Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

COMPLETE PRESENTATION AND HILBERT SERIES OF THE MIXED BRAID MONOID $MB_{1,3}$

ZAFFAR IQBAL[1], MUHAMMAD MOBEEN MUNIR[2*], MALEEHA AYUB[1], ABDUL RAUF NIZAMI[3]

Abstract. Hilbert series is the simplest way of finding dimension and degree of an algebraic variety defined explicitly by polynomial equations. The mixed braid groups were introduced by Sofia Lambropoulou in 2000. In this paper we compute the complete presentation and the Hilbert series of the canonical words of the mixed braid monoid $MB_{1,3}$.

Keywords: braid group, mixed braid monoid, canonical words, complete presentation, Hilbert series.
AMS Mathematics Subject Classification (2010): 20F36, 20F05, 13D40.

1. Introduction

The braid group B_{n+1} for the Euclidean space consisting on $n+1$ strands is given by the following Artin's presentation [1]:

$$B_{n+1} = \left\langle z_1, z_2, \ldots, z_n \;\middle|\; \begin{array}{l} z_i z_j = z_j\, z_i \text{ if } \mid i-j \mid \geq 2 \\ z_{i+1}\, z_i\, z_{i+1} = z_i\, z_{i+1}\, z_i \text{ if } 1 \leq i \leq n-1 \end{array} \right\rangle .$$

Elements of B_{n+1} are expressed in the generators $z_1, z_2, \ldots, z_n$ and their inverses. The presentation of braid monoid MB_{n+1} is similar to the presentation of B_{n+1}. In [17] Lambropoulou gave the presentation of the mixed braid monoid $B_{m,n}$. Before this presentation she gave the presentation of $B_{1,n}$ in [16]. In this paper we compute the Hilbert series of $B_{1,3}$.

Definition 1. [17] *The mixed braid group $B_{m,n}$ of $m+n$ strands is defined as:*

$$B_{m,n} = \left\langle \begin{array}{c} \alpha_1, \ldots, \alpha_m, \\ \beta_1, \ldots, \beta_{n-1} \end{array} \;\middle|\; \begin{array}{l} \beta_r \beta_s = \beta_s\, \beta_r \;\; if |r-s| \geq 2 \\ \beta_{r+1}\, \beta_r\, \beta_{r+1} = \beta_r\, \beta_{r+1}\, \beta_r \; if\, 1 \leq r \leq n-1 \\ \alpha_p\, \beta_s = \beta_s\, \alpha_p \; if \; s \geq 2, \;\; 1 \leq p \leq m \\ \alpha_p\, (\beta_1\, \alpha_q\, \beta_1^{-1}) = (\beta_1\, \alpha_q\, \beta_1^{-1})\, \alpha_p \; if \; q < p \end{array} \right\rangle .$$

[1] Department of Mathematics, University of Gujrat-Pakistan, email: zaffar.iqbal@uog.edu.pk, maleeha.ayub003@gmail.com,
[2*] Department of Mathematics, University of the Punjab Lahore-Pakistan, email: mmunir.math@pu.edu.pk,
[3] Faculty of Information Technology, University of Central Punjab, Lahore-Pakistan, email: arnizami@ucp.edu.pk.

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

In the mixed braid group $B_{m,n}$, the first index m denotes the strings which make the identity braid of m strings and the next n strings shows the braiding by itself and with m strings. The mixed braid group $B_{m,n}$ is a subgroup of the Artin braid group B_{m+n}. The associated Dynkin diagram for $B_{m,n}$ is given in [17]:

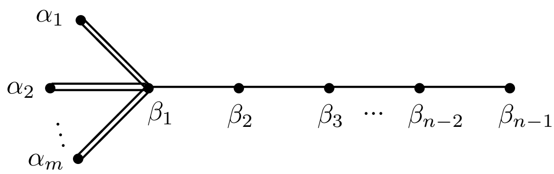

Figure 1

In the above diagram, the double lines represent the relation of length 4, while relation of length 3 is represented by the single line. However, if the there is no line among the generators, then they commute. Hence the Dynkin diagram for $MB_{1,2}$ reduces to:

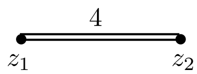

Figure 2

Therefore we have

$$MB_{1,2}=\left\langle z_1, z_2 \,\middle|\, z_2\, z_1\, z_2\, z_1 = z_1\, z_2\, z_1\, z_2 \right\rangle.$$

The complete structure and Hilbert series for $MB_{1,2}$ are computed in [7]. This motivated us to compute the Hilbert series of $MB_{1,3}$, where the Dynkin diagram for $MB_{1,3}$ is given in Figure 3.

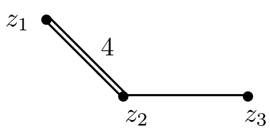

Figure 3

Therefore we have the following presentation of $MB_{1,3}$:

$$MB_{1,3} = \left\langle z_1, z_2, z_3 \,\middle|\, z_3\, z_2\, z_3 = z_2\, z_3\, z_2,\; z_2\, z_1\, z_2\, z_1 = z_1\, z_2\, z_1\, z_2,\; z_3\, z_1 = z_1\, z_3 \right\rangle.$$

In this case we have three Artin's relations namely, $R_0 : z_3\, z_1 = z_1\, z_3$, $R_1 : z_2\, z_1\, z_2\, z_1 = z_1\, z_2\, z_1\, z_2$ and $R_2 : z_3\, z_2\, z_3 = z_2\, z_3\, z_2$. The following is an example of a braid in $B_{1,3}$.

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

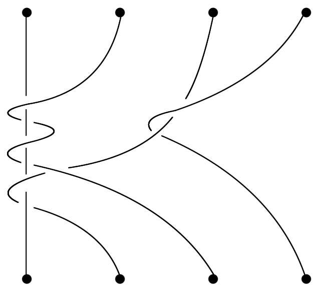

Figure 4

In [8], Zafar et al. constructed a linear system for the braid monoid MB_{n+1} and computed the Hilbert series for the braid monoids MB_3 and MB_4. The growth series of binomial edge ideals was computed by Kumar and Sarkar in [14]. In [8], growth series of the graded algebra of real regular functions on the symplectic quotient associated to an SU2-module has been given.

