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THE w-CORE–EP INVERSE IN RINGS WITH INVOLUTION

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Abstract. The main goal of this paper is to present two new classes of generalized inverses in order to extend the concepts of the (dual) core–EP inverse and the (dual) w-core inverse. Precisely, we introduce the w-core–EP inverse and its dual for elements of a ring with involution. We characterize the (dual) w-core–EP invertible elements and develop several representations of the w-core–EP inverse and its dual in terms of different well-known generalized inverses. Using these results, we get new characterizations and expressions for the core–EP inverse and its dual. We apply the dual w-core–EP inverse to solve certain operator equations and give their general solutions forms.

1. Introduction

Let $\mathcal{R}$ be an associative ring with unit 1. For $a \in \mathcal{R}$, we define the following kernel ideals $a^\circ = \{x \in \mathcal{R} : ax = 0\}$, ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$, and image ideals $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$, $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$.

An element $a \in \mathcal{R}$ is Drazin invertible if there exists $x \in \mathcal{R}$ such that

$$xax = x, \qquad ax = xa \qquad \text{and} \qquad a^k = a^{k+1}x, \tag{1.1}$$

for some nonnegative integer k. The Drazin inverse x of a is unique (if it exists) and denoted by a^D (see [7]). It is known that the Drazin inverse was defined in a semigroup [7] and in a semigroup without the identity we have $k > 0$, while for a semigroup with identity we have $k \geq 0$ and for $k = 0$ we define $a^0 = 1$. The smallest above mentioned k is called the Drazin index of a and denoted by $\mathrm{ind}(a)$. Recall that a^D double commutes with a, that is, $ay = ya$ implies $a^D y = ya^D$. For $\mathrm{ind}(a) = 1$, a is group invertible and its group inverse is denoted by $a^\#$. Notice that $a^\#$ satisfies $a^\# a a^\# = a^\#$, $a^\# a = aa^\#$ and $aa^\# a = a$. It is well known that $a^\#$ exists if and only if $a \in a^2\mathcal{R} \cap \mathcal{R}a^2$ if and only if $a\mathcal{R} = a^2\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}a^2$ [7, 25]. The sets $\mathcal{R}^D$ and $\mathcal{R}^\#$ involve all Drazin invertible and all group invertible elements of $\mathcal{R}$, respectively.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, i.e. $(a^*)^* = a$, $(a + b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$, for all $a, b \in \mathcal{R}$. An element $p \in \mathcal{R}$ is orthogonal projector if $p^2 = p = p^*$. Significant results related to orthogonal

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projector can be seen in [16]. An element $a \in \mathcal{R}$ is Moore–Penrose invertible if there exists $x \in \mathcal{R}$ satisfying the so-called Penrose equations [26]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa.$$

The Moore–Penrose inverse x of a is uniquely determined (if it exists) and denoted by $x = a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

An element $x \in R$ is a $\{1\}$-inverse of $a \in R$ if $axa = a$ and, in this case, we say that a is regular. An element $x \in R$ is a $\{1,3\}$-inverse (or $\{1,4\}$-inverse) of a if $axa = a$ and $(ax)^* = ax$ ($axa = a$ and $(xa)^* = xa$). The symbol $a\{1,3\}$ (or $a\{1,4\}$) stands for the set of all $\{1,3\}$-inverses ($\{1,4\}$-inverses) of a. The set of all $\{1,3\}$-invertible ($\{1,4\}$-invertible) elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{\{1,3\}}$ ($\mathcal{R}^{\{1,4\}}$). An interesting class of $\{1\}$-inverses was studied in [4].

The notion of inverse along one element introduced by Mary [19] is important because a number of well-known generalized inverses such as group inverse, Drazin inverse and Moore-Penrose inverse, are special cases of this inverse. For $d \in \mathcal{R}$, an element $a \in \mathcal{R}$ is invertible along d if there exists $x \in \mathcal{R}$ satisfying

$$xad = d = dax \qquad \text{and} \qquad x \in d\mathcal{R} \cap \mathcal{R}d.$$

The inverse x of a along d is unique (if it exists) and denoted by $a^{||d}$ [19]. According to [19, 20], $a \in \mathcal{R}^\#$ if and only if $a^{||a}$ exists if and only if $1^{||a}$ exists. In addition, $a^\# = a^{||a}$ and $1^{||a} = aa^\#$. Also, $a \in \mathcal{R}^D$ if and only $a^{||a^k}$ exists, for some positive integer k; and $a \in \mathcal{R}^\dagger$ if and only if $a^{||a^*}$ exists. Furthermore, $a^D = a^{||a^k}$ and $a^\dagger = a^{||a^*}$. More results about inverse along one element can be found in [2, 3, 21, 35].

The core–EP inverse was introduced in [27] for a square matrix over an arbitrary field, as an extension of the core inverse given in [1]. The core–EP inverse for elements of a ring was defined in [10] in the following way. Let $a \in \mathcal{R}$. Then a is core–EP (or pseudo core) invertible if there exists an element $x \in \mathcal{R}$ such that

$$ax^2 = x, \qquad xa^{k+1} = a^k \qquad \text{and} \qquad (ax)^* = ax,$$

for some positive integer k. The core–EP inverse of a is unique (if it exists) and denoted by $a^{\textcircled{\tiny D}}$. The smallest positive integer k in the definition of the core–EP inverse, is called the pseudo core index of a and denoted by $I(a)$, either equals the Drazin index $\mathrm{ind}(a)$ of a if $\mathrm{ind}(a) > 0$, or is 1 if $\mathrm{ind}(a) = 0$ (see [10, Theorem 2.3] and observe that Gao et al. defined the Drazin index of a as the smallest positive integer k that satisfies (1.1)). Notice that a is core–EP invertible if and only if there exist a^D and $(a^k)^{(1,3)} \in a^k\{1,3\}$, for $k \geq \mathrm{ind}(a)$ [10, Theorem 2.3]. In addition, $a^{\textcircled{\tiny D}} = a^D a^k (a^k)^{(1,3)}$. The dual core–EP inverse $a_{\textcircled{\tiny D}}$ of a was introduced as the unique solution of equations $x^2a = x$, $a^{k+1}x = a^k$ and $(xa)^* = xa$, for some positive integer k. In a special case that $\mathrm{ind}(a) = 1$, the core–EP inverse of a becomes the core inverse $a^{\textcircled{\tiny \#}} = a^\# a a^\dagger$ [1], and the dual core–EP inverse coincides to the dual core inverse $a_{\textcircled{\tiny \#}} = a^\dagger a a^\#$.

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Recently, the core inverse and core–EP inverse were studied in numerous papers [5, 11, 13, 14, 15, 17, 30, 32, 36]. For instance, different properties and representations of the core–EP inverse were proved in [8, 9, 17, 18, 22, 31]; limit representations for the core-EP inverse were given in [32]; continuity of core-EP inverse was investigated in [12]; iterative method for computing core-EP inverse was proved in [28, 29]. The core–EP inverse was extended for operators on Hilbert spaces in [23, 24] and for tensors in [30].

Two new classes of generalized inverses were recently presented in [39]. Precisely, the w-core inverse and its dual for elements of a ring with involution, were introduced in [39] as generalizations of the core inverse and dual core inverse, respectively. We now state the definition of the w-core inverse. Let $a, w \in \mathcal{R}$. Then a is w-core invertible if there exists an element $x \in \mathcal{R}$ such that

$$awx^2 = x, \qquad xawa = a \qquad \text{and} \qquad (awx)^* = awx.$$

If such x exists, it is the uniquely determined w-core inverse of a [39] and denoted by $a_w^{\textcircled{\#}}$. Note that 1-core inverse of a coincides with the core inverse of a, i.e. $a_1^{\textcircled{\#}} = a^{\textcircled{\#}}$. Some significant results about the w-core inverse can be found in [38].

Motivated by a number of researches and popularity of the core–EP inverse and a recent investigation about the w-core inverse, the aim of this paper is to introduce a new class of generalized inverses which includes the core–EP inverse and the w-core inverse. In particular, we present the w-core–EP inverse and its dual for elements of a ring with involution. In this way, we define two new wider classes of generalized inverses, extending the notions of the core–EP inverse, the w-core inverse and their duals. Various characterizations for the existence of the w-core–EP inverse and its dual are established as well as corresponding representations involving the inverse of w along a corresponding element, group inverse, Drazin inverse, $\{1, 3\}$-inverse and $\{1, 4\}$-inverse of adequate elements. Using these results, we obtain new characterizations and representations of the core–EP inverse and its dual. Applying the dual w-core–EP inverse, we solve several operator equations and give the forms of their general solutions.

