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CLUSTER ALGEBRAS OF TYPE $\mathbb{A}_{n-1}$ THROUGH PERMUTATION GROUP S_n

KODJO ESSONANA MAGNANI

Abstract. Flip of triangulations appears in the definition of cluster algebras of Fomin and Zelevinsky. We give in this article an interpretation of mutation in the sense of permutation using triangulation of convex polygon. Using this, we establish a link between cluster variables and permutation mutations in the case of cluster algebras of type $\mathbb{A}$.

1. Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in [7, 8]. They are a class of commutative algebras which was shown to be connected to various areas of mathematics like combinatorics, Lie theory, Poisson geometry, Teichmüller theory, mathematical physics and representation theory of algebras. A cluster algebra is generated by a set of variables, called *cluster variables*, obtained recursively by a combinatorial process known as *mutation* starting from a set of initial cluster variables.

Triangulating a convex polygon plays a central role in the theory of cluster algebras. Consider a triangulation T of a convex polygon which is the partition of its interior into triangles by non-intersecting diagonals [2, 3, 9]. Each diagonal d in the triangulation T is the diagonal of some quadrilateral. A new triangulation T' is obtained by replacing the diagonal d with the other diagonal of that quadrilateral. This well-known process is called *flip*. The cluster variables are in natural bijection with the diagonals of a convex polygon [1] and flip of diagonals corresponds to a mutation of cluster variables [5].

The present work is motivated by the correspondence between triangulations and permutations in [10]. Our objective here is to show how one can mutate a permutation in the permutation groups S_n.

In this paper, we establish a link between the mutation of cluster variables and the mutation of a permutation in the permutation groups S_n. For this, taking a permutation we show how to enumerate all diagonals composing the corresponding triangulation. This correspondence allows us to prove that the cluster algebra associated with a permutation in S_n is of type $\mathbb{A}_{n-1}$.

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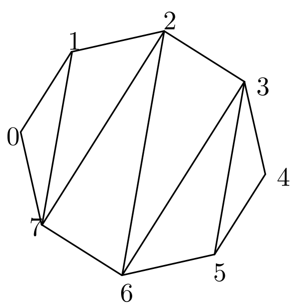
The article is organized as follows. In section 2, we recall some basic notions on triangulations and permutations and set some results. In section 3, we establish a link between cluster algebras of type $\mathbb{A}_{n-1}$ and permutation groups S_n.

2. Triangulations and permutations

Let n be an integer $n \geqslant 1$. Let P_{n+2} be a convex polygon with $(n+2)$ vertices labelled $0, 1, \dots$ to $(n+1)$ in clockwise order. The partition of the interior of P_{n+2} into triangles by non-crossing diagonals is called a triangulation of P_{n+2}. The partition uses $(n-1)$ diagonals. The set of triangulations of P_{n+2} will be denoted T_n, its cardinality will be denoted t_n. It is well known that t_n is Catalan number $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$, $n \geqslant 1$, see [9].

Example 2.1. Let $n = 6$ and P_8 a convex octagon with vertices labelled $0, 1, 2, 3, 4, 5, 6, 7$. We can have the following triangulation:



We denote by S_n the group of permutations of $\{1, 2, \dots, n\}$and denoting elementary transpositions $(i, i+1)$for $1 \leqslant i \leqslant n-1$. Let w be the set of words on $\{1, 2, \dots, n\}$. A word in w is said to be standard if its letters are pairwise distinct. The set of standard words of length n in w will be identify with S_n. In this way a permutation σ in S_n will be represented by the word such that $\sigma = a_1 a_2 \dots a_n \in w$, where $a_i = \sigma(i)$ for all $1 \leqslant i \leqslant n$. Note that the left multiplication of the word σ by elementary transposition $\tau = (i, i+1)$ results in the word where the $i - th$ and $(i+1)\,th$ letters of σ are permuted. Indeed, if $\sigma = u a_i a_{i+1} v$ where $u = a_1 a_2 \dots a_{i-1}$ and $v = a_{i+2} \dots a_n$ (with convention that u or v is empty if $i = 1$ or $i = n-1$, respectively), then $\tau\sigma = u a_{i+1} a_i v$.

Example 2.2. Let σ be in S_6 such that $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix}$. The word associated with the permutation σ is $\sigma = 314265$.

