta(\chi)$ is called a *Verma module* of $\mathbf{H}_k$ (see [6, §5.4.A]: in this reference, $\Delta(\chi)$ is denoted by $\Delta(E_\chi^\#)$). Let $\mathbf{H}_k^{\mathrm{reg}}$ denote the localization of $\mathbf{H}_k$ at $\mathbf{P}_{\mathrm{reg}} = \mathbb{C}[V_{\mathrm{reg}}/W] \otimes \mathbb{C}[V^*/W]$. By [13, Prop. 4.11], we have an isomorphism $\mathbb{C}[V_{\mathrm{reg}} \times V^*] \rtimes W \simeq \mathbf{H}_k^{\mathrm{reg}}$. We denote by $\Delta^{\mathrm{reg}}(\chi)$ the localization of $\Delta(\chi)$ at $\mathbf{H}_k^{\mathrm{reg}}$. So, by restriction to $\mathbb{C}[V_{\mathrm{reg}} \times V^*]$, the localized Verma module $\Delta^{\mathrm{reg}}(\chi)$ might be viewed as a W-equivariant coherent sheaf on $V_{\mathrm{reg}} \times V^*$. We also view $e\Delta(\chi)$ as a coherent sheaf on $\mathcal{Z}_k$, so that $e\Delta^{\mathrm{reg}}(\chi)$ may be viewed as a coherent sheaf on $(V_{\mathrm{reg}} \times V^*)/W$. If $p \in \mathcal{Z}_k$ (or if $(v, v^*) \in V_{\mathrm{reg}} \times V^*$), we denote by $e\Delta(\chi)_p$ (respectively $e\Delta(\chi)_{W\cdot(v,v^*)} = e\Delta^{\mathrm{reg}}(\chi)_{W\cdot(v,v^*)}$, respectively $\Delta^{\mathrm{reg}}(\chi)_{v,v^*}$) the restriction of $e\Delta(\chi)$ (respectively of $e\Delta(\chi)$ or $e\Delta^{\mathrm{reg}}(\chi)$, respectively $\Delta^{\mathrm{reg}}(\chi)$) at the point p (respectively $W \cdot (v, v^*) \in (V_{\mathrm{reg}} \times V^*)/W \simeq (\mathcal{Z}_k)_{\mathrm{reg}}$, respectively (v, v^*)). It follows from the definition that the support of $e\Delta(\chi)$ is contained in $\Upsilon_k^{-1}(V/W \times 0)$, and recall that, through the isomorphism $\mathcal{Z}_k^{\mathrm{reg}} \simeq (V_{\mathrm{reg}} \times V^*)/W$, $\Upsilon_k^{-1}(V_{\mathrm{reg}}/W \times 0)$ is not necessarily contained in $(V_{\mathrm{reg}} \times \{0\})/W$.

Lemma 3.1. *Let $\chi \in \mathrm{Irr}(W)$ and let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$. Then $e\Delta(\chi)_p \neq 0$ if and only if $\chi \in \mathfrak{F}_p$.*

Proof. Let $\mathfrak{p}_0$ denote the maximal ideal of the algebra $\mathbf{P} = \mathbb{C}[\mathcal{P}]$ consisting of functions which vanish at 0. Then $\Delta(\chi)/\mathfrak{p}_0\Delta(\chi)$ is a representation of the *restricted rational Cherednik algebra* $\mathbf{H}_k/\mathfrak{p}_0\mathbf{H}_k$ which coincides with the *baby Verma module* defined by Gordon [16, §4.2]. As $\mathcal{Z}_k^{\mathbb{C}^\times} = \Upsilon_k^{-1}(0)$, the result follows from the very definition of Calogero-Moser families in terms of baby Verma modules and the fact that it is equivalent to the definition given in §1.D. □

3.B. **Bialynicki-Birula decomposition.** We denote by $\mathcal{Z}_k^{\mathrm{att}}$ the *attracting set* of $\mathcal{Z}_k$ for the action of $\mathbb{C}^\times$, namely

$$\mathcal{Z}_k^{\mathrm{att}} = \{p \in \mathcal{Z}_k \mid \lim_{\xi \to 0} {}^\xi p \text{ exists}\}.$$

Recall from [6, Chap. 9] the following facts:

Proposition 3.2. *With the above notation, we have:*

(a) *The map $\lim : \mathcal{Z}_k^{\mathrm{att}} \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$, $p \mapsto \lim_{\xi \to 0} {}^\xi p$ is a morphism of varieties.*
(b) $\mathcal{Z}_k^{\mathrm{att}} = \Upsilon_k^{-1}(V/W \times \{0\})$.