We shortly describe the content of this paper. In Section 2, we define the w-core–EP inverse and investigate necessary and sufficient conditions for the existence of the w-core–EP inverse and its representations. New characterizations and expressions of the core–EP inverse are also given. The dual w-core–EP inverse is studied in Section 3 as well as the dual core–EP inverse. Section 4 contains applications of the dual w-core–EP inverse in solving some operator matrix equations.

2. The w-core–EP inverse

In order to extend the notions of the core–EP inverse and the w-core inverse, we define the w-core–EP inverse in a ring $\mathcal{R}$ with involution.

Definition 2.1. Let $a, w \in \mathcal{R}$. Then a is called w-core–EP invertible if there exists an element $x \in \mathcal{R}$ such that

$$awx^2 = x, \qquad x(aw)^{k+1}a = (aw)^k a \qquad \text{and} \qquad (awx)^* = awx,$$

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for some nonnegative integer k. In this case, x is a w-core–EP inverse of a.

Remark that, for $k = 0$ in the above definition, the w-core–EP inverse becomes the w-core inverse. Notice that the 1-core-EP inverse is equal to the core–EP inverse. Thus, core–EP invertible and w-core invertible elements are w-core-EP invertible. The smallest nonnegative integer k in the definition of the w-core–EP inverse, is called the w-core–EP index of a and denoted by $i_w(a)$.

Theorem 2.2. *Let $a, w \in \mathcal{R}$. Then a has at most one w-core-EP inverse.*

Proof. If x is the w-core–EP inverse of a, then $awx^2 = x$, $x(aw)^{k+1}a = (aw)^k a$ and $(awx)^* = awx$, for some nonnegative integer k. We have $x(aw)^{k+2} = (aw)^{k+1}$ and thus $x = (aw)^{\textcircled{D}}$. □

Since the w-core–EP inverse of a is unique, if it exists, by Theorem 2.2, we use the symbol $a_w^{\textcircled{D}}$ to denote the w-core–EP inverse of a.

Although core–EP invertible elements are w-core-EP invertible, the converse is not true in general. In the next example, we give a w-core-EP invertible element which is not core-EP invertible.

Example 2.3. Let $\mathcal{R} = \mathbb{Z}$ be the ring of all integers. For $a = 2$ and $w = 0$, we conclude that a is w-core–EP invertible with $a_w^{\textcircled{D}} = 0$. However, a is not Drazin invertible in $\mathbb{Z}$ and so it is not core–EP invertible.

Several necessary and sufficient conditions for the existence of the w-core–EP inverse, are established now.

Theorem 2.4. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*

(ii) *there exists an element $x \in \mathcal{R}$ such that*

$$awx^2 = x, \quad x(aw)^{k+1}a = (aw)^k a, \quad xawx = x,$$

$$awx(aw)^k a = (aw)^k a \quad \text{and} \quad (awx)^* = awx,$$

for some nonnegative integer k;

(iii) *there exists an element $x \in \mathcal{R}$ such that*

$$awx(aw)^k a = (aw)^k a, \quad (aw)^k a\mathcal{R} = x\mathcal{R} \quad \text{and} \quad \mathcal{R}x = \mathcal{R}\left((aw)^k a\right)^*,$$

for some nonnegative integer k;

(iv) *there exists an element $x \in \mathcal{R}$ such that*

$$awx(aw)^k a = (aw)^k a \quad \text{and} \quad (aw)^k a\mathcal{R} = x\mathcal{R} = x^*\mathcal{R},$$

for some nonnegative integer k;

(v) *there exists an element $x \in \mathcal{R}$ such that*

$$awx(aw)^k a = (aw)^k a \quad \text{and} \quad (aw)^k a\mathcal{R} = x\mathcal{R} \supseteq x^*\mathcal{R},$$

for some nonnegative integer k;

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(vi) *there exists an element* $x \in \mathcal{R}$ *such that*

$$awx(aw)^k a = (aw)^k a, \quad {}^\circ\left((aw)^k a\right) = {}^\circ x \quad \text{and} \quad x^\circ = \left(((aw)^k a)^*\right)^\circ,$$

for some nonnegative integer k*;*

(vii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$awx(aw)^k a = (aw)^k a, \quad {}^\circ\left((aw)^k a\right) = {}^\circ x \quad \text{and} \quad x^\circ \supseteq \left(((aw)^k a)^*\right)^\circ,$$

for some nonnegative integer k*;*

(viii) *there exists an element* $x \in \mathcal{R}$ *such that*

$$awx^2 = x, \; x(aw)^{k+1}a = (aw)^k a, \; awx = (aw)^n x^n \text{ and } (awx)^* = awx,$$

for some nonnegative integer k *and all/some positive integer* n*.*

Proof. (i) ⇒ (ii): Assume that x is the w-core–EP inverse of a. Thus, for some nonnegative integer k, $awx^2 = x$, $x(aw)^{k+1}a = (aw)^k a$ and $(awx)^* = awx$. Then

$$\begin{aligned} x &= awx^2 = (aw)^2x^3 = \cdots = (aw)^{k+1}x^{k+2} = ((aw)^k a)wx^{k+2} \\ &= x(aw)^{k+1}awx^{k+2} = xawx \end{aligned}$$

and

$$(aw)^k a = x(aw)^{k+1}a = awx^2(aw)^{k+1}a = awx(aw)^k a.$$

(ii) ⇒ (iii): Using $x(aw)^{k+1}a = (aw)^k a$ and $awx^2 = x$, we have

$$(aw)^k a\mathcal{R} = x(aw)^{k+1}a\mathcal{R} \subseteq x\mathcal{R} = awx^2\mathcal{R} = (aw)^k awx^{k+2}\mathcal{R} \subseteq (aw)^k a\mathcal{R}.$$

Thus, $(aw)^k a\mathcal{R} = x\mathcal{R}$. The assumptions $awx(aw)^k a = (aw)^k a$ and $(awx)^* = awx$ imply

$$\mathcal{R}\left((aw)^k a\right)^* = \mathcal{R}\left(awx(aw)^k a\right)^* = \mathcal{R}\left((aw)^k a\right)^* awx \subseteq \mathcal{R}x.$$

Since $awx = (aw)^{k+1}x^{k+1}$, then

$$\begin{aligned} x &= xawx = x(awx)^* = x\left((aw)^{k+1}x^{k+1}\right)^* \\ &= x\left((aw)^k awx^{k+1}\right)^* = x\left(wx^{k+1}\right)^*\left((aw)^k a\right)^*, \end{aligned}$$

which gives $\mathcal{R}x \subseteq \mathcal{R}\left((aw)^k a\right)^*$. Hence, $\mathcal{R}x = \mathcal{R}\left((aw)^k a\right)^*$.

(iii) ⇒ (iv) ⇒ (v): It is evident.

(v) ⇒ (i): From $awx(aw)^k a = (aw)^k a$ and $(aw)^k a\mathcal{R} = x\mathcal{R}$, we get, for some $u \in \mathcal{R}$,

$$x = (aw)^k au = awx\left((aw)^k au\right) = awx^2.$$

The hypothesis $(aw)^k a\mathcal{R} \supseteq x^*\mathcal{R}$ yields, for some $y \in \mathcal{R}$,

$$x = y\left((aw)^k a\right)^* = y\left(awx(aw)^k a\right)^* = y\left((aw)^k a\right)^*(awx)^* = x(awx)^*.$$

Further, $awx = awx(awx)^*$ implies that $(awx)^* = awx$ and so $x = xawx$. Because $(aw)^k a = xv$, for some $v \in \mathcal{R}$, we have $(aw)^k a = xv = xaw(xv) = x(aw)^{k+1}a$. Therefore, $x = a_w^{\textcircled{\scriptsize D}}$.

(iii) ⇒ (vi) ⇒ (vii): These implications are obvious.

(vii) ⇒ (i): The condition $awx(aw)^k a = (aw)^k a$ gives $1 - awx \in {}^\circ\left((aw)^k a\right) = {}^\circ x$. Thus, $(1-awx)x = 0$, i.e. $x = awx^2$. Since $((aw)^k a)^*(awx)^* = ((aw)^k a)^*$, then $1 - (awx)^* \in \left(((aw)^k a)^*\right)^\circ \subseteq x^\circ$. Hence, $x = x(awx)^*$ yields $awx = awx(awx)^* =$

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$(awx)^*$ and $x = xawx$. Now, $1 - xaw \in {}^\circ x = {}^\circ\big((aw)^k a\big)$ implies $(aw)^k a = x(aw)^{k+1}a$. So, x is the w-core–EP inverse of a.