The triangulations of a fixed convex plane $(n+2)$-gon P_{n+2} will be now associated with permutations in S_n. The way to associate a permutation with a triangulation is described in the following example:

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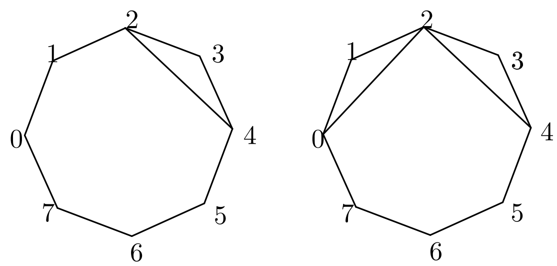
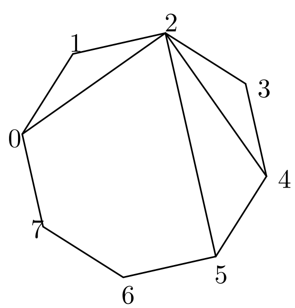
Example 2.3. Let $n = 6$ and P_8 a convex octagon with vertices labelled in clockwise from 0 to 7.

Let $\sigma = 314265 \in S_6$. We associate with σ the triangulation T of P_8 constructed by the following procedure. We read the word σ from left to right.

- The first letter being 3, gives rise to diagonal joining its two neighbors in P_8, namely 2 and 4. Cutting vertex 3 from P_8, the newly added diagonal$\{2, 4\}$ gives rise to a 7-gon P_8' on the vertices $0, 1, 2, 4, 5, 6, 7$.
- The next letter in σ is 1 and gives rise to the diagonal $\{0, 2\}$ joining its two neighbours in P_8'. Now we cut vertex 1 from P_8' and denote P_8'' the resulting hexagon on vertices $0, 2, 4, 5, 6, 7$.



- The third letter of σ is 4 and gives rise to the diagonal $\{2, 5\}$ between its neighbours in P_8''



- The fourth letter of σ is 2 and gives rise to the diagonal $\{0, 5\}$ between its two neighbours in the vertex sequence $0, 2, 5, 6, 7$.
- Finally, the fifth letter of σ is 6 and gives rise to the diagonal $\{5, 7\}$ joining its neighbours in the square on $0, 5, 6, 7$.

The resulting triangulation T of P_8 has inner diagonals $\{2, 4\}, \{0, 2\}, \{2, 5\}, \{0, 5\}, \{5, 7\}$.

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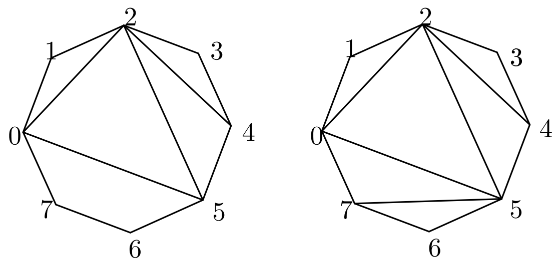

The degree of a vertex i in P_{n+2} is the number of edges of T which are incident to i. A vertex of degree exactly 2 is called *ear* in T.

The words in S_n obtained with the cutting procedure will be call the readings of the triangulation T.

According to [10, Lemma 13] the map $t\colon S_n \longrightarrow T_n$ is surjective. Then each word will be associated with a triangulation. The following example gives the readings of such triangulation T in T_n.

Example 2.4. The readings of the triangulation $T = t\,(\sigma = 314265)$ obtained in Example 2.3 are;

$$\begin{array}{lll} \sigma = 314265 & \sigma_5 = 134265 & \sigma_{10} = 613425 \\ \sigma_1 = 341625 & \sigma_6 = 136425 & \sigma_{11} = 341265 \\ \sigma_2 = 346125 & \sigma_7 = 163425 & \sigma_{12} = 134625 \\ \sigma_3 = 316425 & \sigma_8 = 634125 & \\ \sigma_4 = 314625 & \sigma_9 = 631425 & \end{array}$$

all in S_6.

Definition 2.5. Let $T \in T_n$ and $t\colon S_n \longrightarrow T_n$ the surjective map. The canonical reading of T is the lexicographically smallest word in the fiber $t^{-1}\,(T)$.

Example 2.6. In Example 2.4 the canonical reading of T is $\sigma_5 = 134265$.

Remark 2.7. It was shown in [10, Lemma 15] that the set of readings of T is exactly the fiber $t^{-1}\,(T)$. It is clear that the canonical reading is unique in $t^{-1}\,(T)$.

Now we are going to define a separation of permutation as in [10].

