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HAAR WAVELET CHARACTERIZATION OF DYADIC LIPSCHITZ REGULARITY

HUGO AIMAR, CARLOS EXEQUIEL ARIAS, AND IVANA GÓMEZ

To Pola Harboure and Roberto Macías

ABSTRACT. We obtain a necessary and sufficient condition on the Haar coefficients of a real function f defined on $\mathbb{R}^+$ for the Lipschitz α regularity of f with respect to the ultrametric $\delta(x, y) = \inf\{|I| : x, y \in I; I \in \mathcal{D}\}$, where $\mathcal{D}$ is the family of all dyadic intervals in $\mathbb{R}^+$ and α is positive. Precisely, $f \in \text{Lip}_\delta(\alpha)$ if and only if $\left| \langle f, h_k^j \rangle \right| \leq C 2^{-(\alpha + \frac{1}{2})j}$, for some constant C , every $j \in \mathbb{Z}$ and every $k = 0, 1, 2, \dots$. Here, as usual $h_k^j(x) = 2^{j/2} h(2^j x - k)$ and $h(x) = \mathcal{X}_{[0, 1/2)}(x) - \mathcal{X}_{[1/2, 1)}(x)$.

Arde de abejas el aguaribay, arde.
Ríen los ojos, los labios, hacia las islas azules
a través de la cortina
de los racimos
pálidos.

Juan L. Ortiz

1. INTRODUCTION

In [4] and [3], see also [2], M. Holschneider and Ph. Tchamitchian provide characterizations of the Lipschitz α regularity of a function in $L^2(\mathbb{R})$ for $0 < \alpha < 1$ in terms of the behaviour of the continuous wavelet transform. The result is that a given function is Lipschitz α if and only if its continuous wavelet transform satisfies a power law in the absolute value of the scale parameter. Here Lipschitz α refers to the classical definition with respect to the usual metric in $\mathbb{R}$, i.e. $|f(x) - f(y)| \leq C |x - y|^\alpha$ for some constant $C > 0$ and every x and y in $\mathbb{R}$. In [1] these results are extended to more general moduli of regularity of functions when the basic wavelet is the Haar wavelet. The method used in [1] provides the tool for

2010 *Mathematics Subject Classification.* Primary. 42C15.

Key words and phrases. wavelets, Lipschitz regularity.

This work was supported by the Ministerio de Ciencia, Tecnología e Innovación-MINCYT in Argentina: Consejo Nacional de Investigaciones Científicas y Técnicas-CONICET (grant PIP-2021-2023-11220200101940CO) and Universidad Nacional del Litoral (grant CAI+D 50620190100070LI)..

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the analysis of pointwise regularity through the discrete wavelet transform associated to dyadic scaling and integer translations of the Haar wavelet. The natural Lipschitz α class, in our setting, is defined through the dyadic distance instead of the usual one.

The result of this paper is contained in the next statement.

Theorem 1.1. *Let f be a real valued function in $L^1_{\text{loc}}(\mathbb{R}^+)$. Let $h_k^j(x) = 2^{j/2}h(2^jx - k)$ where $h(x) = \mathcal{X}_{[0,1/2)}(x) - \mathcal{X}_{[1/2,1)}(x)$, $j \in \mathbb{Z}$, $k = 0, 1, 2, \dots$, and $\langle f, h_k^j \rangle = \int_{\mathbb{R}^+} f(x)h_k^j(x)dx$. Let α be any positive number. Then, the boundedness of the sequence*

$$\left\{ 2^{(\alpha + \frac{1}{2})j} \left| \langle f, h_k^j \rangle \right| : j \in \mathbb{Z}, k = 0, 1, 2, \dots \right\}$$

is equivalent to the essential boundedness of the quotients

$$\frac{|f(x) - f(y)|}{\delta^\alpha(x, y)}, \quad x \neq y,$$

where $\delta(x, y) = \inf\{|I| : x, y \in I; I \in \mathcal{D}\}$ with $\mathcal{D}$ the family of all dyadic intervals in $\mathbb{R}^+$.

In Section 2 we introduce the basic facts and notation and Section 3 is devoted to the proof of Theorem 1.1.