In [10], authors computed the Hilbert series of braid monoid MB_4 in band generators. In [12], authors constructed a linear system of canonical words of finite dimensional generalized Hecke algebras $H(Q_m, 3)$, where $Q_m = x^m - 1, m \in \{3, 4, 5\}$ and computed its Hilbert series. In [18] Saito computed the growth series of Artin monoids. In [15] Mairesse and Mathéus gave the growth series of Artin groups of dihedral type. In [11] we computed the Hilbert series of the Artin monoids $M(I_2(p))$, where $M(I_2(4))$ is isomorphic to $MB_{1,2}$ and $MB_{1,2}$ is isomorphic to the Artin monoid of type B_2. In this paper we construct a similar kind of linear system to compute the Hilbert series of $MB_{1,3}$ which is isomorphic to Artin monoid of type B_3.

2. Complete Presentation of $MB_{1,3}$

To get a canonical form of a word in an algebra the diamond lemma by G. Bergman [2] is extremely useful. To understand the notions of ambiguities and canonical words, we start with his terminology.

Definition 2. [2] *Let $\alpha_1 = ut$ and $\alpha_2 = tv$ be two words consisting of the left hand sides of two relations R_i and R_j in $MB_{1,3}$ then the word of the form utv is said to be an ambiguity and we denote it by $R_i - R_j$.*

A word containing a sub-word of left side of any relation of braid monoid is called a *reducible word* and a word that does not contain any sub-word of left side of any relation is called an *irreducible (or canonical) word*.

Definition 3. [5] *Let G be a finitely generated group and S be a finite set of generators of G. The word length $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.*

The diamond lemma [2] says that a set of relations is complete if all the ambiguities are solved. We call a complete set of relations in $MB_{1,3}$ a *complete presentation*

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

of $MB_{1,3}$. The other names for the complete presentation are being used as Gröbner bases, presentation with solvable ambiguities and rewriting system etc. We find the system of linear equations of the canonical words of $MB_{1,3}$ and solve this system which consequently leads to the Hilbert series of $MB_{1,3}$.

In a relation in $MB_{1,3}$ we place the equivalent words on left hand side which are greater in length-lexicographic ordering [9] (we choose a natural total order $z_1 < z_2 < \cdots < z_n$ between the generators). For example the words $z_2z_1z_2z_1$ and $z_1z_2z_1z_2$ are equivalent in the mixed braid monoid $MB_{1,3}$. Hence we write $z_2z_1z_2z_1 = z_1z_2z_1z_2$ as the basic braid relation. We use the notation $R_j^{(4)}$ to express j^{th} generalized relation in $MB_{1,3}$. The words $Xz_2 \times_2 z_2Y$ and $Xz_2z_1 \times_{21} z_2z_1Y$ denote the products Xz_2Y and Xz_2z_1Y, respectively.

The ambiguity utv has two resolutions, namely $(ut)v$ and $u(tv)$. Let $w = utv$. Then by $L(w)$ we mean the canonical form of $(ut)v$ and by $R(w)$ we mean the canonical form of $u(tv)$. If $L(w)$ and $R(w)$ are identical then the ambiguity is solvable. If $L(w)$ and $R(w)$ differ by lexicographic order then we get a new relation in $MB_{1,3}$.

Theorem 1. *The complete presentation of $MB_{1,3}$ is given by*

$\langle z_1, z_2, z_3\,|z_3\,z_1 = z_1\,z_3, z_3\,z_2\,z_3 = z_2\,z_3\,z_2,\ z_2\,z_1\,z_2\,z_1 = z_1\,z_2\,z_1\,z_2,\ R_1^{(4)}, \ldots, R_{11}^{(4)}\rangle$,

where

1. $R_1^{(4)}: z_2z_1^{n+1}z_2z_1z_2 = z_1z_2z_1z_2^2z_1^n$
2. $R_2^{(4)}: z_3z_2z_1^nz_3 = z_2z_3z_2z_1^n$
3. $R_3^{(4)}: z_3z_2^nz_3z_2 = z_2z_3z_2^2z_3^n$
4. $R_4^{(4)}: z_3z_2z_1^nz_2^{n_1}z_3z_2 = z_2z_3z_2z_1^nz_2z_3^{n_1}$
5. $R_5^{(4)}: z_3z_2z_1z_2^nz_3z_2 = z_2z_3z_2z_1z_2z_3^n$
6. $R_6^{(4)}: z_3z_2^nz_1^{n_1}z_3z_2z_1z_2 = z_2z_3z_2^2z_1z_3^nz_2z_1^{n_1}$
7. $R_7^{(4)}: z_3z_2z_1^nz_2^{n_1}z_1^{n_2}z_3z_2z_1z_2 = z_2z_3z_2z_1^nz_2z_1z_3^{n_1}z_2z_1^{n_2}$
8. $R_8^{(4)}: z_3z_2z_1z_2^nz_1^{n_1}z_3z_2z_1z_2 = z_2z_1z_3z_2z_1z_2z_3^nz_2z_1^{n_1}$
9. $R_9^{(4)}: z_3(z_2^nz_1^{n_1}z_2^{n_2}z_1^{n_3}\cdots)z_3z_2z_1z_2z_3 = z_2z_3z_2^2z_1z_3^nz_2z_1^{n_1}z_3(z_2^{n_2}z_1^{n_3}\cdots)$
10. $R_{10}^{(4)}: z_3z_2(z_1^nz_2^{n_1}z_1^{n_2}z_2^{n_3}z_1^{n_4}\cdots)z_3z_2z_1z_2z_3 = z_2z_3z_2z_1^nz_2z_1z_3^{n_1}z_2z_1^{n_2}z_3(z_2^{n_3}z_1^{n_4}\cdots)$
11. $R_{11}^{(4)}: z_3z_2z_1(z_2^nz_1^{n_1}z_2^{n_2}\cdots)z_3z_2z_1z_2z_3 = z_2z_1z_3z_2z_1z_2z_3^nz_2z_1^{n_1}z_3(z_2^{n_2}z_1^{n_3}\cdots)$,

where $n, n_1, n_2, n_3, \cdots \in \mathbb{N}$.

Proof. In this proof we use the inductive argument. We compute the relations by solving the ambiguities involving the relations R_0, R_1 and R_2 and the new relations.