Definition 2.8. Let $\sigma = a_1 a_2 \dots a_n \in S_n$. We say that σ is separated if it may be factored as $\sigma = \sigma_1 \sigma_2 a_n$, where the subwords σ_1, σ_2 have the property that $a_i < a_n$ for every letter a_i in σ_1 and $a_j > a_n$ for every letter a_j in σ_2.

Example 2.9. Let $\sigma = 134265 \in S_6$. σ is a separation with $\sigma_1 = 1342$, $\sigma_2 = 6$ and $a_n = a_6 = 5$.

Remark 2.10. The separation of a permutation is unique if it exists. Let $\sigma = \sigma_1 \sigma_2 a_n$ be a separation, the triangulation $T = t\,(\sigma)$ associated with σ is of the form:

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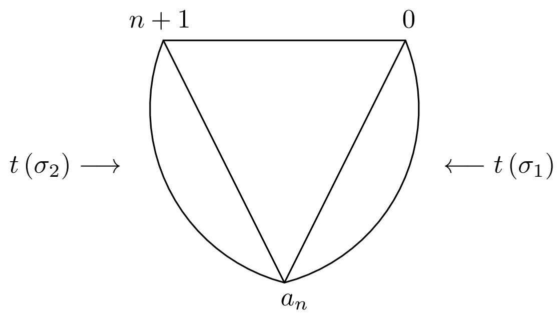

We know that $a_n \in \{1, \dots, n\}$, then we can rewrite a separation as follows: $\sigma = \sigma_1\sigma_2 i$ with $i \in \{1, \dots, n\}$.

For a fixed i in $\{1, \dots, n\}$ the following theorem gives the number of separation ended by i.

Theorem 2.11. *Let $n \geq 1$ be an integer, $i \in \{1, \dots, n\}$and S_n the permutation group. Let $\sigma = \sigma_1\sigma_2 i$ be a separation.*

1. *The number of separation in S_n ended by i is Δ_i such that*

$$\Delta_i = \frac{1}{i}\frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}$$

2. $\displaystyle\sum_{i=1}^{n} \Delta_i = \frac{1}{n+1}\binom{2n}{n}$

Before giving the proof of the above theorem, let us state the following lemma:

Lemma 2.12. *Let $n \geq 1$ be an integer. We have*

$$\sum_{i=1}^{n} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i} = \frac{1}{2}\binom{2n}{n}$$

Proof. The assertion holds for $n = 1$, $n = 2$, $\displaystyle\frac{1}{1}\binom{0}{0}\binom{0}{0} = 1 = \frac{1}{2}C_2^1$

$$\sum_{i=1}^{2} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(2-i)}{2-i} = \frac{1}{1}\binom{0}{0}\binom{2}{1} + \frac{1}{2}\binom{2}{1}\binom{0}{0} = 2+1 = 3$$
$$= \frac{1}{2}\binom{4}{2} = \frac{1}{2}.\frac{4\,.\,3}{2\,.\,1} = 3$$

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So fix $n \in \mathbb{N}$ and assume that the assertion is valid for every $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n+1-i)}{n+1-i} &= \sum_{i=1}^{n}\left[\frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n-i+1)}{n-i+1}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&= \sum_{i=1}^{n}\left[\frac{2}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)+1}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
\text{because } \binom{n}{p} = \frac{n}{p}\binom{n-1}{p-1} \ (1) & \\
&= \sum_{i=1}^{n}\left[\frac{2}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)+1}{n-i+1}\right] + \frac{1}{n+1}\binom{2n}{n} \\
\text{because } \binom{n}{p} = \binom{n}{n-p} \ (2) & \\
&= \sum_{i=1}^{n}\left[\frac{2}{i}\binom{2(i-1)}{i-1}\frac{2(n-i)+1}{n-i+1}\binom{2(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
\text{using } (1) & \\
&= \sum_{i=1}^{n}\left[\frac{2}{i}\left(2-\frac{1}{n-i+1}\right)\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&= \sum_{i=1}^{n}\left[\left[\frac{4}{i}-\frac{2}{i(n-i+1)}\right]\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&= \sum_{i=1}^{n}\left[\left[\frac{4}{i}-\frac{2}{n+1}\left(\frac{1}{i}+\frac{1}{n-i+1}\right)\right]\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\right] + \\
&\quad + \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

Due to fact that

$$\sum_{i=1}^{n} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i} = \sum_{i=1}^{n}\frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}$$