(i) $\Leftrightarrow$ (viii): These equivalence is clear. □

In the case that $k = 0$, notice that Theorem 2.4 recovers [39, Theorem 2.6] related to w-core invertible elements.

Remark 2.5. Let $a, b, c \in \mathcal{R}$. An element $x \in \mathcal{R}$ is a (b, c)-inverse of a if $xax = x$, $x\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}c$. The (b, c)-inverse of a is unique, if it exists, and denoted by $a^{||(b,c)}$ [6]. By Theorem 2.4(iii), for $a, w \in \mathcal{R}$, we have that a is w-core–EP invertible if and only if aw is $((aw)^k a, ((aw)^k a)^*)$-invertible, for some nonnegative integer k. In this case, $a_w^{\textcircled{\dagger}} = (aw)^{||((aw)^k a,((aw)^k a)^*)}$.

Applying Theorem 2.4 for $w = 1$, we get new characterizations for core–EP invertible elements.

Remark 2.6. Let $a \in \mathcal{R}$. Then a is core–EP invertible if and only if there exists an element $x \in \mathcal{R}$ such that, for some nonnegative integer k and all/some positive integer n, one of the following equivalent statements holds:

(i) $ax^2 = x,\ xa^{k+2} = a^{k+1},\ xax = x,\ axa^{k+1} = a^{k+1}$ *and* $(ax)^* = ax$;
(ii) $axa^{k+1} = a^{k+1}, a^{k+1}\mathcal{R} = x\mathcal{R}$ *and* $\mathcal{R}x = \mathcal{R}(a^{k+1})^*$;
(iii) $axa^{k+1} = a^{k+1}$ *and* $a^{k+1}\mathcal{R} = x\mathcal{R} = x^*\mathcal{R}$;
(iv) $axa^{k+1} = a^{k+1}$ *and* $a^{k+1}\mathcal{R} = x\mathcal{R} \supseteq x^*\mathcal{R}$;
(v) $axa^{k+1} = a^{k+1},\ {}^\circ(a^{k+1}) = {}^\circ x$ *and* $x^\circ = ((a^{k+1})^*)^\circ$;
(vi) $axa^{k+1} = a^{k+1},\ {}^\circ(a^{k+1}) = {}^\circ x$ *and* $x^\circ \supseteq ((a^{k+1})^*)^\circ$;
(vii) $ax^2 = x,\ xa^{k+2} = a^{k+1}$ *and* $(ax)^* = ax = a^n x^n$.

By [39, Theorem 2.11], a is w-core invertible if and only if there exist $w^{||a}$ and $a^{(1,3)} \in a\{1, 3\}$. In this case, $a_w^{\textcircled{\#}} = w^{||a}a^{(1,3)}$. We can develop a representation of the w-core–EP inverse in terms of the inverse along a corresponding element and $\{1, 3\}$-inverse, generalizing [39, Theorem 2.11] for the w-core inverse.

Theorem 2.7. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*
(ii) *there exist $w^{||(aw)^k a}$ and $\big((aw)^k a\big)^{(1,3)} \in \big((aw)^k a\big)\{1, 3\}$, for some nonnegative integer k;*
(iii) *there exist $w^{||(aw)^k a}$ and $\big((aw)^{k+1}\big)^{(1,3)} \in \big((aw)^{k+1}\big)\{1, 3\}$, for some nonnegative integer k;*
(iv) *there exist $w^{||(aw)^k a}$ and $\big((aw)^{k+1}a\big)^{(1,3)} \in \big((aw)^{k+1}a\big)\{1, 3\}$, for some nonnegative integer k.*

In addition, if any of statements (i)–(iv) *holds, then, for some nonnegative integer k and $\big((aw)^k a\big)^{(1,3)} \in \big((aw)^k a\big)\{1, 3\}$,*

$$a_w^{\textcircled{\dagger}} = (aw)^k w^{||(aw)^k a}\big((aw)^k a\big)^{(1,3)}.$$

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Proof. (i) $\Rightarrow$ (ii): Let x be the w-core–EP inverse of a. Then $awx^2 = x$, $x(aw)^{k+1}a = (aw)^k a$ and $(awx)^* = awx$, for some nonnegative integer k. Because

$$\begin{aligned}(aw)^k a &= x(aw)^{k+1}a = x\left((aw)^k a\right) wa = x^2(aw)^{k+1}awa = x^2(aw)^k awawa = \dots\\ &= x^{k+1}(aw)^k aw(aw)^k a \in \mathcal{R}\left((aw)^k a\right) w \left((aw)^k a\right)\end{aligned}$$

and

$$\begin{aligned}(aw)^k a &= x(aw)^{k+1}a = awx^2(aw)^{k+1}a = \dots = (aw)^{2k+2}x^{2k+3}(aw)^{k+1}a\\ &= (aw)^k aw(aw)^k awx^{2k+3}(aw)^{k+1}a \in \left((aw)^k a\right) w \left((aw)^k a\right) \mathcal{R},\end{aligned}$$

by [20, Theorem 2.2], we deduce that $w \in \mathcal{R}^{||(aw)^k a}$. Furthermore, from

$$\begin{aligned}(aw)^k a &= (aw)^k aw(aw)^k awx^{2k+3}(aw)^{k+1}a\\ &= \left((aw)^k a\right) w(aw)^k awx^{2k+3}aw \left((aw)^k a\right)\end{aligned} \tag{2.1}$$

and

$$(aw)^k awx^{k+1} = awx = (awx)^* = \left((aw)^k awx^{k+1}\right)^*,$$

we observe that $(aw)^k a \in \mathcal{R}^{(1,3)}$.

(ii) $\Rightarrow$ (i): Suppose that $x = (aw)^k w^{||(aw)^k a} \left((aw)^k a\right)^{(1,3)}$, for some nonnegative integer k and $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$. Notice that

$$(aw)^k a = (aw)^k aww^{||(aw)^k a} = (aw)^{k+1}w^{||(aw)^k a}$$

and $(aw)^k a = w^{||(aw)^k a}w(aw)^k a = w^{||(aw)^k a}(wa)^{k+1}$. Since $w^{||(aw)^k a} = (aw)^k au = v(aw)^k a$, for some $u, v \in \mathcal{R}$, we can get $w^{||(aw)^k a} = (aw)^k a \left((aw)^k a\right)^{(1,3)} w^{||(aw)^k a}$ and $w^{||(aw)^k a} = w^{||(aw)^k a} \left((aw)^k a\right)^{(1,3)} (aw)^k a$. Therefore,

$$awx = (aw)^{k+1}w^{||(aw)^k a} \left((aw)^k a\right)^{(1,3)} = (aw)^k a \left((aw)^k a\right)^{(1,3)}$$

gives $(awx)^* = awx$ and

$$\begin{aligned}awx^2 &= (aw)^k a \left((aw)^k a\right)^{(1,3)} (aw)^k w^{||(aw)^k a} \left((aw)^k a\right)^{(1,3)}\\ &= \left[(aw)^k a \left((aw)^k a\right)^{(1,3)} (aw)^k a\right] (wa)^k u \left((aw)^k a\right)^{(1,3)}\\ &= (aw)^k[a(wa)^k u] \left((aw)^k a\right)^{(1,3)} = (aw)^k w^{||(aw)^k a} \left((aw)^k a\right)^{(1,3)}\\ &= x.\end{aligned}$$

According to [20, Theorem 2.1], we have

$$w^{||(aw)^k a} = ((aw)^{k+1})^{\#}(aw)^k a = (aw)^k((aw)^{k+1})^{\#}a.$$

So,

$$\begin{aligned}x(aw)^{k+1}a &= (aw)^k w^{||(aw)^k a} \left((aw)^k a\right)^{(1,3)} (aw)^{k+1}a\\ &= (aw)^k \left[w^{||(aw)^k a} \left((aw)^k a\right)^{(1,3)} (aw)^k a\right] wa = (aw)^k w^{||(aw)^k a}wa\\ &= (aw)^k((aw)^{k+1})^{\#}(aw)^k awa = [(aw)^k((aw)^{k+1})^{\#}a](wa)^{k+1}\\ &= w^{||(aw)^k a}(wa)^{k+1} = (aw)^k a\end{aligned}$$

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and $x = a_w^{\textcircled{D}}$.