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(c) *If $\mathcal{I}$ is an irreducible component of $\mathcal{Z}_k^{\mathrm{att}}$, then $\mathcal{I}$ is $\mathbb{C}^\times$-stable and $\Upsilon_k(\mathcal{I}) = V/W \times \{0\}$ and $\lim(\mathcal{I})$ is a single point.*
(d) *If $\chi \in \mathrm{Irr}(W)$, then the support of $e\Delta(\chi)$ is a union of irreducible components of $\mathcal{Z}_k^{\mathrm{att}}$.*
(e) *If $\mathcal{I}$ is an irreducible component of $\mathcal{Z}_k^{\mathrm{att}}$, then there exists $\chi \in \mathrm{Irr}(W)$ such that the support of $e\Delta(\chi)$ contains $\mathcal{I}$.*

Proof. (a) is classical (see for instance [6, §9.1]). For (b), see [6, Lem. 9.3.2]. (c) is explained at the end of [6, §9.3]. For (d) and (e), see [6, (8.1.3) and Prop. 9.3.3]. ☐

We characterize points of $\mathcal{Z}_k^{\mathbb{C}^\times}$ belonging to $(\mathcal{Z}_k^\tau)_{\max}$ in terms of Verma modules:

Lemma 3.3. *Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ and assume that $\tau(p) = p$. Then $p \in (\mathcal{Z}_k^\tau)_{\max}$ if and only if there exist $\chi \in \mathfrak{F}_p^k$ and $(v, v^*) \in V_{\mathrm{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W\cdot(v,v^*)} \neq 0$.*

Proof. Let $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}$ denote the attracting set of $(\mathcal{Z}_k^\tau)_{\max}$. Then (2.4) and Proposition 3.2(b) imply that $\Upsilon_k((\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}) = (V/W)^\tau \times \{0\}$. Since Υ_k is a finite morphism, the same arguments used in [6, Chap. 9] to prove the Proposition 3.2 above yields the following statements:

(a) The map $\lim : (\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}} \longrightarrow (\mathcal{Z}_k^\tau)_{\max}^{\mathbb{C}^\times}$, $p \mapsto \lim_{\xi\to 0} {}^\xi p$ is a morphism of varieties.
(b) $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}} = (\mathcal{Z}_k^\tau)_{\max} \cap \Upsilon_k^{-1}((V/W)^\tau \times \{0\})$.
(c) If $\mathcal{I}$ is an irreducible component of $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}$, then $\mathcal{I}$ is $\mathbb{C}^\times$-stable and $\Upsilon_k(\mathcal{I}) = (V/W)^\tau \times \{0\}$ and $\lim(\mathcal{I})$ is a single point.

Assume that $p \in (\mathcal{Z}_k^\tau)_{\max}$. Let $\mathcal{I}$ be an irreducible component of $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}} \cap \lim^{-1}(p)$. Then $\mathcal{I}$ is contained in an irreducible component $\mathcal{I}'$ of $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}$. Since $\lim(\mathcal{I}')$ is a single point by (c), we have $\lim(\mathcal{I}') = \{p\}$ and so $\mathcal{I} = \mathcal{I}'$. Still by (c), this says that $\Upsilon_k(\mathcal{I}) = (V/W)^\tau \times \{0\}$. So let $q \in \mathcal{I}$ be such that $\Upsilon_k(q) \in (V_{\mathrm{reg}}/W)^\tau \times \{0\}$.

Now, let $\mathcal{J}$ be an irreducible component of $\mathcal{Z}_k^{\mathrm{att}}$ containing $\mathcal{I}$. By Proposition 3.2(e), there exists $\chi \in \mathrm{Irr}(W)$ such that the support of $e\Delta(\chi)$ contains $\mathcal{J}$. In particular, $e\Delta(\chi)_p \neq 0$ and so $\chi \in \mathfrak{F}_p$ by Lemma 3.1. But also $e\Delta(\chi)_q \neq 0$. Since $q \in (\mathcal{Z}_k^\tau)_{\max}$ and $\Upsilon_k(q) \in V_{\mathrm{reg}}/W$, it follows that there exists $(v, v^*) \in V_{\mathrm{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W\cdot(v,v^*)} \neq 0$, as desired.