1. In [11] we computed the first relation (for $p = 4$) $R_1^{(4)}$, which is given by

$$R_1^{(4)}:\ z_2z_1^{n+1}z_2z_1z_2 = z_1z_2z_1z_2^2z_1^n.$$

2. For an ambiguity $R_2 - R_0 = z_3z_2z_3z_1 = w_1(say)$, we have

$$R(w_1) = z_3z_2\underline{z_3z_1} = z_3z_2z_1z_3,\ L(w_1) = \underline{z_3z_2z_3}z_1 = z_2z_3z_2z_1.$$

Hence we have a relation $R_{w_1} : z_3z_2z_1z_3 = z_2z_3z_2z_1$. Again by solving a new ambiguity $R_{w_1} - R_0 = z_3z_2z_1z_3z_1 = w_2$ we have

$$R(w_2) = z_3z_2z_1\underline{z_3z_1} = z_3z_2z_1^2z_3,\ L(w_2) = \underline{z_3z_2z_1z_3}z_1 = z_2z_3z_2z_1z_2$$

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

which gives another relation $R_{w_2} : z_3z_2z_1^2z_3 = z_2z_3z_2z_1z_2$. By continuing the same process we have the general relation

$$R_2^{(4)} :\ z_3z_2z_1^n z_3 = z_2z_3z_2z_1^n.$$

3. In the ambiguity $R_2 - R_2 = z_3z_2z_3z_2z_3 = w_3$, we have

$$R(w_3) = z_3z_2\underline{z_3z_2z_3} = z_3z_2^2z_3z_2,\ L(w_3) = \underline{z_3z_2z_3}z_2z_3 = z_2z_3z_2^2z_3.$$

Hence we have a relation $R_{w_3} : z_3z_2^2z_3z_2 = z_2z_3z_2^2z_3$. Therefore in general we have

$$R_3^{(4)} :\ z_3z_2^n z_3z_2 = z_2z_3z_2^2z_3^n.$$

4. Successive ambiguities of $R_2^{(4)}$ and R_2 leads to the relation,

$$R_4^{(4)} :\ z_3z_2z_1^n z_2^{n_1} z_3z_2 = z_2z_3z_2z_1^n z_2z_3^{n_1}.$$

5. By solving $R_{w_1} - R_2 = z_3z_2z_1z_3z_2z_3$ and generalizing we have,

$$R_5^{(4)} :\ z_3z_2z_1z_2^n z_3z_2 = z_2z_3z_2z_1z_2z_3^n.$$

6. $R_6^{(4)} :\ z_3z_2^n z_1^{n_1} z_3z_2z_1z_2 = z_2z_3z_2^2z_1z_3^n z_2z_1^{n_1}$ is obtained by solving the ambiguity of the relations $R_3^{(4)}$ and R_1.

7. Solving the ambiguities formed by $R_4^{(4)}$ and R_1, we get

$$R_7^{(4)} :\ z_3z_2z_1^n z_2^{n_1} z_1^{n_2} z_3z_2z_1z_2 = z_2z_3z_2z_1^n z_2z_1z_3^{n_1} z_2z_1^{n_2}.$$

8. Successive ambiguities of $R_5^{(4)}$ and R_1 leads to the relation,

$$R_8^{(4)} :\ z_3z_2z_1z_2^n z_1^{n_1} z_3z_2z_1z_2 = z_2z_1z_3z_2z_1z_2z_3^n z_2z_1^{n_1}.$$

9. Now, solving the ambiguity formed by $R_6^{(4)}$, $R_4^{(4)}$ and R_0 we have,

$$R_9^{(4)} :\ z_3(z_2^n z_1^{n_1} z_2^{n_2} z_1^{n_3}\cdots)z_3z_2z_1z_2z_3 = z_2z_3z_2^2z_1z_3^n z_2z_1^{n_1} z_3(z_2^{n_2} z_1^{n_3}\cdots).$$

10. The relation

$$R_{10}^{(4)} :\ z_3z_2(z_1^n z_2^{n_1} z_1^{n_2} z_2^{n_3} z_1^{n_4}\cdots)z_3z_2z_1z_2z_3 = z_2z_3z_2z_1^n z_2z_1z_3^{n_1} z_2z_1^{n_2} z_3(z_2^{n_3} z_1^{n_4}\cdots)$$

is obtained by solving the ambiguities formed by $R_7^{(4)}$, $R_4^{(4)}$ and R_0.

11. Successively solving the ambiguities formed by $R_8^{(4)}$, $R_4^{(4)}$ and R_0 we get,

$$R_{11}^{(4)} :\ z_3z_2z_1(z_2^n z_1^{n_1} z_2^{n_2} z_1^{n_3}\cdots)z_3z_2z_1z_2z_3 = z_2z_1z_3z_2z_1z_2z_3^n z_2z_1^{n_1} z_3(z_2^{n_2} z_1^{n_3}\cdots).$$

All other ambiguities all solvable. Hence we have the complete set of relations.

3. Hilbert Series of $MB_{1,3}$

Definition 4. [5] *Let M be a group or a monoid and a_n be the number of elements of M of word length n. Then the Hilbert series of M for arbitrary variable t is denoted by $H_M(t)$ and is defined by $H_M(t) = \sum\limits_{n=0}^{\infty} a_n t^n$.*