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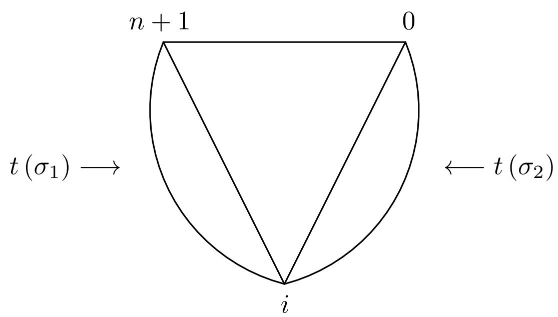
we obtain

$$\begin{aligned}
\sum_{i=1}^{n+1}\frac{1}{i}\binom{2\,(i-1)}{i-1}\binom{2\,(n+1-i)}{n+1-i} &= \sum_{i=1}^{n}\left[\left[\frac{4}{i}-\frac{2}{n+1}\left(\frac{2}{i}\right)\right]\binom{2\,(i-1)}{i-1}\binom{2\,(n-i)}{n-i}\right]+\\
&\quad+\frac{1}{n+1}\binom{2n}{n}\\
&= \sum_{i=1}^{n}\left[\left(\frac{2n}{n+1}\cdot\frac{2}{i}\right)\binom{2\,(i-1)}{i-1}\binom{2\,(n-i)}{n-i}\right]+\frac{1}{n+1}\binom{2n}{n}\\
&= \frac{2n}{n+1}\binom{2n}{n}+\frac{1}{n+1}\binom{2n}{n}\\
&= \frac{2n+1}{n+1}\binom{2n}{n}\\
&= \binom{2n+1}{n+1} \quad \text{applying (1)}\\
&= \binom{2n+1}{n} \quad \text{applying (2)}\\
&= \frac{n+1}{2(n+1)}\binom{2\,(n+1)}{n+1} \quad \text{applying (1)}\\
&= \frac{1}{2}\binom{2\,(n+1)}{n+1} \qquad \square.
\end{aligned}$$

Proof of the Theorem 2.11.

1. Let $\sigma = \sigma_1\sigma_2 i$ be a separation, $i \in \{2, \ldots, (n-1)\}$. The triangulation $T = t(\sigma)$ associated with σ is of the form



$t(\sigma_1)$ is a triangulation of $(i+1)$-gon. The number of triangulations of $(i+1)$-gon is the Catalan number $t_{i-1} = \frac{1}{i}\binom{2\,(i-1)}{i-1}$.

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Considering $t(\sigma_2)$, we will see that $t(\sigma_2)$ is a triangulation of $(n-i+2)$-gon which number of triangulations is $t_{n-i} = \dfrac{1}{n-i+1}\dbinom{2(n-i)}{n-i}$.

Due to the fact that the triangle with vertices $0, i, n+1$ is fixed in $T = t(\sigma)$, all order separations ended by i are obtained by making flips in $t(\sigma_1)$ and $t(\sigma_2)$. Then, combining this, we obtain that the number Δ_i of separations in S_n ended by i is: $\Delta_i = t_{i-1}t_{n-i}$. Thus

$$\Delta_i = \frac{1}{i}\frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}$$

The cases that $i=1$ and $i=n$, the triangulation is associated with $(n+1)$-gon. In these cases the number of triangulations is the same and coincide with t_{n-1}. So the formula of Δ_i in above can be apply for $i=1$ and $i=n$. Indeed i occur $\{1, 2, \ldots, n\}$.

2. Now let us compute $\sum_{i=1}^{n} \Delta_i$

$$\begin{aligned}\sum_{i=1}^{n} \Delta_i &= \sum_{i=1}^{n} \frac{1}{i}\frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i} \\ &= \sum_{i=1}^{n} \frac{1}{n+1}\left[\frac{1}{i} + \frac{1}{n-i+1}\right]\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\end{aligned}$$

Due to the fact that

$$\sum_{i=1}^{n} \frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i} = \sum_{i=1}^{n} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}$$

$$\sum_{i=1}^{n} \Delta_i = \frac{1}{n+1}\sum_{i=1}^{n} \frac{2}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}$$

and by Lemma 2.12 we have

$$\sum_{i=1}^{n} \Delta_i = \frac{1}{n+1}\binom{2n}{n} \qquad \square.$$

3. Cluster algebras and Permutations

In this section we establish the connection between cluster algebra of type $\mathbb{A}_{n-1}$ and permutation groups S_n.