Conversely, assume that there exist both $\chi \in \mathfrak{F}_p^k$ and $(v, v^*) \in V_{\mathrm{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W\cdot(v,v^*)} \neq 0$. Let $\mathcal{I}$ be an irreducible component of $\mathcal{Z}_k^{\mathrm{att}}$ contained in the support of $e\Delta(\chi)$. Then $p \in \mathcal{I}$ and so $p = \lim W\cdot(v, v^*)$. Since $W\cdot(v, v^*) \in (\mathcal{Z}_k^\tau)_{\max}$ by the definition of $(\mathcal{Z}_k^\tau)_{\max}$, this implies that $p \in (\mathcal{Z}_k^\tau)_{\max}$, as desired. ☐

4. Gaudin algebra

4.A. **Definition.** We recall here the definition of *Gaudin algebra* [6, §8.3.B]. First, let $\mathbb{C}[V_{\mathrm{reg}}][W]$ denote the group algebra of W over the algebra $\mathbb{C}[V_{\mathrm{reg}}]$ (and not the

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semi-direct product $\mathbb{C}[V_{\mathrm{reg}}] \rtimes W$). For $y \in V$, let

$$\mathcal{D}_y^k = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\alpha_s} s \ \in \mathbb{C}[V_{\mathrm{reg}}][W].$$

Now, let $\mathrm{Gau}_k(W)$ be the sub-$\mathbb{C}[V_{\mathrm{reg}}]$-algebra of $\mathbb{C}[V_{\mathrm{reg}}][W]$ generated by the $\mathcal{D}_y^k$'s (where y runs over V): it will be called the *Gaudin algebra* (*with parameter* k) associated with W.

Let $\mathbb{C}(V)$ denote the function field of V (which is the fraction field of $\mathbb{C}[V]$ or of $\mathbb{C}[V_{\mathrm{reg}}]$) and let $\mathbb{C}(V)\mathrm{Gau}_k(W)$ denote the subalgebra $\mathbb{C}(V) \otimes_{\mathbb{C}[V_{\mathrm{reg}}]} \mathrm{Gau}_k(W)$ of the group algebra $\mathbb{C}(V)[W]$. Recall [6, §8.3.B] that

$$\mathrm{Gau}_k(W) \textit{ is a commutative algebra,} \tag{4.1}$$

but that $\mathbb{C}(V)\mathrm{Gau}_k(W)$ is generally non-split, as shown by the examples treated in [2, §4] and [19].

4.B. **Generalized eigenspaces.** If $v \in V_{\mathrm{reg}}$, we denote by $\mathcal{D}_y^{k,v}$ the specialization of $\mathcal{D}_y^k$ at v, namely $\mathcal{D}_y^{k,v}$ is the element of the group algebra $\mathbb{C}W$ equal to

$$\mathcal{D}_y^{k,v} = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} s.$$

Now, if $v^* \in V^*$ and if M is a $\mathbb{C}W$-module, we define M^{k,v,v^*} to be the common generalized eigenspace of the operators $\mathcal{D}_y^{k,v}$ for the eigenvalue $\langle y, v^* \rangle$, for y running over V. Namely,

$$M^{k,v,v^*} = \{m \in M \mid \forall\, y \in V,\ (\mathcal{D}_y^{k,v} - \langle y, v^* \rangle \,\mathrm{Id}_M)^{\dim(M)}(m) = 0\}.$$

Then

$$M = \bigoplus_{v^* \in V^*} M^{k,v,v^*}, \tag{4.2}$$

since $\mathrm{Gau}_k(W)$ is commutative.

Lemma 4.3. *Let $\chi \in \mathrm{Irr}(W)$ and let $(v, v^*) \in V_{\mathrm{reg}} \times V^*$. Then the following are equivalent:*

(1) $e\Delta(\chi)_{W\cdot(v,v^*)} \neq 0$.
(2) $\Delta^{\mathrm{reg}}(\chi)_{v,v^*} \neq 0$.
(3) $E_\chi^{k,v,v^*} \neq 0$.