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

We use the complete presentation of $MB_{1,3}$ to compute the Hilbert series. Let $A^{(m+n)}$ and $B^{(m+n)}$ denote the set of all canonical and reducible words in $MB_{m,n}$, respectively. In particular assume that $A_{\mu}^{(m+n)}$ and $B_{\mu,\nu}^{(m+n)}$ denote the set of all canonical and reducible words in $MB_{m,n}$, respectively, where μ is related to prefix of a word while ν is suffix of the word. For example, $A_{j(j-1)\dots k}^{(n+m)}$ denotes collection of all canonical words in $MB_{m,n}$ that starts with $z_j z_{j-1}\dots z_k$ and $B_{j,v}^{(m+n)}$, $B_{j(j+1),v}^{(m+n)}$ denote the collection of all reducible words that starts with $z_{(m+n)-1}z_{(m+n)-2}\cdots z_j$ and $z_{(m+n)-1}z_{(m+n)-2}\cdots z_1z_2\cdots z_j$, respectively, and v is a word in the generators $z_1,\dots,z_n$. The set $B_{*,v}^{(m+n)}$ denotes all the reducible words starting with any word and ending in the generators $z_1,\dots,z_n$. Hence in $MB_{1,3}$ we have the following set of reducible words:
$B_{1,2}^{(4)}=\{z_2z_1z_2z_1\}$, $B_{1,212}^{(4)}=\{z_2z_1^{n+1}z_2z_1z_2\}$, $B_{2,3}^{(4)}=\{z_3z_2z_3\}$, $B_{1,3}^{(4)}=\{z_3z_2z_1^nz_3\}$,
$B_{2,32}^{(4)}=\{z_3z_2^nz_3z_2\}$, $B_{1,32}^{(4)}=\{z_3z_2z_1^nz_2^{n_1}z_3z_2\}$, $B_{12,32}^{(4)}=\{z_3z_2z_1z_2^nz_3z_2\}$,
$B_{2,3212}^{(4)}=\{z_3z_2^nz_1^{n_1}z_3z_2z_1z_2\}$, $B_{1,3212}^{(4)}=\{z_3z_2z_1^nz_2^{n_1}z_1^{n_2}z_3z_2z_1z_2\}$,
$B_{12,3212}^{(4)}=\{z_3z_2z_1z_2^nz_1^{n_1}z_3z_2z_1z_2\}$, $B_{2,32123}^{(4)}=\{z_3(z_2^nz_1^{n_1}z_2^{n_2}z_1^{n_3}\cdots)z_3z_2z_1z_2z_3\}$,
$B_{1,32123}^{(4)}=\{z_3z_2(z_1^nz_2^{n_1}z_1^{n_2}z_2^{n_3}z_1^{n_4}\cdots)z_3z_2z_1z_2z_3\}$,
$B_{12,32123}^{(4)}=\{z_3z_2z_1(z_2^nz_1^{n_1}z_2^{n_2}\cdots)z_3z_2z_1z_2z_3\}$.
Assuming that $Q_{\mu,\nu}^{(m+n)}(t)$ denotes Hilbert series of $B_{\mu,\nu}^{(m+n)}$ and $P_{\mu}^{(m+n)}(t)$ denotes Hilbert series of $A_{\mu}^{(m+n)}$. If $A_{*}^{(m+n)}$ denotes a set of canonical words in $MB_{m,n}$, then $\Sigma A_{*}^{(m+n)}$ denotes the same set of canonical words with each index increased by 1. For example for $A_1^{(2)}=\{z_1,z_1^2,z_1^3,\dots\}$, we have $\Sigma A_1^{(2)}=\{z_2,z_2^2,z_2^3,\dots\}$. Therefore
$$P_1^{(2)}=t+t^2+t^3+\cdots=\frac{t}{1-t}.$$

Lemma 1. [7] *The following equations hold for the canonical words in $MB_{1,2}$*
1. $P_1^{(3)}(t)=\frac{t}{(1-t)(1-t-t^2-t^3)}$
2. $P_2^{(3)}(t)=\frac{t(1+t+t^2)}{(1-t-t^2-t^3)}$
3. $P_{21}^{(3)}(t)=\frac{t^2(1+t)}{(1-t-t^2-t^3)}$
4. $P_{212}^{(3)}(t)=\frac{t^3}{(1-t-t^2-t^3)}$.

Corollary 1. [7] *The Hilbert series for the canonical words in $MB_{1,2}$ is*
$$H_M^{(3)}(t)=\frac{1}{(1-t)(1-t-t^2-t^3)}.$$

Now, we have to find $P_1^{(4)}(t)$, $P_2^{(4)}(t)$ and $P_3^{(4)}(t)$ for the computation of Hilbert series $H_M^{(4)}(t)$ of $MB_{1,3}$.

Lemma 2. *The following equations hold for the reducible words in $MB_{1,3}$*
1. $Q_{1,2}^{(4)}=t^4$
2. $Q_{1,212}^{(4)}=\frac{t^6}{1-t}$
3. $Q_{2,3}^{(4)}=t^3$
4. $Q_{1,3}^{(4)}=\frac{t^4}{1-t}$

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

5. $Q^{(4)}_{\mu,3} = \frac{t^3}{1-t}$
6. $Q^{(4)}_{2,32} = \frac{t^5}{1-t}$
7. $Q^{(4)}_{1,32} = \frac{t^6}{(1-t)^2}$
8. $Q^{(4)}_{12,32} = \frac{t^6}{1-t}$
9. $Q^{(4)}_{*,32} = \frac{t^5}{(1-t)^2}$
10. $Q^{(4)}_{2,3212} = \frac{t^8}{(1-t)^2}$
11. $Q^{(4)}_{1,3212} = \frac{t^{10}}{(1-t)^3}$
12. $Q^{(4)}_{12,3212} = \frac{t^{10}}{(1-t)^2}$
13. $Q^{(4)}_{*,3212} = \frac{t^8(1-t+2t^2-t^3)}{(1-t)^3}$
14. $Q^{(4)}_{2,32123} = \frac{t^{10}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}$
15. $Q^{(4)}_{1,32123} = \frac{2t^{13}}{(1-t-t^2-t^3)(1-t)^3}$
16. $Q^{(4)}_{12,32123} = \frac{t^{12}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}$
17. $Q^{(4)}_{*,32123} = \frac{t^{10}(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3}$.

Proof. We proceed the prove by considering tail-wise reducible words. Here for all the reducible words we use the decompositions.
1. We have only one word that starts and ends with z_2z_1, i.e., $B^{(4)}_{1,2} = \{z_2z_1z_2z_1\}$. Hence we have $Q^{(4)}_{1,2} = t^4$.
2. Since $B^{(4)}_{1,212} = \{z_2z_1^{n+1}z_2z_1z_2\} = \{z_2z_1\} \times A^{(2)}_1 \times \{z_2z_1z_2\}$. Hence $Q^{(4)}_{1,212} = \frac{t^6}{1-t}$.
3. For $B^{(4)}_{2,3} = \{z_3z_2z_3\}$ we have $Q^{(4)}_{2,3} = t^3$.
4. Similarly $B^{(4)}_{1,3} = \{z_3z_2z_1^nz_3\} = \{z_3z_2z_1\} \times_1 A^{(2)}_1 \times \{z_3\}$. Therefore $Q^{(4)}_{1,3} = \frac{t^4}{1-t}$.
5. As there are two types of reducible words whose tail is z_3 that is $B^{(4)}_{*,3} = B^{(4)}_{2,3} \sqcup B^{(4)}_{1,3}$. Hence