3.1. **Mutations in permutation group S_n.** Consider a diagonal d in a triangulation T. This diagonal d is a diagonal of some quadrilateral. Then there is a new triangulation T' which is obtained by replacing the diagonal d with the order diagonal of that quadrilateral. This process is called a flip. It is well known that flips are the mutations in triangulations [1, 5]. According to [2, Theorem 2.11] there exists a bijection between the set of triangulations and the set of canonical readings which are separations. Before state how to mutate separations, we will define the passive classes.

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Definition 3.1. Let $\sigma = ua_ia_{i+1}v$ and $\gamma = ua_{i+1}a_iv$ be two permutations of S_n. The word σ and γ are in the same passive class if and only if the factor v contains a letter p in w which is between the letters a_i and a_{i+1} that is $a_i < p < a_{i+1}$ or $a_{i+1} < p < a_i$.

Example 3.2. Considering the readings of T in Example 2.4, we can say that $\sigma_3 = 316425$ and $\sigma_4 = 314625$ are in the same passive class by taking $u = 31$, $v = 25$ and $p = 5$.

It results from [10, Proposition 16] that two permutations being in the same passivity class represent the same triangulation T in T_n. Then by Remark 2.7 the fiber $t^{-1}(T)$ represents a passivity class. It is now clear to see that each canonical reading represents a passivity class.

Now we are ready to define how to mutate a word σ in the permutation group S_n.

Definition 3.3. Let $\sigma = uxyv$ and $\gamma = uyxv$ be two permutations of S_n. The permutation γ will be called a mutation of σ, $\mu_{xy}(\sigma) = \gamma$, if and only if there is no letter p in v which is between the letters x and y.

Over all to mutate a permutation is to exchange his two consecutive letters under the above definition condition.

Example 3.4. Let $\sigma = 1234$ in S_4. We have $\mu_{23}(\sigma) = 1324$ and
$\mu_{13}(\mu_{23}(\sigma)) = \mu_{13}(1324) = 3124$.
$\mu_{13} \circ \mu_{23}(\sigma)$ is not defined because $\mu_{13} \circ \mu_{23}(\sigma)$ and $\mu_{23}(\sigma)$ are in the same passivity class. Then the words 1324 and 3124 represent the same triangulation.
$\mu_{24}(1324) = 1342$ is defined but $\mu_{13}(1342)$ is not defined because it stays in the same passivity class.

3.2. **Permutations and diagonals.** It is clear that each permutation corresponds to a triangulation. Then, taking a permutation we need to recognize the diagonals composing the triangulation.

Let $\sigma = a_1a_2\cdots a_n$ be a permutation of S_n. We know that σ corresponds to a triangulation, so we need to enumerate all diagonals composing this triangulation through the permutation σ. For this end, we go step by step on the left to right in reading of σ. Recall that a diagonal joins two vertices of the convex polygon P_{n+2}. Then we denote a diagonal joining the vertices i and j by $\{i,j\}$. Each a_k in σ is an element of the set $\{1, 2, \ldots, n\}$. The vertices of the convex polygon P_{n+2} are labelled $0, 1, \ldots, n+1$ in clockwise.

Now take a_k in σ and rewrite σ as follows: $\sigma = Ua_kV$ where U and V are view as $U = \{a_1, a_2, \ldots, a_{k-1}\}$ and $V = \{a_{k+1}, a_{k+2}, \ldots, a_n\}$. Looking a_k as an element of the ordered set $L = \{0, 1, \ldots, (n+1)\}$, we construct two subsets of L as follows:

W_1 is the subset of L such that for all $p \in W_1$ we have $p < a_k$ and W_2 the subset of L such that for all $q \in W_2$ we have $q > a_k$.

For a fixed a_k in σ we can construct the vertices of the corresponding diagonal as follows:

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The label of the first vertex is equal to

$$\begin{cases} \max(W_1 \cap V) & \text{if } W_1 \cap V \text{ is not empty} \\ 0 & \text{if } W_1 \cap V \text{ is empty} \end{cases}$$

The label of the second vertex is equal to

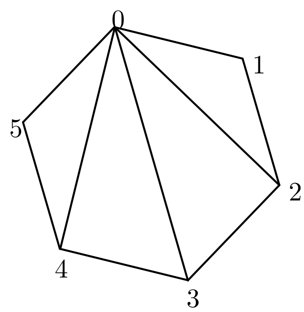
$$\begin{cases} \min(W_2 \cap V) & \text{if } W_2 \cap V \text{ is not empty} \\ (n+1) & \text{if } W_2 \cap V \text{ is empty} \end{cases}$$

Then we get the two vertices ending the diagonal. Indeed, taking the permutation $\sigma = a_1 a_2 \cdots a_n$ and starting the procedure by a_1 and ending by a_{n-1}, we obtain all diagonals composing the triangulation T associated with the permutation σ.