(ii) $\Rightarrow$ (iii): By [40, Lemma 2.2], recall that $u \in \mathcal{R}^{\{1,3\}}$ if and only if $u \in \mathcal{R}u^*u$. Since $(aw)^k a \in \mathcal{R}^{\{1,3\}}$, for some nonnegative integer k, then $(aw)^k a \in \mathcal{R}\left((aw)^k a\right)^* (aw)^k a$, which yields $(aw)^{k+1} \in \mathcal{R}\left((aw)^k a\right)^* (aw)^{k+1}$. Notice that, by the equivalence (i) $\Leftrightarrow$ (ii), (2.1) holds. Hence, $(aw)^k a \in (aw)^{k+1}\mathcal{R}$ which gives $\left((aw)^k a\right)^* \in \mathcal{R}\left((aw)^{k+1}\right)^*$. Now $(aw)^{k+1} \in \mathcal{R}\left((aw)^{k+1}\right)^* (aw)^{k+1}$ implies that $(aw)^{k+1} \in \mathcal{R}^{\{1,3\}}$.

(iii) $\Rightarrow$ (iv): Because $(aw)^{k+1} \in \mathcal{R}^{\{1,3\}}$ gives $(aw)^{k+1} \in \mathcal{R}\left((aw)^{k+1}\right)^* (aw)^{k+1}$, then $(aw)^{k+1}a \in \mathcal{R}\left((aw)^{k+1}\right)^* (aw)^{k+1}a$. Also, $(aw)^k a = (aw)^{k+1} w^{||(aw)^k a}$ and $w^{||(aw)^k a} = (aw)^k au$, for some $u \in \mathcal{R}$, imply

$$(aw)^{k+1} = (aw)^k aw = (aw)^{k+1} w^{||(aw)^k a} w = (aw)^{k+1} a (wa)^k uw.$$

Therefore, $\left((aw)^{k+1}\right)^* \in \mathcal{R}\left((aw)^{k+1}a\right)^*$ yields $(aw)^{k+1}a \in \mathcal{R}\left((aw)^{k+1}a\right)^* (aw)^{k+1}a$ and so $(aw)^{k+1}a \in \mathcal{R}^{\{1,3\}}$.

(iv) $\Rightarrow$ (ii): Since $w^{||(aw)^k a}$ exists, by [20, p. 1132], $(aw)^k a$ is regular. Using $\left((aw)^{k+1}a\right)^{(1,3)} \in \left((aw)^{k+1}a\right)\{1,3\}$, for some nonnegative integer k, we have $(aw)^k awa\left((aw)^{k+1}a\right)^{(1,3)} = (aw)^{k+1}a\left((aw)^{k+1}a\right)^{(1,3)}$ and thus $\left((aw)^k a\right)\{1,3\} \neq \emptyset$. $\square$

Remark 2.8. It is clear that the representation of the w-core–EP inverse given in Theorem 2.7 does not depend from the choice of $\{1,3\}$-inverse. Indeed, for $x, y \in \left((aw)^k a\right)\{1,3\}$, we have that $(aw)^k ax = (aw)^k ay$ and $w^{||(aw)^k a} = w^{||(aw)^k a} y (aw)^k a$ which imply $w^{||(aw)^k a} x = w^{||(aw)^k a} y (aw)^k ax = (w^{||(aw)^k a} y (aw)^k a) y = w^{||(aw)^k a} y$ and $(aw)^k w^{||(aw)^k a} x = (aw)^k w^{||(aw)^k a} y$.

As a consequence of Theorem 2.7, we obtain the following characterization of a core–EP invertible element and its expression based on the inverse along an element and $\{1,3\}$-inverse.

Corollary 2.9. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is core–EP invertible;*
(ii) *there exist $1^{||a^{k+1}}$ and $(a^{k+1})^{(1,3)} \in (a^{k+1})\{1,3\}$, for some nonnegative integer k;*
(iii) *there exist $1^{||a^{k+1}}$ and $\left(a^{k+2}\right)^{(1,3)} \in \left(a^{k+2}\right)\{1,3\}$, for some nonnegative integer k.*

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer k and $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$,*

$$a^{\textcircled{D}} = a^k 1^{||a^{k+1}} (a^{k+1})^{(1,3)}.$$

By Theorem 2.7 and some properties of inverse along an element proved in [20], we can provide more characterizations of w-core–EP invertible elements.

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Theorem 2.10. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*
(ii) *$(aw)^k a \in (aw)^{k+1}\mathcal{R}$ and there exist $\left((aw)^{k+1}\right)^{\#}$ and $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$, for some nonnegative integer k;*
(iii) *$(aw)^k a \in \mathcal{R}(wa)^{k+1}$ and there exist $\left((wa)^{k+1}\right)^{\#}$ and $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$, for some nonnegative integer k;*
(iv) *$(aw)^k a \in (aw)^{2k+1}a\mathcal{R} \cap \mathcal{R}(aw)^{2k+1}a$ and there exists $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$, for some nonnegative integer k.*

In addition, if any of statements (i)–(iv) *holds, then, for some nonnegative integer k and $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$,*

$$a_w^{\textcircled{\dagger}} = \left((aw)^{k+1}\right)^{\#} (aw)^{2k} a \left((aw)^k a\right)^{(1,3)} = (aw)^{2k} a \left((wa)^{k+1}\right)^{\#} \left((aw)^k a\right)^{(1,3)}.$$

Proof. This proof is evident by Theorem 2.7 and [20, Theorem 2.1 and Theorem 2.2]. □

For $w = 1$ in Theorem 2.10, we get the next result.

Corollary 2.11. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is core–EP invertible;*
(ii) *$a^{k+1} \in \mathcal{R}^{\#} \cap \mathcal{R}^{(1,3)}$, for some nonnegative integer k.*

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer k and $(a^{k+1})^{(1,3)} \in a^{k+1}\{1,3\}$,*

$$a^{\textcircled{\dagger}} = (a^{k+1})^{\#} a^{2k+1} (a^{k+1})^{(1,3)} = a^k (a^{k+1})^{\textcircled{\#}}.$$

We can show that a is core–EP invertible element if and only if a is a-core–EP invertible.

Theorem 2.12. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is core–EP invertible;*
(ii) *a is a-core–EP invertible.*

Proof. Since a is core–EP invertible, by Corollary 2.11, $a^{k+1} \in \mathcal{R}^{\#} \cap \mathcal{R}^{(1,3)}$, for some nonnegative integer k. Then $a^{2k+2} = (a^{k+1})^2 \in \mathcal{R}^{\#}$ and $a^{2k+1} = a^k a^{k+1} = a^k (a^{k+1})^2 (a^{k+1})^{\#} \in a^{2k+2}\mathcal{R}$. For $y \in a^{k+1}\{1,3\}$, the equalities $a^{k+1} y a^{k+1} = a^{k+1}$ and $a^{k+1} y = (a^{k+1} y)^*$ imply $a^{2k+1} a (a^{k+1})^{\#} y a^{2k+1} = a^{2k+1}$ and

$$a^{2k+1} a (a^{k+1})^{\#} y = a^{k+1} y = (a^{k+1} y)^* = (a^{2k+1} a (a^{k+1})^{\#} y)^*,$$

i.e. $a(a^{k+1})^{\#} y \in a^{2k+1}\{1,3\}$. Using Theorem 2.10, we deduce that a is a-core–EP invertible.

If a is a-core EP-invertible, then, by Theorem 2.7, $a^{||a^{2k+1}}$ exists and $a^{2k+1} \in \mathcal{R}^{(1,3)}$, for some nonnegative integer k. Since $a^{||a^{2k+1}}$ exists, then $a \in R^D$ with $\mathrm{ind}(a) \le 2k+1$. So, by [10, Theorem 2.3], a is core-EP invertible. □

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Consequently, when $w = a$ in Theorem 2.4, Theorem 2.7 and Theorem 2.10, we present a list of characterizations for core–EP invertible element using Theorem 2.12.