$$Q^{(4)}_{*,3} = t^3 + \frac{t^4}{1-t} = \frac{t^3}{1-t}.$$

6. The decomposition $B^{(4)}_{2,32} = \{z_3z_2^nz_3z_2\} = \{z_3z_2\} \times \Sigma A^{(2)}_1 \times \{z_3z_2\}$ gives $Q^{(4)}_{2,32} = \frac{t^5}{1-t}$.
7. The set $B^{(4)}_{1,32} = \{z_3z_2z_1^nz_2^{n_1}z_3z_2\} = \{z_3z_2z_1\} \times_1 A^{(2)}_1 \times \Sigma A^{(2)}_1 \times \{z_3z_2\}$ gives the relation $Q^{(4)}_{1,32} = \frac{t^6}{(1-t)^2}$.
8. Similarly $B^{(4)}_{12,32} = \{z_3z_2z_1z_2^nz_3z_2\} = \{z_3z_2z_1z_2\} \times_2 \Sigma A^{(2)}_1 \times \{z_3z_2\}$. Therefore $Q^{(4)}_{12,32} = \frac{t^6}{1-t}$.
9. Using complete presentation, we have two different types of reducible words ending with z_3z_2 (as $B^{(4)}_{12,32}$ is a subword of $B^{(4)}_{1,32}$ for $n=1$), i.e., $B^{(4)}_{*,32} = B^{(4)}_{2,32} \sqcup B^{(4)}_{1,32}$. Hence, we have

$$Q^{(4)}_{*,32} = Q^{(4)}_{2,32} + Q^{(4)}_{1,32} = \frac{t^5}{1-t} + \frac{t^6}{(1-t)^2} = \frac{t^5}{(1-t)^2}.$$

10. As $B^{(4)}_{2,3212} = \{z_3z_2^nz_1^{n_1}z_3z_2z_1z_2\} = \{z_3z_2\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\}$. Hence we have $Q^{(4)}_{2,3212} = \frac{t^8}{(1-t)^2}$.
11. $B^{(4)}_{1,3212} = \{z_3z_2z_1^nz_2^{n_1}z_1^{n_2}z_3z_2z_1z_2\} = \{z_3z_2z_1\} \times A^{(2)}_1 \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\}$ gives $Q^{(4)}_{1,3212} = \frac{t^{10}}{(1-t)^3}$.
12. $B^{(4)}_{12,3212} = \{z_3z_2z_1z_2^nz_1^{n_1}z_3z_2z_1z_2\} = \{z_3z_2z_1\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\}$ gives

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

the relation $Q^{(4)}_{12,3212} = \frac{t^{10}}{(1-t)^2}$.

13. Using reduced complete presentation, we have three types of reducible words ending with $z_3z_2z_1z_2$, i.e., $B^{(4)}_{*,3212} = B^{(4)}_{2,3212} \sqcup B^{(4)}_{1,3212} \sqcup B^{(4)}_{12,3212}$. Hence we get

$$\begin{aligned} Q^{(4)}_{*,3212} &= Q^{(4)}_{2,3212} + Q^{(4)}_{1,3212} + Q^{(4)}_{12,3212} \\ &= \frac{t^8}{(1-t)^2} + \frac{t^{10}}{(1-t)^3} + \frac{t^{10}}{(1-t)^2} \\ &= \frac{t^8(1-t+2t^2-t^3)}{(1-t)^3}. \end{aligned}$$

14. The word $B^{(4)}_{2,32123} = \{z_3(z_2^n z_1^{n_1} z_2^{n_2} z_1^{n_3}\cdots)z_3z_2z_1z_2z_3\} = \{z_3z_2\} \times A^{(3)}_2 \times \{z_3z_2z_1z_2z_3\}$ can be written as $\{z_3z_2\} \times \Sigma A^{(2)}_1 \times \{z_3z_2\} \times \{z_1z_2z_3\}$ as well as $\{z_3z_2\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\} \times \{z_3\}$. In this case we have reducible subwords, which will be subtracted. Hence we have

$$B^{(4)}_{2,32123} = \{z_3z_2\} \times A^{(3)}_2 \times \{z_3z_2z_1z_2z_3\} \backslash \Big((B^{(4)}_{2,32} \times \{z_1z_2z_3\}) \sqcup (B^{(4)}_{2,3212} \times \{z_3\})\Big).$$

For which we have

$$Q^{(4)}_{2,32123} = t^7 P^3_2 - \frac{t^8}{1-t} - \frac{t^9}{(1-t)^2}$$

Using Lemma 1 we have

$$\begin{aligned} Q^{(4)}_{2,32123} &= \frac{t^8(1+t+t^2)}{1-t-t^2-t^3} - \frac{t^8}{1-t} - \frac{t^9}{(1-t)^2} \\ &= \frac{t^{10}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}. \end{aligned}$$

15. As we have $B^{(4)}_{1,32123} = z_3z_2(z_1^n z_2^{n_1} z_1^{n_2} z_2^{n_3} z_1^{n_4}\cdots)z_3z_2z_1z_2z_3$. Hence we can write

$B^{(4)}_{1,32123} = \{z_3z_2z_1\} \times A^{(3)}_1 \times \{z_3z_2z_1z_2z_3\}$. Using the above argument we have $B^{(4)}_{1,32123} = \{z_3z_2z_1\} \times A^{(3)}_1 \times \{z_3z_2z_1z_2z_3\} \backslash \Big((\{z_3z_2z_1\} \times A^{(2)}_1 \times \{z_2z_1z_2\} \times_{212} \times A^{(3)}_{212} \times \{z_3z_2z_1z_2z_3\}) \sqcup (\{z_3z_2z_1\} \times A^{(2)}_1 \times \{z_3\} \times \{z_2z_1z_2z_3\}) \sqcup (\{z_3z_2z_1\} \times A^{(2)}_1 \times \Sigma A^{(2)}_1 \times \{z_3z_2\} \times \{z_1z_2z_3\}) \sqcup (\{z_3z_2z_1\} \times A^{(2)}_1 \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\} \times \{z_3\})\Big)$. Hence

$$\begin{aligned} Q^{(4)}_{1,32123} &= t^8 P^{(3)}_1 - \frac{t^9}{1-t} P^{(3)}_{212} - \frac{t^9}{1-t} - \frac{t^{10}}{(1-t)^2} - \frac{t^{11}}{(1-t)^3} \\ &= \frac{2t^{13}}{(1-t-t^2-t^3)(1-t)^3}. \end{aligned}$$