Example 3.5. Let σ_1 and σ_2 be in S_4 such that $\sigma_1 = 1234$ and $\sigma_2 = 4231$.

First let us give all diagonals composing the triangulation T_1 associated with σ_1. Here the set $L = \{0, 1, 2, 3, 4, 5\}$ and $a_1 = 1$, $a_2 = 2$, $a_3 = 3$. For $a_1 = 1$ we have $U = \emptyset$, $V = \{2, 3, 4\}$, $W_1 = \{0\}$, $W_2 = \{2, 3, 4, 5\}$.
$W_1 \cap V = \{0\} \cap \{2, 3, 4\} = \emptyset$ then the first vertex of the diagonal is 0.
$W_2 \cap V = \{2, 3, 4, 5\} \cap \{2, 3, 4\} = \{2, 3, 4\}$
$\min(W_2 \cap V) = \min\{2, 3, 4\} = 2$, then the second vertex of the diagonal is 2. Thus the diagonal obtained is $\{0, 2\}$.
For $a_2 = 2$ we get the diagonal $\{0, 3\}$
For $a_3 = 3$ we get the diagonal $\{0, 4\}$
Then the diagonals composing the triangulation T_1 are $\{0, 2\}$, $\{0, 3\}$, $\{0, 4\}$.

T_1:



Now we give the diagonals composing the triangulation T_2 associated with σ_2. Here we have $L = \{0, 1, 2, 3, 4, 5\}$, $a_1 = 4$, $a_2 = 2$, $a_3 = 3$.
Starting by $a_1 = 4$ we have $V = \{1, 2, 3\}$, $W_1 = \{0, 1, 2, 3\}$, $W_2 = \{5\}$
$W_1 \cap V = \{1, 2, 3\} \neq \emptyset$, $\max(W_1 \cap V) = 3$ then the first vertex of the diagonal is 3.
$W_2 \cap V = \emptyset$, then the second vertex of the diagonal is 5. So the diagonal obtained is $\{3, 5\}$.
For $a_2 = 2$ we get the diagonal $\{1, 3\}$
For $a_3 = 3$ we get the diagonal $\{1, 5\}$
Then the diagonals composing the triangulation T_2 are $\{1, 3\}$, $\{1, 5\}$ and $\{3, 5\}$.

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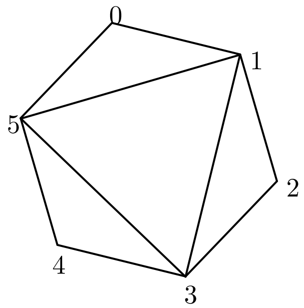
T_2:



Let $\sigma = uxyv$ be a canonical reading in S_n. It is well known that (according to Definition 3.3) $\mu_{xy}(\sigma) = uyxv$. This action corresponds to a flip of diagonal. To see this, we need some statements.

Note that a diagonal $\{i, j\}$ in a triangulation is a diagonal of some quadrilateral such that i and j are two vertices of the four vertices of this quadrilateral. Assume that the diagonal $\{i, j\}$ is the corresponding diagonal of x in the word $\sigma = uxyv$. To get the other vertices we need the following definition.

Definition 3.6. Let α, β be two vertices of a polygon P_{n+2}. The vertices are said to be adjacent if they are related by a diagonal or a side of polygon P_{n+2}.

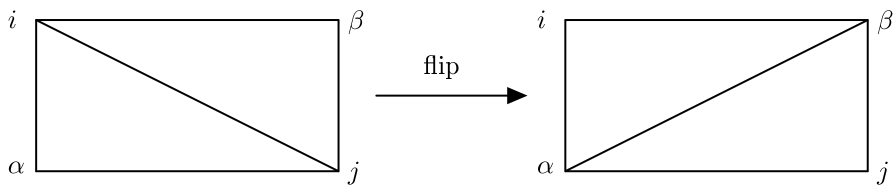
Note E_α the set of the vertices which are adjacent to vertex α.

$$E_\alpha = \{k, 0 \leqslant k \leqslant n+1 / \{\alpha, k\} \in T\}$$

It is clear that two other vertices of the quadrilateral, such that $\{i, j\}$ is one of the diagonal, compose the second diagonal and are obtained by $E_i \cap E_j$. The intersection $E_i \cap E_j$ gives two vertices because the triangulation T is of non intersecting diagonals.

$$E_i \cap E_j = \{\alpha, \beta\},\ \alpha, \beta \in \{0, 1, \ldots, (n+1)\}.$$

Then the flip of the diagonal $\{i, j\}$ in the triangulation T gives the diagonal $\{\alpha, \beta\}$ with a new triangulation T'.