Corollary 2.13. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is core–EP invertible;*

(ii) *there exist $a^{||a^{2k+1}}$ and $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k;*

(iii) *there exists an element $x \in \mathcal{R}$ such that*
$$a^2x^2 = x, \qquad xa^{2k+3} = a^{2k+1} \qquad \text{and} \qquad (a^2x)^* = a^2x,$$
for some nonnegative integer k;

(iv) *there exists an element $x \in \mathcal{R}$ such that*
$$a^2x^2 = x,\ xa^{2k+3} = a^{2k+1},\ xa^2x = x,\ a^2xa^{2k+1} = a^{2k+1} \text{ and } (a^2x)^* = a^2x,$$
for some nonnegative integer k;

(v) *there exists an element $x \in \mathcal{R}$ such that*
$$a^2xa^{2k+1} = a^{2k+1}, \quad a^{2k+1}\mathcal{R} = x\mathcal{R} \quad \text{and} \quad \mathcal{R}x = \mathcal{R}(a^{2k+1})^*,$$
for some nonnegative integer k;

(vi) *there exists an element $x \in \mathcal{R}$ such that*
$$a^2xa^{2k+1} = a^{2k+1} \quad \text{and} \quad a^{2k+1}a\mathcal{R} = x\mathcal{R} = x^*\mathcal{R},$$
for some nonnegative integer k;

(vii) *there exists an element $x \in \mathcal{R}$ such that*
$$a^2xa^{2k+1} = a^{2k+1} \quad \text{and} \quad a^{2k+1}\mathcal{R} = x\mathcal{R} \supseteq x^*\mathcal{R},$$
for some nonnegative integer k;

(viii) *there exists an element $x \in \mathcal{R}$ such that*
$$a^2xa^{2k+1} = a^{2k+1}, \quad {}^\circ(a^{2k+1}) = {}^\circ x \quad \text{and} \quad x^\circ = \big((a^{2k+1})^*\big)^\circ,$$
for some nonnegative integer k;

(ix) *there exists an element $x \in \mathcal{R}$ such that*
$$a^2xa^{2k+1} = a^{2k+1}, \quad {}^\circ(a^{2k+1}) = {}^\circ x \quad \text{and} \quad x^\circ \supseteq \big((a^{2k+1})^*\big)^\circ,$$
for some nonnegative integer k;

(x) *$a^{2k+1} \in a^{2k+2}\mathcal{R}$ and there exist $(a^{2k+2})^\#$ and $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k;*

(xi) *$a^{2k+1} \in \mathcal{R}a^{2k+2}$ and there exist $(a^{2k+2})^\#$ and $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k;*

(xii) *$a^{2k+1} \in a^{4k+3}\mathcal{R} \cap \mathcal{R}a^{4k+3}$ and there exists $(a^{2k+1})^{(1,3)} \in a^{2k+1}\{1,3\}$ for some nonnegative integer k.*

In addition, if any of statements (i)–(xii) *holds, then, for some nonnegative integer k and $(a^{2k+1})^{(1,3)} \in (a^{2k+1})\{1,3\}$,*
$$a_a^{\textcircled{\dagger}} = a^{2k}a^{||a^{2k+1}}(a^{2k+1})^{(1,3)} = (a^{2k+1})^\# a^{4k+1}(a^{2k+1})^{(1,3)}.$$

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It is interesting to observe that a is w-core–EP invertible is equivalent to aw is core–EP invertible.

Theorem 2.14. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*
(ii) *aw is core–EP invertible;*
(iii) *there exist $(aw)^D$ and $\left((aw)^k\right)^{(1,3)} \in (aw)^k\{1,3\}$, for $k \geq \mathrm{ind}(aw)$;*
(iv) *there exist $(aw)^D$ and the unique orthogonal projector $p \in \mathcal{R}$ such that $p\mathcal{R} = (aw)^k a\mathcal{R}$, for $k \geq \mathrm{ind}(aw)$.*

In addition, if any of statements (i)–(ii) *holds, then $i_w(a) \leq I(aw) \leq i_w(a)+1$ and, for $\left((aw)^k a\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$,*

$$a_w^{\textcircled{\dagger}} = (aw)^{\textcircled{\dagger}} = (aw)^D p = (aw)^D (aw)^k a \left((aw)^k a\right)^{(1,3)}.$$

Proof. (i) $\Rightarrow$ (ii): It is clear by Theorem 2.2.

(ii) $\Rightarrow$ (i): If x is the core–EP inverse of aw, then $awx^2 = x$, $x(aw)^{k+1} = (aw)^k$ and $(awx)^* = awx$, for some positive integer k. Because $x(aw)^{k+1}a = (aw)^k a$, we conclude that x is the w-core–EP inverse of a.

(ii) $\Leftrightarrow$ (iii): This equivalence follow by [10, Theorem 2.3].

(iii) $\Rightarrow$ (iv): For $k \geq \mathrm{ind}(aw)$ and $\left((aw)^k\right)^{(1,3)} \in (aw)^k\{1,3\}$, we observe that $y = w(aw)^D \left((aw)^k\right)^{(1,3)} \in \left((aw)^k a\right)\{1,3\}$ by

$$(aw)^k ay = (aw)^k aw(aw)^D \left((aw)^k\right)^{(1,3)} = (aw)^k \left((aw)^k\right)^{(1,3)}$$

and

$$(aw)^k ay(aw)^k a = (aw)^k \left((aw)^k\right)^{(1,3)} (aw)^k a = (aw)^k a.$$

Set $p = (aw)^k ay$. Hence, $p = p^* = p^2$ and $p\mathcal{R} = (aw)^k ay\mathcal{R} = (aw)^k a\mathcal{R}$.

To prove the uniqueness of p, let two orthogonal projectors p and p_1 satisfy $p\mathcal{R} = (aw)^k a\mathcal{R} = p_1\mathcal{R}$. Then $p = p_1 p$ and $p_1 = pp_1$ gives $p = p^* = (p_1 p)^* = pp_1 = p_1$.

(iv) $\Rightarrow$ (i): Because there exist $(aw)^D$ and the unique orthogonal projector $p \in \mathcal{R}$ such that $p\mathcal{R} = (aw)^k a\mathcal{R}$, for $k \geq \mathrm{ind}(aw)$, we have $p = (aw)^k au$, for some $u \in \mathcal{R}$, and $(aw)^k a = p(aw)^k a$. Therefore, $(aw)^k a = (aw)^k au(aw)^k a$ and $((aw)^k au)^* = p = (aw)^k au$, that is, $(aw)^k a \in \mathcal{R}^{(1,3)}$. We now observe that $p = (aw)^k au = (aw)^k a \left((aw)^k a\right)^{(1,3)} (aw)^k au = (aw)^k a \left((aw)^k a\right)^{(1,3)} p$, where $\left((aw)^k a\right)^{(1,3)} \in (aw)^k\{1,3\}$. So,

$$p = p^* = p(aw)^k a \left((aw)^k a\right)^{(1,3)} = (aw)^k a \left((aw)^k a\right)^{(1,3)}.$$

Denote by $x = (aw)^D p = (aw)^D (aw)^k a \left((aw)^k a\right)^{(1,3)}$. From

$$awx = (aw(aw)^D (aw)^k) a \left((aw)^k a\right)^{(1,3)} = (aw)^k a \left((aw)^k a\right)^{(1,3)} = p,$$

$$\begin{aligned} awx^2 &= px = \left[(aw)^k a \left((aw)^k a\right)^{(1,3)} (aw)^k a\right] w((aw)^D)^2 a \left((aw)^k a\right)^{(1,3)} \\ &= (aw)^k (aw)^D a \left((aw)^k a\right)^{(1,3)} = x \end{aligned}$$

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and $x(aw)^{k+1}a = (aw)^D p(aw)^{k+1}a = (aw)^D(aw)^{k+1}a = (aw)^k a$, we deduce that x is the w-core–EP inverse of a. □

As a consequence of Theorem 2.14 and [37, Theorem 4.4], we develop one more representation for the w-core–EP inverse.

Corollary 2.15. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible;*
(ii) *$\mathcal{R} = \mathcal{R}(aw)^k \oplus {}^\circ((aw)^k) = \mathcal{R}((aw)^k)^* \oplus {}^\circ((aw)^k)$, for some positive integer k;*
(iii) *$\mathcal{R} = (aw)^k\mathcal{R} \oplus ((aw)^k)^\circ = \mathcal{R}((aw)^k)^* \oplus {}^\circ((aw)^k)$, for some positive integer k.*

In addition, if any of statements (i)–(iii) holds, then $a_w^{\textcircled{D}} = (aw)^{2k-1}b^2a^ks^*$, where $b, s \in \mathcal{R}$, $c \in ((aw)^k)^\circ$ and $t \in {}^\circ((aw)^k)$ such that $(aw)^kb + c = s((aw)^k)^* + t = 1$.

Proof. (i) ⇔ (ii) ⇔ (iii): These equivalences follow by Theorem 2.14 and [37, Theorem 4.4]. □

Under the assumption $(aw)^k a \in \mathcal{R}^\dagger$, we prove that the w-core–EP inverse of a is equal to the inverse of aw along $(aw)^k a((aw)^k a)^*$.