16. Similarly, as $B^{(4)}_{12,32123} = z_3z_2z_1(z_2^n z_1^{n_1} z_2^{n_2}\cdots)z_3z_2z_1z_2z_3$. Therefore we can write

$B^{(4)}_{12,32123} = \{z_3z_2z_1z_2\} \times A^{(3)}_2 \times \{z_3z_2z_1z_2z_3\} \backslash \Big((\{z_3z_2z_1z_2\} \times \Sigma A^{(2)}_1 \times \{z_3z_2\} \times \{z_1z_2z_3\}) \sqcup (\{z_3z_2z_1z_2\} \times \Sigma A^{(2)}_1 \times A^{(2)}_1 \times \{z_3z_2z_1z_2\} \times \{z_3\})\Big)$. Hence (using Lemma

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

1) we have

$$\begin{aligned} Q^{(4)}_{12,32123} &= t^9 P^{(3)}_2 - \frac{t^{10}}{1-t} - \frac{t^{11}}{(1-t)^2} \\ &= \frac{t^{12}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2}. \end{aligned}$$

17. We have three types of reducible words ending with $z_3z_2z_1z_2z_3$, i.e., $B^{(4)}_{*,32123} = B^{(4)}_{2,32123} \sqcup B^{(4)}_{1,32123} \sqcup B^{(4)}_{12,32123}$. Therefore we get

$$\begin{aligned} Q^{(4)}_{*,32123} &= Q^{(4)}_{2,32123} + Q^{(4)}_{1,32123} + Q^{(4)}_{12,32123} \\ &= \frac{t^{10}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} + \frac{2t^{13}}{(1-t-t^2-t^3)(1-t)^3} \\ &+ \frac{t^{12}(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} \\ &= \frac{t^{10}(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3}. \end{aligned}$$

For the computations of the Hilbert series of $MB_{1,3}$, we have a following linear system for the canonical words.

Lemma 3. *The following equations hold for the canonical words in $MB_{1,3}$:*
1. $P^{(4)}_1 = P^{(3)}_1 + P^{(3)}_1 P^{(4)}_3$
2. $P^{(4)}_2 = P^{(3)}_2 + P^{(3)}_2 P^{(4)}_3$
3. $P^{(4)}_{21} = P^{(3)}_{21} + P^{(3)}_{21} P^{(4)}_3$
4. $P^{(4)}_{212} = P^{(3)}_{212} + P^{(3)}_{212} P^{(4)}_3$
5. $P^{(4)}_3 = t + tP^{(4)}_3 + P^{(4)}_{32}$
6. $P^{(4)}_{32} = tP^{(4)}_2 - \frac{t^2}{1-t} P^{(4)}_3 - \frac{t^3(1+t-t^2)}{(1-t)^2} P^{(4)}_{32} - \frac{t^4(1-t+2t^2-t^3)}{(1-t)^3} P^{(4)}_{3212} - \frac{t^5(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3} P^{(4)}_{32123}$
7. $P^{(4)}_{321} = tP^{(4)}_{21} - \frac{t^3}{1-t} P^{(4)}_3 - \frac{t^4}{(1-t)^2} P^{(4)}_{32} - \frac{t^6(2-t)}{(1-t)^3} P^{(4)}_{3212} - \frac{t^7(1+t+t^2-t^3)}{(1-t-t^2-t^3)(1-t)^3} P^{(4)}_{32123}$
8. $P^{(4)}_{3212} = tP^{(4)}_{212} - \frac{t^4}{1-t} P^{(4)}_{32} - \frac{t^6}{(1-t)^2} P^{(4)}_{3212} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} P^{(4)}_{32123}$
9. $P^{(4)}_{32123} = t^4 P^{(4)}_3 - t^4 P^{(4)}_{32}$.

Proof. The canonical words may starts from z_1, z_2, z_2z_1, $z_2z_1z_2$, z_3, z_3z_2, $z_3z_2z_1$, $z_3z_2z_1z_2$ or $z_3z_2z_1z_2z_3$. By $\sqcup$ we mean the disjoint union of sets.
1) For the canonical words starting from z_1, we have the decomposition of the form $A^{(4)}_1 = A^{(3)}_1 \sqcup (A^{(3)}_1 \times A^{(4)}_3)$. The associated Hilbert series becomes

$$P^{(4)}_1 = P^{(3)}_1 + P^{(3)}_1 P^{(4)}_3.$$

2) The canonical words starting with z_2 have the form $A^{(4)}_2 = A^{(3)}_2 \sqcup (A^{(3)}_2 \times A^{(4)}_3)$. Hence

$$P^{(4)}_2 = P^{(3)}_2 + P^{(3)}_2 P^{(4)}_3.$$

3) The decomposition $A^{(4)}_{21} = A^{(3)}_{21} \sqcup (A^{(3)}_{21} \times A^{(4)}_3)$ gives

$$P^{(4)}_{21} = P^{(3)}_{21} + P^{(3)}_{21} P^{(4)}_3.$$

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

4) The decomposition $A_{212}^{(4)} = A_{212}^{(3)} \sqcup (A_{212}^{(3)} \times A_3^{(4)})$ gives

$$P_{212}^{(4)}(t) = P_{212}^{(3)} + P_{212}^{(3)} P_3^{(4)}.$$

5) The canonical words starting with z_3 can be written as $A_3^{(4)} = \{z_3\} \sqcup (\{z_3\} \times A_3^{(4)}) \sqcup A_{32}^{(4)}$. Therefore the corresponding the Hilbert series is