Proof. The equivalence between (1) and (2) follows from the Morita equivalence between $\mathbb{C}[V_{\mathrm{reg}} \times V^*]^W$ and $\mathbb{C}[V_{\mathrm{reg}} \times V^*] \rtimes W$ proved in [6, Lem. 3.1.8(b)]. Now, as a $(\mathbb{C}[V_{\mathrm{reg}}] \rtimes W)$-module, $\Delta^{\mathrm{reg}}(\chi) \simeq \mathbb{C}[V_{\mathrm{reg}}] \otimes E_\chi$. and the action of $y \in V \subset \mathbb{C}[V^*]$ is given by the operator $-\mathcal{D}_y^k \in \mathbb{C}[V_{\mathrm{reg}}][W]$ (see [6, §8.3.B]). Now, $\Delta^{\mathrm{reg}}(\chi)_{v,v^*} \simeq E_\chi$ as a $\mathbb{C}$-vector space: on this vector space, the action of an element $f \in \mathbb{C}[V_{\mathrm{reg}}]$ is given by multiplication by $f(v)$ while the action of an element $y \in V$ is given by the operator $\langle y, v^* \rangle \,\mathrm{Id}_{E_\chi} - \mathcal{D}_y^{k,v}$ (see [6, Theo. 4.1.7]). This shows the equivalence between (2) and (3). □

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4.C. **Proof of Theorem A (i.e. Theorem 2.8).** Let $\chi \in \mathrm{Irr}(W)$ be τ-stable and such that $\tilde{\chi}(\tau) \neq 0$ and let $v \in V_{\mathrm{reg}}^{\tau}$. By Lemmas 3.1 and 4.3, it is sufficient to show that there exists $v^* \in V^{*\tau}$ such that $E_\chi^{k,v,v^*} \neq 0$.

For this, let $\mathcal{E}$ denote the set of $v^* \in V^*$ such that $E_\chi^{k,v,v^*} \neq 0$. Then it follows from (4.2) that

$$(*) \qquad E_\chi = \bigoplus_{v^* \in \mathcal{E}} E_\chi^{k,v,v^*}.$$

Since $\tau(v) = v$, we have

$${}^{\tau}\mathcal{D}_y^{k,v} = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} \tau s \tau^{-1} = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \tau^{-1}(\alpha_s) \rangle}{\langle v, \tau^{-1}(\alpha_s) \rangle} s = \mathcal{D}_{\tau(y)}^{k,v}.$$

Consequently,

$${}^{\tau}E_\chi^{k,v,v^*} = E_\chi^{k,v,\tau(v^*)}.$$

But $\tilde{\chi}(\tau) = \mathrm{Tr}(\tau, E_\chi) \neq 0$, so τ must fix at least one of the generalized eigenspaces in the decomposition $(*)$. In other words, this implies that there exists $v^* \in \mathcal{E}$ such that $\tau(v^*) = v^*$, as desired. The proof is complete.

5. Complements: further conjectures, examples

5.A. **Conjectures.** The variety $\mathcal{Z}_k$ is endowed with a Poisson structure [13, §1] and so the variety of fixed points $\mathcal{Z}_k^\tau$ inherits a Poisson structure too, as well as all its irreducible components. Recall from Springer Theorem 1.6 that W^τ is a reflection group for its action on V^τ, so we can define a set of pairs $\aleph_\tau$ for the pair (V^τ, W^τ) as well as $\aleph$ has been defined for the pair (V, W) and, for each parameter $l \in \mathbb{C}^{\aleph_\tau}$, we can define a Calogero-Moser space $\mathcal{Z}_l(V^\tau, W^\tau)$. The following conjecture is a particular case of [3, Conj. B] (see [3] for a discussion about the cases where this conjecture is known to hold):

Conjecture 5.1. *Recall that τ is regular. Then there exist a linear map $\lambda : \mathbb{C}^{\aleph} \to \mathbb{C}^{\aleph_\tau}$ and, for each $k \in \mathbb{C}^{\aleph}$, a $\mathbb{C}^\times$-equivariant isomorphism of Poisson varieties*

$$\iota_k : (\mathcal{Z}_k^\tau)_{\max} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau).$$

If the existence of such a $\mathbb{C}^\times$-equivariant isomorphism $\iota_k : (\mathcal{Z}_k^\tau)_{\max} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau)$ is known but if it is not known that it preserves the Poisson structure, then we will say that "Conjecture 5.1$^-$ holds".