Theorem 2.16. *Let $a, w \in \mathcal{R}$ such that $(aw)^k a \in \mathcal{R}^\dagger$, for some nonnegative integer k. Then the following statements are equivalent:*

(i) *a is w-core–EP invertible with $i_w(a) = k$;*
(ii) *aw is invertible along $(aw)^k a((aw)^k a)^*$.*

In addition, if any of statements (i)–(ii) holds, then $a_w^{\textcircled{D}} = (aw)^{||(aw)^k a((aw)^k a)^*}$.

Proof. (i) ⇒ (ii): For $d = (aw)^k a((aw)^k a)^*$ and $x = a_w^{\textcircled{D}}$, we have

$$xawd = x(aw)^{k+1}a((aw)^k a)^* = (aw)^k a((aw)^k a)^* = d$$

and

$$dawx = (awxd)^* = (awx(aw)^k a((aw)^k a)^*)^* = ((aw)^k a((aw)^k a)^*)^* = d^* = d.$$

Applying Theorem 2.4 and the hypothesis $(aw)^k a \in \mathcal{R}^\dagger$, it is clear that $x \in (aw)^k a\mathcal{R} \cap \mathcal{R}\left((aw)^k a\right)^* = (aw)^k a\left((aw)^k a\right)^* \mathcal{R} \cap \mathcal{R}(aw)^k a\left((aw)^k a\right)^* = d\mathcal{R} \cap \mathcal{R}d$. So, we deduce that $x = (aw)^{||(aw)^k a((aw)^k a)^*}$.

(ii) ⇒ (i): Let $x = (aw)^{||(aw)^k a((aw)^k a)^*}$ and $d = (aw)^k a((aw)^k a)^*$. Then $x\mathcal{R} = d\mathcal{R} = (aw)^k a\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}d = \mathcal{R}((aw)^k a)^*$. We observe that $(aw)^k a((aw)^k a)^* = d = dawx = (aw)^k a((aw)^k a)^* awx$ and $x = du = (aw)^k a((aw)^k a)^* u$, for some $u \in \mathcal{R}$, which imply

$$\begin{aligned} awx &= (aw)^{k+1}a((aw)^k a)^* u = (aw)^k a((aw)^k a)^\dagger((aw)^{k+1}a((aw)^k a)^* u) \\ &= (aw)^k a((aw)^k a)^\dagger awx = [((aw)^k a)^\dagger]^*((aw)^k a)^\dagger((aw)^k a((aw)^k a)^* awx) \\ &= [((aw)^k a)^\dagger]^*((aw)^k a)^\dagger(aw)^k a((aw)^k a)^* = (aw)^k a((aw)^k a)^\dagger. \end{aligned}$$

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Thus, $(awx)^* = awx$. Since

$$\begin{aligned}((aw)^k a)^* &= ((aw)^k a)^\dagger((aw)^k a((aw)^k a)^*) = ((aw)^k a)^\dagger (aw)^k a((aw)^k a)^* awx \\ &= ((aw)^k a)^* awx,\end{aligned}$$

we get $(aw)^k a = awx(aw)^k a$. By Theorem 2.4, we conclude that $x = a_w^{\textcircled{\dagger}}$. □

We also verify that a is w-core–EP invertible implies that $awa_w^{\textcircled{\dagger}}a$ is w-core invertible.

Theorem 2.17. *Let $a, w \in \mathcal{R}$. If a is w-core–EP invertible, then $awa_w^{\textcircled{\dagger}}a$ is w-core invertible and*

$$(awa_w^{\textcircled{\dagger}}a)_w^{\textcircled{\#}} = a_w^{\textcircled{\dagger}}.$$

Proof. Suppose that a is w-core–EP invertible and $a' = awa_w^{\textcircled{\dagger}}a$. Then $aw(a_w^{\textcircled{\dagger}})^2 = a_w^{\textcircled{\dagger}}$, $a_w^{\textcircled{\dagger}}(aw)^{k+1}a = (aw)^k a$ and $(awa_w^{\textcircled{\dagger}})^* = awa_w^{\textcircled{\dagger}}$, for some nonnegative integer k. Now, $a'wa_w^{\textcircled{\dagger}} = aw(a_w^{\textcircled{\dagger}}awa_w^{\textcircled{\dagger}}) = awa_w^{\textcircled{\dagger}}$, which yields $(a'wa_w^{\textcircled{\dagger}})^* = (awa_w^{\textcircled{\dagger}})^* = awa_w^{\textcircled{\dagger}} = a'wa_w^{\textcircled{\dagger}}$ and $a'w(a_w^{\textcircled{\dagger}})^2 = aw(a_w^{\textcircled{\dagger}})^2 = a_w^{\textcircled{\dagger}}$. Furthermore, by

$$\begin{aligned}a_w^{\textcircled{\dagger}}a'wa' &= (a_w^{\textcircled{\dagger}}awa_w^{\textcircled{\dagger}})aw(awa_w^{\textcircled{\dagger}})a = a_w^{\textcircled{\dagger}}aw(aw)^{k+1}(a_w^{\textcircled{\dagger}})^{k+1}a \\ &= (a_w^{\textcircled{\dagger}}(aw)^{k+1}a)w(a_w^{\textcircled{\dagger}})^{k+1}a = (aw)^k aw(a_w^{\textcircled{\dagger}})^{k+1}a \\ &= awa_w^{\textcircled{\dagger}}a = a',\end{aligned}$$

we deduce that $(awa_w^{\textcircled{\dagger}}a)_w^{\textcircled{\#}} = a_w^{\textcircled{\dagger}}$. □

3. The dual w-core–EP inverse

This section is dedicated to investigation of the dual w-core–EP inverse.

Definition 3.1. Let $a, w \in \mathcal{R}$. Then a is called dual w-core–EP invertible if there exists an element $x \in \mathcal{R}$ such that

$$x^2wa = x, \qquad (aw)^{k+1}ax = (aw)^k a \qquad \text{and} \qquad (xwa)^* = xwa,$$

for some nonnegative integer k. In this case, x is a dual w-core–EP inverse of a.

When $k = 0$ in the above definition, the dual w-core–EP inverse coincides to the dual w-core inverse. Also, the dual 1-core-EP inverse is the dual core–EP inverse, i.e. dual core–EP invertible elements are w-core-EP invertible. The smallest nonnegative integer k in the definition of the dual w-core–EP inverse, is called the dual w-core–EP index of a and denoted by $i'_w(a)$.

As Theorem 2.2, we can check the following result.

Theorem 3.2. *Let $a, w \in \mathcal{R}$. Then a has at most one dual w-core-EP inverse.*

Thus, if the dual w-core-EP inverse of a exists, it is unique and denoted by $a_{\textcircled{\dagger},w}$.

Lemma 3.3. *Let $a, w \in \mathcal{R}$. Then a is dual w-core–EP invertible if and only if a^* is w^*-core–EP invertible. In addition, $(a_{\textcircled{\dagger},w})^* = (a^*)_{w^*}^{\textcircled{\dagger}}$ and $i'_w(a) = i_{w^*}(a^*)$.*

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Proof. Note that x is the dual w-core-EP inverse of a if and only if $x^2wa = x$, $(aw)^{k+1}ax = (aw)^k a$ and $(xwa)^* = xwa$, for some nonnegative integer k, which is equivalent to $a^*w^*(x^*)^2 = x^*$, $x^*(a^*w^*)^{k+1}a^* = (a^*w^*)^k a^*$ and $(a^*w^*x^*)^* = a^*w^*x^*$, for some nonnegative integer k, that is, x^* is the w^*-core–EP inverse of a^*. ☐

Remark that, for $w = 1$, Lemma 3.3 recovers the well-known fact that a is dual core–EP invertible if and only if a^* is core–EP invertible [10]. In this case, $(a_{\textcircled{\scriptsize D}})^* = (a^*)^{\textcircled{\scriptsize D}}$.

Using Theorem 2.4 and Lemma 3.3, we can present the next characterizations of dual w-core–EP invertible elements.