Remark 3.7. According to the cutting procedure in section 2, letter x in the word $\sigma = uxyv$ is a vertex of the quadrilateral having $\{i, j\}$ as one of the diagonals. Thus x is one of the vertices of $E_i \cap E_j$.

This allows us to give the following theorem.

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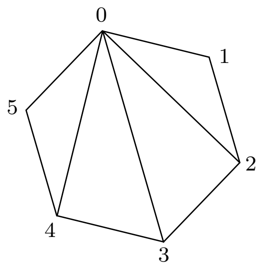
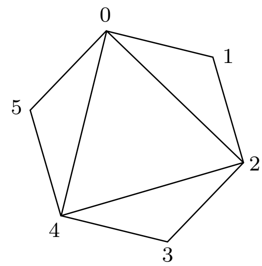
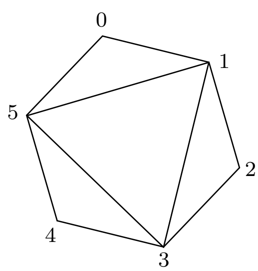
Theorem 3.8. *The mutation of permutations as defined in definition 3.3 corresponds to a flip diagonals of triangulation.* □

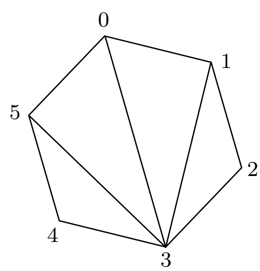
Example 3.9. 1. Let $\sigma_1 = 1234$ be a word in S_4 and $\mu_{23}(\sigma_1) = 1324$

Permutations	$\sigma_1 = 1234$	$\mu_{23}(\sigma_1) = 1324$
Corresponding diagonals	$\{0,2\}$, $\{0,3\}$, $\{0,4\}$	$\{0,2\}$, $\{2,4\}$, $\{0,4\}$
Triangulations	T_1: 	T_1': 

The diagonal corresponding to 2 in σ_1 is $\{0,3\}$ $E_0 = \{1,2,3,4,5\}$, $E_3 = \{2,4,0\}$, $E_3 \cap E_0 = \{2,4\}$
Then the diagonal $\{2,4\}$ replace the diagonal $\{3,0\}$ in the new triangulation T_1'.

2. Let $\sigma_2 = 4231$ be a word in S_4.
The corresponding diagonals are $\{3,5\}$, $\{1,3\}$, $\{1,5\}$. The diagonal corresponding to 3 in σ_2 is $\{1,5\}$
$E_1 = \{3,5,0,2\}$, $E_5 = \{3,1,0,4\}$, $E_3 \cap E_5 = \{3,0\}$
In the triangulation T_2 associated with σ_2, we make a flip on the diagonal $\{1,5\}$ and we get the new diagonal $\{0,3\}$ in the new triangulation T_2'.

T_2: 

T_2': 

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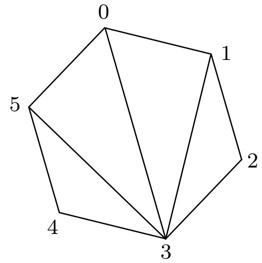
Let T_2'' be the corresponding triangulation of $\mu_{31}(\sigma_2) = 4213$. According to the cutting procedure in section 2 we have the following triangulation

T_2'': 

. It is clear that $T_2'' = T_2'$

3.3. **Cluster algebras and Permutations.** A cluster algebra is generated by a set of variables, called cluster variables, obtained recursively by a combinatorial process known as mutation starting from a set of initial cluster variables [7, 8].