$$P_3^{(4)} = t + tP_3^{(4)} + P_{32}^{(4)}.$$

6) By taking the product of z_3 on left side of the set of canonical words starting with z_2, we may have reducible words of any one of the form $B_{\mu,\nu}^{(4)}$. In order to get canonical words starting with $z_3 z_2$, we have to get rid of above mentioned reducible words from the $\{z_3\} \times A_2^{(4)}$. Therefore,
$A_{32}^{(4)} = \{z_3\} \times A_2^{(4)} \backslash \Big((B_{*,3}^{(4)} \times_3 A_3^{(4)}) \sqcup (B_{*,32}^{(4)} \times_{32} A_{32}^{(4)}) \sqcup (B_{*,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \sqcup (B_{*,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \Big)$. Hence we have
$P_{32}^{(4)} = tP_2^{(4)} - \frac{t^2}{1-t} P_3^{(4)} - \frac{t^3}{(1-t)^2} P_{32}^{(4)} - \frac{t^4(1-t+2t^2-t^3)}{(1-t)^3} P_{3212}^{(4)} - \frac{t^5(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)}$.
Equivalently we have
$tP_2^{(4)} - \frac{t^2}{1-t} P_3^{(4)} - (1 + \frac{t^3}{(1-t)^2}) P_{32}^{(4)} - \frac{t^4(1-t+2t^2-t^3)}{(1-t)^3} P_{3212}^{(4)} - \frac{t^5(1-t+2t^2+t^4-t^5)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)} = 0$.
7) Similarly we can write
$A_{321}^{(4)} = \{z_3 z_2 z_1\} \times_{21} A_{21}^{(4)} \backslash \Big((B_{1,3}^{(4)} \times_3 A_3^{(4)}) \sqcup (B_{1,32}^{(4)} \times_{32} A_{32}^{(4)}) \sqcup (B_{1,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \sqcup (B_{12,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \sqcup (B_{1,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \sqcup (B_{12,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \Big)$.
Therefore we get
$P_{321}^{(4)} = tP_{21}^{(4)} - \frac{t^3}{1-t} P_3^{(4)} - \frac{t^4}{(1-t)^2} P_{32}^{(4)} - \frac{t^6}{(1-t)^3} P_{3212}^{(4)} - \frac{t^6}{(1-t)^2} P_{3212}^{(4)} - \frac{2t^8}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} P_{32123}^{(4)}$ or
$tP_{21}^{(4)} - \frac{t^3}{1-t} P_3^{(4)} - \frac{t^4}{(1-t)^2} P_{32}^{(4)} - P_{321}^{(4)} - \frac{t^6(2-t)}{(1-t)^3} P_{3212}^{(4)} - \frac{t^7(1+t+t^2-t^3)}{(1-t-t^2-t^3)(1-t)^3} P_{32123}^{(4)} = 0$.
8) Similarly we have
$A_{3212}^{(4)} = \{z_3 z_2 z_1 z_2\} \times_{212} A_{212}^{(4)} \backslash \Big((B_{12,32}^{(4)} \times_{32} A_{32}^{(4)}) \sqcup (B_{12,3212}^{(4)} \times_{3212} A_{3212}^{(4)}) \sqcup (B_{12,32123}^{(4)} \times_{32123} A_{32123}^{(4)}) \Big)$. The corresponding Hilbert series becomes

$$P_{3212}^{(4)} = tP_{212}^{(4)} - \frac{t^4}{1-t} P_{32}^{(4)} - \frac{t^6}{(1-t)^2} P_{3212}^{(4)} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} P_{32123}^{(4)}$$

or

$$tP_{212}^{(4)} - \frac{t^4}{1-t} P_{32}^{(4)} - \frac{t^6+t^2-2t+1}{(1-t)^2} P_{3212}^{(4)} - \frac{t^7(1+t^2)}{(1-t-t^2-t^3)(1-t)^2} P_{32123}^{(4)} = 0.$$

9) The decomposition $A_{32123}^{(4)} = \{z_3 z_2 z_1 z_2 z_3\} \times_3 A_3^{(4)} \backslash (\{z_3 z_2 z_1 z_2 z_3 z_2\} \times_{32} A_{32}^{(4)})$ gives $P_{32123}^{(4)} = t^4 P_3^{(4)} - t^4 P_{32}^{(4)}$ or $t^4 P_3^{(4)} - t^4 P_{32}^{(4)} - P_{32123}^{(4)} = 0$,

Finally we have our main result.

Theorem 2. *The Hilbert series of $MB_{1,3}$ is*

$$H_M^{(4)}(t) = \frac{1}{(1-t)(1-2t-t^2+t^4+t^5+t^6+t^7+t^8)}.$$

Proof. Let $T_1 = 1 - t - t^2 - t^3$ and $T_2 = 1 - 2t - t^2 + t^4 + t^5 + t^6 + t^7 + t^8$. Then solving the linear system given in Theorem 3 we have the augmented matrix of the

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

system

$$\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & \frac{-t}{(1-t)T_1} & 0 & 0 & 0 & 0 & \frac{t}{(1-t)T_1} \\
0 & 1 & 0 & 0 & \frac{-t(1+t+t^2)}{T_1} & 0 & 0 & 0 & 0 & \frac{t(1+t+t^2)}{T_1} \\
0 & 0 & 1 & 0 & \frac{-t^2(1+t)}{T_1} & 0 & 0 & 0 & 0 & \frac{t^2(1+t)}{T_1} \\
0 & 0 & 0 & 1 & \frac{-t^3}{T_1} & 0 & 0 & 0 & 0 & \frac{t^3}{T_1} \\
0 & 0 & 0 & 0 & 1-t & -1 & 0 & 0 & 0 & t \\
0 & t & 0 & 0 & \frac{-t^2}{1-t} & -\frac{(1-t)^2+t^3}{(1-t)^2} & 0 & \frac{-t^4(1-t+2t^2-t^3)}{(1-t)^3} & \frac{-t^5(1-t+2t^2+t^4-t^5)}{(1-t)^3T_1} & 0 \\
0 & 0 & t & 0 & \frac{-t^3}{1-t} & \frac{-t^4}{(1-t)^2} & -1 & \frac{-t^6(2-t)}{(1-t)^3} & \frac{-t^7(1+t+t^2-t^3)}{(1-t)^3T_1} & 0 \\
0 & 0 & 0 & t & 0 & \frac{-t^4}{1-t} & 0 & -\frac{(1-t)^2+t^6}{(1-t)^2} & \frac{-t^7(1+t^2)}{(1-t)^2T_1} & 0 \\
0 & 0 & 0 & 0 & t^4 & -t^4 & 0 & 0 & -1 & 0
\end{array}\right]$$

The solution gives the following values:
$P_1^{(4)} = \frac{t}{(1-t)(T_2)}$, $P_2^{(4)} = \frac{t(1+t+t^2)}{T_2}$, $P_3^{(4)} = \frac{t(1-t^2-t^3-t^4-t^5-t^6-t^7)}{T_2}$, $P_{21}^{(4)} = \frac{t^2(1+t)}{T_2}$,
$P_{212}^{(4)} = \frac{t^3}{T_2}$, $P_{32}^{(4)} = \frac{t^2(1-t^3-t^4-t^5-t^6)}{T_2}$, $P_{321}^{(4)} = \frac{t^3(1-t^2-t^3-t^4-t^5)}{T_2}$, $P_{3212}^{(4)} = \frac{t^4(1-t^2-t^3-t^4)}{T_2}$,
$P_{32123}^{(4)} = \frac{t^5(1-t-t^2-t^3)}{T_2}$.