Assume now that Conjecture 5.1$^-$ holds and keep its notation. Then ι_k restricts to a map $\iota_k : (\mathcal{Z}_k)_{\max}^{\mathbb{C}^\times} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$. If $p \in \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_{\iota_k(p)}^{(\tau)}$ the corresponding Calogero-Moser $\lambda(k)$-family of W^τ. The next conjecture, still inspired by the representation theory of finite reductive groups (see again [4, Ex. 12.9] for some explanations), makes Conjecture B more precise:

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Conjecture 5.2. *Recall that τ is regular and assume that Conjecture 5.1$^-$ holds. If $p \in (\mathcal{Z}_k^\tau)^{\mathbb{C}^\times}_{\max}$, then*

$$\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 = \sum_{\psi \in \mathfrak{F}^{(\tau)}_{\iota_k(p)}} \psi(1)^2.$$

Note that this last conjecture is compatible with the fact that

$$\sum_{\chi \in \operatorname{Irr}(W)^\tau} |\tilde{\chi}(\tau)|^2 = |W^\tau| = \sum_{\psi \in \operatorname{Irr}(W^\tau)} \psi(1)^2,$$

where the first equality follows from the second orthogonality relation for characters applied to $W\langle\tau\rangle$. Indeed, $\sum_{\theta \in \operatorname{Irr}(W\langle\tau\rangle)} |\theta(\tau)|^2 = |C_{W\langle\tau\rangle}(\tau)|$ and $\theta(\tau) \neq 0$ implies that θ is an extension of a τ-invariant character χ of W by [18, Theo. 6.11]: the equality then follows from the fact that $|\theta(\tau)|$ depends only on χ and that there are $|W\langle\tau\rangle|/|W|$ extensions of χ by [18, Coro. 6.17].

5.B. **Roots of unity.** We consider in this subsection a particular (but very important) case of the general situation studied in this paper. We fix a natural number $d \geq 1$ and a primitive d-th root of unity ζ_d. The group of d-th roots of unity is denoted by $\boldsymbol{\mu}_d$. An element $w \in W$ is called *ζ_d-regular* if the element $\zeta_d^{-1} w$ of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ is regular. In other words, w is ζ_d-regular if and only if its ζ_d-eigenspace meets V_{reg}. The existence of a ζ_d-regular element is not guaranteed: we say that d is a *regular number* of W if such an element exists.

Hypothesis. *We assume in this subsection that d is a regular number of W. We denote by w_d a ζ_d-regular element and we also set $\tau_d = \zeta_d^{-1} w_d$, so that τ_d is a regular element of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$.*

Recall from [21, Theo. 4.2(iv)] that w_d is uniquely defined up to conjugacy. Note that

$$V^{\tau_d} = \operatorname{Ker}(w_d - \zeta_d \operatorname{Id}_V), \qquad W^{\tau_d} = C_W(w_d) \qquad \text{and} \qquad \mathcal{Z}_k^{\tau_d} = \mathcal{Z}_k^{\boldsymbol{\mu}_d}. \tag{5.1}$$

Since τ_d induces an inner automorphism of W, all the irreducible characters are τ_d-stable. Moreover, if $\chi \in \operatorname{Irr}(W)$, then $\tilde{\chi}(\tau_d) = \xi\chi(w_d)$ for some root of unity ξ, so $|\tilde{\chi}(\tau_d)|^2 = |\chi(w_d)|^2$. This allows to reformulate both Theorem A and Conjecture B in this case:

Conjecture 5.4. *Recall that d is a regular number. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$. Then p belongs to $(\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max}$ if and only if $\sum_{\chi \in \mathfrak{F}_p} |\chi(w_d)|^2 \neq 0$.*

Theorem 5.5. *Recall that d is regular. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\sum_{\chi \in \mathfrak{F}_p} |\chi(w_d)|^2 \neq 0$. Then p belongs to $(\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max}$.*