Let T be any triangulation of polygon P_{n+2}. According to [4, section 3.4,4.1] the cluster algebra $\mathcal{A}(T)$ associated with the triangulation T of the polygon P_{n+2} is constructed as follows:

Start with convex polygon P_{n+2} and choose a triangulation T by non intersecting diagonals. Labels these diagonals $x_{i,j}$, $i, j \in \{0, 1, \ldots, (n+1)\}$, $i \notin \{j-1, j, j+1\}$ and the sides by $y_{i,i+1}$ and vertices of P_{n+2} by $0, 1, \ldots, (n+1)$. The labels for the remaining diagonals are obtained by flipping diagonals. The cluster algebra $\mathcal{A}(T)$ is the subalgebra of the rational functions field in $x_{i,j}$ and $y_{i,i+1}$ generated by all the labels in P_{n+2}. The cluster algebra associated with a triangulation of a polygon P_{n+2} depends only on $(n+2)$ and is of type $\mathbb{A}_{n-1}$ because this cluster algebra has rank $(n-1)$.

In this text the variables $y_{i,i+1}$ are taken to be equal to 1.

The cluster algebra associated with a polygon P_{n+2} can be identified with the coordinate ring $\mathbb{C}\left[Gr_{2,(n+2)}\right]$ the Grassmannian $Gr_{2,(n+2)}$ of 2-planes in a $(n+2)$-dimensional vector space [6]. The coordinate ring $\mathbb{C}\left[Gr_{2,(n+2)}\right]$ is generated by the three-term Plücker coordinates $p_{i,j}$ for $0 \leqslant i < j \leqslant (n+1)$. The relations among the Plücker coordinate are generated by the three-term Plücker relations: for any $0 \leqslant i < j < k < l \leqslant (n+1)$, one has

$$P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}.$$

By assigning the value of P_{ij} to the variable $x_{i,j}$ associated with the diagonal $\{i, j\}$ of the triangulation T of the polygon P_{n+2}, the the three-term Plücker relations correspond to exchange relation in $\mathcal{A}(T)$. We state the following.

Theorem 3.10. *Let $n \geq 1$ be an integer, let σ be a canonical reading in the permutation group S_n and T the triangulation associated with σ. The cluster algebras $\mathcal{A}(T)$ of type $\mathbb{A}_{n-1}$ and $\mathcal{A}(\sigma)$ coincide.*

Before giving the proof of Theorem 3.10, let us give the analogous of [6, Corollary 5.3.6].

Corollary 3.11. *Cluster variables in a seed pattern of type $\mathbb{A}_{n-1}$ can be labelled by diagonals of convex $(n+2)-$gon P_{n+2} so that*

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- *Clusters correspond to canonical readings,*
- *Flips correspond to mutations of permutations*

Cluster variables labelled by diagonals are distinct, so there are altogether $\frac{(n-1)(n+2)}{2}$ *cluster variables and* $\frac{1}{n+1}\binom{2n}{n}$ *seeds.* □

Proof of the Theorem 3.10.

Each canonical reading produces $(n-1)$ diagonals which correspond to a seed. According to Theorem 2.11 there are $\frac{1}{n+1}\binom{2n}{n}$ canonical readings, so by Corollary 3.11 these canonical readings correspond to seeds and the mutation of canonical readings correspond to flip which correspond to cluster mutation. □

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References

[1] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations arising from clusters (An case). *Trans. Amer. Math. Soc.* **358** (2006), 1347-1364.

[2] J. Conway, H. Coxeter, Triangulated polygons and frieze patterns, *The Mathematical Gazette* 57 (400) (1973) 87-94.

[3] J. Conway, H. Coxeter, Triangulated polygons and frieze patterns, *The Mathematical Gazette* 57 (401) (1973) 175-183.

[4] S. Fomin, N. Reading, Root systems and generalized associahedra *Geometric Combinatorics*, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI (2007), pp. 63-131

[5] S. Fomin, M. Shapiro and D. Thurston. Cluster algebras and triangulated surfaces. Part I: Cluster complexes. *Acta Mathematica* **201.1** (2008): 83-146.

[6] S. Fomin, L. Williams and A. Zelevinsky, Introduction to cluster algebras. chapters 4-5. arXiv preprint arXiv:1707.07190. 2017 Jul 22.

[7] S. Fomin and A. Zelevinsky, Cluster algebras I. Foundations, *J. Amer. Math. Soc.* **15** (2002), no. 2, 497-529 (electronic)

[8] S. Fomin and A. Zelevinsky, Cluster algebras II: Finite type classification, *Invent. Math.* **154**, (2003), 63-121.

[9] F. Hurtado and M. Noy, 1999. Graph of triangulations of a convex polygon and tree of triangulations. *Computational Geometry,* **13(3)**, pp.179-188.

[10] E. Shalom, and C. Lecouvey, Signed permutations and the four color theorem, *Expositiones Mathematicae* **27.4** (2009), 313-340.

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