All the canonical words in $MB_{1,3}$ are expressed as $A^{(4)} = \{e\} \sqcup A_1^{(4)} \sqcup A_2^{(4)} \sqcup A_3^{(4)}$. Hence the corresponding Hilbert series is given by

$$\begin{aligned}
H_M^{(4)}(t) &= 1 + P_1^{(4)}(t) + P_2^{(4)}(t) + P_3^{(4)}(t) \\
&= 1 + \frac{t}{(1-t)(T_2)} + \frac{t(1+t+t^2)}{T_2} + \frac{t(1-t^2-t^3-t^4-t^5-t^6-t^7)}{T_2} \\
&= \frac{1}{(1-t)(1-2t-t^2+t^4+t^5+t^6+t^7+t^8)} \\
&= 1 + 3t + 8t^2 + 20t^3 + 48t^4 + 112t^5 + 263t^6 + \cdots + a_k^{(4)} t^k + \cdots,
\end{aligned}$$

where $a_k^{(4)}$ is arbitrary constant.

Definition 5. *Let $\{a_k\}_{k\geq 1}$ be a sequence of positive numbers and r be a positive real number then the growth rate r of the sequence $\{a_k\}_{k\geq 1}$ is defined as*

$$r = \overline{\lim_k} \exp\left(\frac{\log a_k}{k}\right).$$

Corollary 2. *The growth rate of $MB_{1,3}$ is 2.29.*

Proof. The Hilbert series in rational form obtained in Theorem 2 can be resolved (approximately) into its partial fraction (using the Maple) as:

$$\begin{aligned}
\frac{1}{(1-t)(1-2t-t^2+t^4+t^5+t^6+t^7+t^8)} &= \frac{0.65564t + 0.51628}{t^2 + 0.98567t + 1.35852} + \frac{0.33333}{1-t} \\
&+ \frac{0.60593t - 0.39941}{t^2 - 0.98615t + 1.49520} + \frac{0.56106t + 0.60272}{t^2 + 2.21096t + 1.45727} + \frac{0.80972}{0.4364 + t}.
\end{aligned}$$

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.3479.

The first four terms have negligible contribution in the approximation of the series, however the last term can be approximated as:

$$\frac{0.809722}{0.43644+t} = 1.8552\{1 + 2.29t + (2.29)^2t^2 + (2.29)^3t^3 + \cdots\}$$

Therefore $a_k^{(4)} \approx 1.8552(2.29)^k$. Hence the growth rate of $MB_{1,3}$ is 2.29.

References

[1] Artin E, Theorie der Zopfe, Abh. Math. Sem. Univ. Hamburg 4 , pp. 47-72, 1926.
[2] Bergman G. The diamond lemma for ring theory. Adv Math 1978; 29: 178-218.
[3] Birman JS. *Braids links and mapping-class groups*, Annals of Math. Studies. Princeton University Press, 12(1974).
[4] Garside FA. *The braid groups and other groups*, Quat. J. Math. Oxford, 2^e ser. 20(1969), 235-254 .
[5] Harpe PD. *Topics in Geometric Group Theory* America: The University of Chicago Press, 2000.
[6] Herbig C and Herden D, and C. Seaton. *Hilbert series associated to symplectic quotients by SU2*, International Journal of Algebra and Computation, vol. 30, no. 7(2020) 1323-1357.
[7] Iqbal Z, Nazir S and Ali U. *Canonical forms and infimum of positive braids*, Algebra Colloquium. Spec.no.1,18(2011),PP.1007-1016.
[8] Iqbal Z. *Combinatorial problems in braids group*, ASSMS, GCU Lahore,2009.
[9] Iqbal Z. *Hilbert series of positive braids*, Algebra Colloquium. Spec.no.1, 18(2011), 1017-1028.
[10] Iqbal Z and Yousaf S. Hilbert series of braid monoid MB_4 in band generators. Turk J Math 2014; 38: 977-984.
[11] Iqbal Z and Ali U. Hilbert series of Artin monoid $M(I_2(p))$, Southeast Asian Bulletin of Mathematics (2013) 37: 475-480.
[12] Iqbal Z. Hilbert series of the finite dimensional generalized Hecke algebras. Turk J Math 2015; 39: 698-705.
[13] Kassel C and Tureav V. *Braid Groups, Graduate Texts in Mathematics 247*, Springer Science and Business Media, New York, 2008. MR 2435235,2009e:20082.
[14] Kumar A and Sarkar R. *Hilbert series of binomial edge ideals*, Communications in Algebra, vol. 47, no. 9(2019) 3830-3841.
[15] Mairesse and Mathéus. Growth series of Artin groups of dihedral type. International Journal of Algebra and Computation 16, 2006; 6: 1087-1107.
[16] Lambropoulou S. *Solid torus links and Hecke algebras of B-type*, Proceedings of the Conference on Quantum Topology, (1994), 225-245.
[17] Lambropoulou S. *Braid structures in knot complements, handle bodies and 3-manifolds*, Word Scientific Press, 24(2000) 274-289.
[18] Saito K. *Growth functions for Artin monoid*, Proc. Japan Acad, 85, Ser. A(2009) 84-88.
[19] Xu PJ. *The genus of closed 3-braid, J. of Knots Theory and its Ramifications* 1 (1992), 303-326.

Submitted: July 7, 2022
Accepted: January 13, 2023
Published (early view): August 22, 2024