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Example 5.6 (Symmetric group)**.** We assume here that $W = \mathfrak{S}_n$ acting on $V = \mathbb{C}^n$ by permutation of the coordinates, for some $n \geq 2$. The canonical basis of $\mathbb{C}^n$ is denoted by $(y_1, \dots, y_n)$. Then there is a unique orbit of hyperplanes, that we denote by Ω, and $e_\Omega = 2$. To avoid too easy cases, we also assume that $k_{\Omega,0} \neq k_{\Omega,1}$ (so that $\mathcal{Z}_k$ is smooth [13, Cor. 1.14]) and that $d \geq 2$. Saying that d is a regular number is equivalent to say that d divides n or $n-1$. Therefore, we will denote by j the unique element of $\{0,1\}$ such that d divides $n-j$ and we set $r = (n-j)/d$. Then w_d is the product of r disjoint cycles of length d, so one can choose for instance

$$w_d = (1, 2, \dots, d)(d+1, d+2, \dots, 2d) \cdots ((r-1)d+1, (r-1)d+2, \dots, rd).$$

Then V^{τ_d} is r-dimensional, with basis $(v_1, \dots, v_r)$ where $v_a = \sum_{b=1}^{d} \zeta_d^{-b} e_{(a-1)d+b}$ and the group $C_W(w_d) \simeq G(d,1,r)$ acts "naturally" as a reflection group on $V^{\tau_d} = \bigoplus_{a=1}^{r} \mathbb{C} v_a$.

We also need some combinatorics. We denote by $\mathrm{Part}(n)$ (resp. $\mathrm{Part}^d(r)$) the set of partitions of n (resp. of d-partitions of r). If $\lambda \in \mathrm{Part}(n)$, we denote by χ_λ the irreducible character of $\mathfrak{S}_n$ (with the convention of [15]: for instance $\chi_n = 1$ and $\chi_{1^n} = \varepsilon$), by $\mathrm{cor}_d(\lambda)$ the d-core of λ, by $\mathrm{quo}_d(\lambda)$ its d-quotient. We let $\mathrm{Part}(n,d)$ denote the set of partitions of n whose d-core is the unique partition of $j \in \{0,1\}$. Then the map

$$\mathrm{quo}_d : \mathrm{Part}(n,d) \longrightarrow \mathrm{Part}^d(r)$$

is bijective. Finally, if $\mu \in \mathrm{Part}^d(r)$, we denote by χ_μ the associated irreducible character of $C_W(w_d) = G(d,1,r)$ (with the convention of [14]). It follows from Murnaghan-Nakayama rule that

$$\chi_\lambda(w_d) \neq 0 \text{ if and only if } \lambda \in \mathrm{Part}(n,d), \tag{5.2}$$

and that

$$\chi_\lambda(w_d) = \pm \chi_{\mathrm{quo}_d(\lambda)}(1) \tag{5.3}$$

for all $\lambda \in \mathrm{Part}(n,d)$ (see for instance [9, Page 47]).

Now, the smoothness of $\mathcal{Z}_k$ implies that the map $\varphi_k : \mathrm{Irr}(\mathfrak{S}_n) \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$ is bijective (so that Calogero-Moser k-families of $\mathfrak{S}_n$ are singleton) and it follows from the main theorem of [5] that Conjecture 5.1 holds (except that we do not know if the isomorphism respects the Poisson structure), so that we have a $\mathbb{C}^\times$-equivariant isomorphism of varieties

$$\iota_k : (\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^{\tau_d}, G(d,1,r))$$

for some explicit $\lambda(k) \in \mathbb{C}^{\aleph_{\tau_d}}$. Moreover, $\mathcal{Z}_{\lambda(k)}(V^{\tau_d}, G(d,1,r))$ is smooth so that the map $\varphi_{\lambda(k)}^{\tau_d} : \mathrm{Irr}(G(d,1,r)) \longrightarrow \mathcal{Z}_{\lambda(k)}(V^{\tau_d}, G(d,1,r))^{\mathbb{C}^\times}$ is bijective (that is, Calogero-Moser $\lambda(k)$-families of $G(d,1,r)$ are singleton). Now, by [5, Theo. 4.21], we have that

$$\varphi_k(\chi_\lambda) \in (\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max} \text{ if and only if } \lambda \in \mathrm{Part}(n,d), \tag{5.4}$$

and that

$$\iota_k(\varphi_k(\chi_\lambda)) = \varphi_{\lambda(k)}^{\tau_d}(\chi_{\mathrm{quo}_d(\lambda)}) \tag{5.5}$$

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for all $\lambda \in \mathrm{Part}(n,d)$. Then (5.2), (5.3), (5.4) and (5.5) show that Conjectures 5.4 and 5.2 hold for the symmetric group. ■

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