Theorem 3.4. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is dual w-core–EP invertible;*

(ii) *there exists an element $x \in \mathcal{R}$ such that*

$$x^2wa = x, \quad (aw)^{k+1}ax = (aw)^k a, \quad xwax = x,$$

$$(aw)^k axwa = (aw)^k a \quad \text{and} \quad (xwa)^* = xwa,$$

for some nonnegative integer k;

(iii) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^k axwa = (aw)^k a, \quad \mathcal{R}(aw)^k a = \mathcal{R}x \quad \text{and} \quad x\mathcal{R} = \left((aw)^k a\right)^* \mathcal{R},$$

for some nonnegative integer k;

(iv) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^k axwa = (aw)^k a \quad \text{and} \quad \left((aw)^k a\right)^* \mathcal{R} = x\mathcal{R} = x^*\mathcal{R},$$

for some nonnegative integer k;

(v) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^k axwa = (aw)^k a \quad \text{and} \quad \left((aw)^k a\right)^* \mathcal{R} = x^*\mathcal{R} \supseteq x\mathcal{R},$$

for some nonnegative integer k;

(vi) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^k axwa = (aw)^k a, \quad \left((aw)^k a\right)^\circ = x^\circ \quad \text{and} \quad {}^\circ x = {}^\circ\left(((aw)^k a)^*\right),$$

for some nonnegative integer k;

(vii) *there exists an element $x \in \mathcal{R}$ such that*

$$(aw)^k axwa = (aw)^k a, \quad \left((aw)^k a\right)^\circ = x^\circ \quad \text{and} \quad {}^\circ x \supseteq {}^\circ\left(((aw)^k a)^*\right),$$

for some nonnegative integer k;

(viii) *there exists an element $x \in \mathcal{R}$ such that*

$$x^2wa = x, \ (aw)^{k+1}ax = (aw)^k a, \ xwa = x^n(wa)^n \text{ and } (xwa)^* = xwa,$$

for some nonnegative integer k and all/some positive integer n.

Consequently, we have the following result concerning dual core–EP invertible elements.

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Corollary 3.5. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is dual core–EP invertible;*

(ii) *there exists an element $x \in \mathcal{R}$ such that*
$$x^2a = x,\ a^{k+2}x = a^{k+1},\ xax = x,\ a^{k+1}xa = a^{k+1} \text{ and } (xa)^* = xa,$$
for some nonnegative integer k;

(iii) *there exists an element $x \in \mathcal{R}$ such that*
$$a^{k+1}xa = a^{k+1}, \quad \mathcal{R}a^{k+1} = \mathcal{R}x \quad \text{and} \quad x\mathcal{R} = (a^{k+1})^*\mathcal{R},$$
for some nonnegative integer k;

(iv) *there exists an element $x \in \mathcal{R}$ such that*
$$a^{k+1}xa = a^{k+1} \quad \text{and} \quad (a^{k+1})^*\mathcal{R} = x\mathcal{R} = x^*\mathcal{R},$$
for some nonnegative integer k;

(v) *there exists an element $x \in \mathcal{R}$ such that*
$$a^{k+1}xa = a^{k+1} \quad \text{and} \quad (a^{k+1})^*\mathcal{R} = x^*\mathcal{R} \supseteq x\mathcal{R},$$
for some nonnegative integer k;

(vi) *there exists an element $x \in \mathcal{R}$ such that*
$$a^{k+1}xa = a^{k+1}, \quad (a^{k+1})^\circ = x^\circ \quad \text{and} \quad {}^\circ x = {}^\circ\left((a^{k+1})^*\right),$$
for some nonnegative integer k;

(vii) *there exists an element $x \in \mathcal{R}$ such that*
$$a^{k+1}xa = a^{k+1}, \quad (a^{k+1})^\circ = x^\circ \quad \text{and} \quad {}^\circ x \supseteq {}^\circ\left((a^{k+1})^*\right),$$
for some nonnegative integer k;

(viii) *there exists an element $x \in \mathcal{R}$ such that*
$$x^2a = x, \quad a^{k+2}x = a^{k+1} \quad \text{and} \quad (xa)^* = xa = x^na^n,$$
for some nonnegative integer k and all/some positive integer n.

Based on $w^{||(aw)^ka}$ and $\left((aw)^ka\right)^{(1,4)}$, we give an expression for the w-core–EP inverse of a.

Theorem 3.6. *Let $a, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is dual w-core–EP invertible;*

(ii) *there exist $w^{||(aw)^ka}$ and $\left((aw)^ka\right)^{(1,4)} \in \left((aw)^ka\right)\{1,4\}$, for some nonnegative integer k;*

(iii) *there exist $w^{||(aw)^ka}$ and $\left((aw)^{k+1}\right)^{(1,4)} \in \left((aw)^{k+1}\right)\{1,4\}$, for some nonnegative integer k;*

(iv) *there exist $w^{||(aw)^ka}$ and $\left((aw)^{k+1}a\right)^{(1,4)} \in \left((aw)^{k+1}a\right)\{1,4\}$, for some nonnegative integer k;*

(v) *$(aw)^ka \in (aw)^{k+1}\mathcal{R}$ and there exist $\left((aw)^{k+1}\right)^\#$ and $\left((aw)^ka\right)^{(1,4)} \in \left((aw)^ka\right)\{1,4\}$, for some nonnegative integer k;*

(vi) *$(aw)^ka \in \mathcal{R}(wa)^{k+1}$ and there exist $\left((wa)^{k+1}\right)^\#$ and $\left((aw)^ka\right)^{(1,4)} \in \left((aw)^ka\right)\{1,4\}$, for some nonnegative integer k;*

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(vii) $(aw)^k a \in (aw)^{2k+1} a\mathcal{R} \cap \mathcal{R}(aw)^{2k+1} a$ *and there exists* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$, *for some nonnegative integer* k.

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer* k *and* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$,

$$\begin{aligned} a_{\textcircled{D},w} &= \left((aw)^k a\right)^{(1,4)} w^{||(aw)^k a}(wa)^k = \left((aw)^k a\right)^{(1,4)} (aw)^{2k} a \left((wa)^{k+1}\right)^{\#} \\ &= \left((aw)^k a\right)^{(1,4)} \left((aw)^{k+1}\right)^{\#} (aw)^{2k} a. \end{aligned}$$

Now, we get new representations for the dual core–EP inverse.

Corollary 3.7. *Let* $a \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) *a is dual core–EP invertible;*
(ii) *there exist* $1^{||a^{k+1}}$ *and* $(a^{k+1})^{(1,4)} \in (a^{k+1})\{1,4\}$ *for some nonnegative integer* k
(iii) *there exist* $1^{||a^{k+1}}$ *and* $\left(a^{k+2}\right)^{(1,4)} \in \left(a^{k+2}\right)\{1,4\}$, *for some nonnegative integer* k;
(iv) $a^{k+1} \in \mathcal{R}^{\#} \cap \mathcal{R}^{(1,4)}$, *for some nonnegative integer* k.

In addition, if any of statements (i)–(ii) *holds, then, for some nonnegative integer* k *and* $(a^{k+1})^{(1,4)} \in (a^{k+1})\{1,4\}$,

$$a_{\textcircled{D}} = (a^{k+1})^{(1,4)} 1^{||a^{k+1}} a^k = (a^{k+1})^{(1,4)} a^{2k+1} (a^{k+1})^{\#} = (a^{k+1})_{\textcircled{\#}} a^k.$$

Theorem 2.7 and Theorem 3.6 imply the following result.

Corollary 3.8. *Let* $a, w \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) *a is both w-core–EP invertible and dual w-core–EP invertible;*
(ii) *there exist* $w^{||(aw)^k a}$ *and* $\left((aw)^k a\right)^{\dagger}$, *for some nonnegative integer* k;
(iii) *there exist* $w^{||(aw)^k a}$ *and* $\left((aw)^{k+1}\right)^{\dagger}$, *for some nonnegative integer* k;
(iv) *there exist* $w^{||(aw)^k a}$ *and* $\left((aw)^{k+1} a\right)^{\dagger}$, *for some nonnegative integer* k.

Clearly, we have the next relation between dual w-core–EP invertibility of a and core–EP invertibility of wa.

Theorem 3.9. *Let* $a, w \in \mathcal{R}$. *Then the following statements are equivalent:*

(i) *a is dual w-core–EP invertible;*
(ii) *wa is dual core–EP invertible;*
(iii) *there exist* $(wa)^D$ *and* $\left((aw)^k\right)^{(1,4)} \in (aw)^k\{1,4\}$, *for* $k \geq \operatorname{ind}(wa)$;
(iv) *there exist* $(wa)^D$ *and the unique orthogonal projector* $p \in \mathcal{R}$ *such that* $p\mathcal{R} = \left((aw)^k a\right)^* \mathcal{R}$, *for* $k \geq \operatorname{ind}(aw)$.

In addition, if any of statements (i)–(ii) *holds, then, for* $\left((aw)^k a\right)^{(1,4)} \in \left((aw)^k a\right)\{1,4\}$,

$$a_{\textcircled{D},w} = (wa)_{\textcircled{D}} = p(wa)^D = \left((aw)^k a\right)^{(1,4)} (aw)^k a (wa)^D.$$

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Theorem 3.10. *Let $a, w \in \mathcal{R}$ such that $(aw)^k a \in \mathcal{R}^\dagger$, for some nonnegative integer k. Then the following statements are equivalent:*

(i) *a is dual w-core–EP invertible with $i'_w(a) = k$;*
(ii) *wa is invertible along $((aw)^k a)^*(aw)^k a$.*

In addition, if any of statements (i)–(ii) *holds, then $a_{\textcircled{\scriptsize D},w} = (wa)^{||((aw)^k a)^*(aw)^k a}$.*

Remark that the dual w-core–EP invertibility of a gives dual w-core invertibility of an adequate element.

Theorem 3.11. *Let $a, w \in \mathcal{R}$. If a is dual w-core–EP invertible, then $aa_{\textcircled{\scriptsize D},w}wa$ is dual w-core invertible and*

$$(aa_{\textcircled{\scriptsize D},w}wa)_{\textcircled{\scriptsize \#},w} = a_{\textcircled{\scriptsize D},w}.$$

We also consider characterizations of dual a^*-core–EP invertibility. Recall that, by [34, Theorem 3.12], $a \in \mathcal{R}$ is Moore–Penrose invertible if and only if $a \in aa^*a\mathcal{R}$ if and only if $a \in \mathcal{R}aa^*a$.

Theorem 3.12. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is dual a^*-core–EP invertible;*
(ii) *$(aa^*)^k a$ is Moore–Penrose invertible, for some nonnegative integer k;*
(iii) *a is a^*-core–EP invertible.*

Proof. (i) $\Rightarrow$ (ii): Since a is dual a^*-core–EP invertible, by Theorem 3.6, $(a^*)^{||(aa^*)^k a}$ exists for some nonnegative integer k. So,

$$(aa^*)^k a \in (aa^*)^k aa^*(aa^*)^k a\mathcal{R} = (aa^*)^{2k+1}a\mathcal{R}$$

which gives $(aa^*)^k a \in (aa^*)^{k+1}(aa^*)^k a\mathcal{R} \subseteq (aa^*)^{k+1}(aa^*)^{2k+1}a\mathcal{R} = (aa^*)^{3k+2}a\mathcal{R}$. According to [34, Theorem 3.12], we deduce that $(aa^*)^k a$ is Moore–Penrose invertible.

(ii) $\Rightarrow$ (iii): If $(aa^*)^k a$ is Moore–Penrose invertible, by [34, Theorem 3.12], $(aa^*)^k a \in (aa^*)^{3k+1}a\mathcal{R} \cap \mathcal{R}(aa^*)^{3k+1}a \subseteq (aa^*)^{2k+1}a\mathcal{R} \cap \mathcal{R}(aa^*)^{2k+1}a$. Thus, $(a^*)^{||(aa^*)^k a}$ exists and, by Theorem 2.7, a is a^*-core–EP invertible.

(iii) $\Rightarrow$ (i): The hypothesis a is a^*-core–EP invertible and Theorem 2.7 implies that $(aa^*)^k a$ is Moore–Penrose invertible as in the part (i) $\Rightarrow$ (ii). Using Theorem 3.6, we conclude that a is dual a^*-core–EP invertible. □

4. Applications of dual w-core–EP inverse

We can investigate solvability of some equations applying the dual w-core–EP inverse. Precisely, we solve some operator equations using the following notations in this section. Let $\mathcal{B}(X, Y)$ be the set of all bounded linear operators from X to Y, where X and Y are arbitrary Hilbert spaces. Especially, $\mathcal{B}(X, X) = \mathcal{B}(X)$. For $W \in \mathcal{B}(Y, X)$ and $A \in \mathcal{B}(X, Y)$, according to [23], remark that Drazin invertibility of WA (or equivalently W-weighted Drazin invertibility of A) implies the existence of $A_{\textcircled{\scriptsize D},W} \in \mathcal{B}(X)$. Notice that, for complex rectangular matrices A and W of appropriated sizes, $A_{\textcircled{\scriptsize D},W}$ always exists.

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Theorem 4.1. *Let $W \in \mathcal{B}(Y,X)$ and $A \in \mathcal{B}(X,Y)$ such that WA is Drazin invertible and $i'_W(A)=k$. For $b \in X$, the equation*

$$(AW)^{k+1}Ax = (AW)^k Ab \tag{4.1}$$

is consistent and its general solution is

$$x = A_{\textcircled{\scriptsize D},W}b + (I - A_{\textcircled{\scriptsize D},W}WA)y, \tag{4.2}$$

for arbitrary $y \in X$.

Proof. Assume that x has the form (4.2). Then

$$(AW)^{k+1}Ax = (AW)^{k+1}AA_{\textcircled{\scriptsize D},W}b + (AW)^{k+1}A(I - A_{\textcircled{\scriptsize D},W}WA)y = (AW)^k Ab,$$

which yields that x is a solution to (4.1).

If x is a solution to (4.1), by the properties of the dual w-core–EP inverse $A_{\textcircled{\scriptsize D},W}$, we obtain

$$\begin{aligned} A_{\textcircled{\scriptsize D},W}b &= A^2_{\textcircled{\scriptsize D},W}WAb = A^{k+2}_{\textcircled{\scriptsize D},W}(WA)^{k+1}b = A^{k+2}_{\textcircled{\scriptsize D},W}W((AW)^k Ab) \\ &= A^{k+2}_{\textcircled{\scriptsize D},W}W(AW)^{k+1}Ax = A^{k+2}_{\textcircled{\scriptsize D},W}(WA)^{k+2}x \\ &= A_{\textcircled{\scriptsize D},W}WAx. \end{aligned}$$

Therefore,

$$x = A_{\textcircled{\scriptsize D},W}b + x - A_{\textcircled{\scriptsize D},W}WAx = A_{\textcircled{\scriptsize D},W}b + (I - A_{\textcircled{\scriptsize D},W}WA)x,$$

i.e. x has the form (4.2). □

In the case that $A_{\textcircled{\scriptsize \#},W}$ exists, we obtain the next result as a particular case of Theorem 4.1 for $k=0$.

Corollary 4.2. *Let $W \in \mathcal{B}(Y,X)$ and $A \in \mathcal{B}(X,Y)$ such that $A_{\textcircled{\scriptsize \#},W}$ exists. For $b \in X$, the equation*

$$AWAx = Ab$$

is consistent and its general solution is

$$x = A_{\textcircled{\scriptsize \#},W}b + (I - A_{\textcircled{\scriptsize \#},W}WA)y,$$

for arbitrary $y \in X$.

When $X = Y$ and $W = I$ in Theorem 4.1 and Corollary 4.2, we get solvability of the following equations in terms of the dual core–EP inverse and dual core inverse.

Corollary 4.3. *Let $W \in \mathcal{B}(Y,X)$ and $A \in \mathcal{B}(X,Y)$ such that WA is Drazin invertible and $i'_W(A)=k$, and let $b \in X$.*

(i) *The equation*

$$A^{k+2}x = A^{k+1}b$$

is consistent and its general solution is

$$x = A_{\textcircled{\scriptsize D}}b + (I - A_{\textcircled{\scriptsize D}}A)y,$$

for arbitrary $y \in X$.

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(ii) *If $A_{\oplus}$ exists, the equation*

$$A^2x = Ab$$

is consistent and its general solution is

$$x = A_{\oplus}b + (I - A_{\oplus}A)y,$$

for arbitrary $y \in X$.

For $W = A^*$ in Theorem 4.1 and Corollary 4.2, we can solve equations $(AA^*)^{k+1}Ax = (AA^*)^kAb$ and $AA^*Ax = Ab$ as special cases.

Corollary 4.4. *Let $A \in \mathcal{B}(X, Y)$ such that A^*A is Drazin invertible and $i'_{A^*}(A) = k$, and let $b \in X$.*

(i) *The equation*

$$(AA^*)^{k+1}Ax = (AA^*)^kAb$$

is consistent and its general solution is

$$x = A_{\textcircled{D},A^*}b + (I - A_{\textcircled{D},A^*}A^*A)y,$$

for arbitrary $y \in X$.

(ii) *If $A_{\oplus,A^*}$ exists and $b \in X$, the equation*

$$AA^*Ax = Ab$$

is consistent and its general solution is

$$x = A_{\oplus,A^*}b + (I - A_{\oplus,A^*}A^*A)y,$$

for arbitrary $y \in X$.

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