2. DYADIC DISTANCE IN $\mathbb{R}^+$ AND THE HAAR SYSTEM

The set of nonnegative real numbers is denoted here by $\mathbb{R}^+$. The family of all dyadic intervals in $\mathbb{R}^+$ is the disjoint union of the classes $\mathcal{D}^j$, $j \in \mathbb{Z}$, where $\mathcal{D}^j = \{I_k^j = [k2^{-j}, (k+1)2^{-j}) : k = 0, 1, 2, \dots\}$ are the dyadic intervals of level j . Notice that with this notation, when j grows the partitions of $\mathbb{R}^+$ get refined and the intervals smaller. Since given two points x and y of $\mathbb{R}^+$ there exists some $j_0 \in \mathbb{Z}$ such that $0 \leq \max\{x, y\} < 2^{-j_0}$, we have that $x, y \in I_0^{j_0}$. Hence, the class of all dyadic intervals $I \in \mathcal{D}$ such that x and y , both, belong to I , is non-empty. So that, if $|E|$ denotes the Lebesgue length of the measurable set E , we have that

$$\delta(x, y) = \inf \{|I| : x, y \in I; I \in \mathcal{D}\}$$

is a well defined nonnegative real number. Even more, δ is an ultrametric in $\mathbb{R}^+$. In other words,

- (i) $\delta(x, y) = 0$ if and only if $x = y$;
- (ii) $\delta(x, y) = \delta(y, x)$, for every x and every y both in $\mathbb{R}^+$;
- (iii) $\delta(x, z) \leq \max\{\delta(x, y), \delta(y, z)\}$ for every x, y, z in $\mathbb{R}^+$.

The triangle inequality follows from the properties of the family $\mathcal{D}$. In fact, given x, y and z in $\mathbb{R}^+$, let $I(x, y)$ and $I(y, z)$ denote the smallest dyadic intervals containing x, y and y, z respectively, then, one of these intervals contains the other because $y \in I(x, y) \cap I(y, z) \neq \emptyset$. Assume $I(x, y) \supseteq I(y, z)$, then $\delta(x, z) \leq |I(x, y)| = \max\{|I(y, z)|, |I(x, y)|\} = \max\{\delta(y, z), \delta(x, y)\}$. In particular, δ is a metric in $\mathbb{R}^+$. Notice that $|x - y| \leq \delta(x, y)$, but $\frac{\delta(x, y)}{|x - y|}$ is unbounded.

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Hence every $\text{Lip}_{\delta}(\alpha)$ function f in the usual sense ($|f(x) - f(y)| \leq C|x - y|^{\alpha}$) is also a $\text{Lip}_{\delta}(\alpha)$ function, i.e.

$$|f(x) - f(y)| \leq C\delta^{\alpha}(x, y)$$

for some constant C and every x and y in $\mathbb{R}^+$. On the other hand, there are $\text{Lip}_{\delta}(\alpha)$ functions which are not Lipschitz(α) in the classical sense. In fact, $\mathcal{X}_I$, $I \in \mathcal{D}$, is in the class $\text{Lip}_{\delta}(1)$. We also observe that in contrast with the class $\text{Lipschitz}(\alpha)$ for every $\alpha > 1$, which is trivial, there exist non constant $\text{Lip}_{\delta}(\alpha)$ functions for every $\alpha > 0$.

Let us now review the basic properties of the Haar system. Set $h_0^0(x) = \mathcal{X}_{[0,1/2)}(x) - \mathcal{X}_{[1/2,1)}(x)$ and $h_k^j(x) = 2^{j/2}h_0^0(2^jx - k)$ for $j \in \mathbb{Z}$ and $k = 0, 1, 2, \dots$. The family $\mathcal{H} = \{h_k^j : j \in \mathbb{Z}, k = 0, 1, 2, \dots\}$ is the Haar system in $\mathbb{R}^+$. It is well known that $\mathcal{H}$ is an orthonormal basis for $L^2(\mathbb{R}^+)$. Since for each $I \in \mathcal{D}$ there is one and only one $h \in \mathcal{H}$ supported in I , we write sometimes h_I to denote the $h \in \mathcal{H}$ supported in $I \in \mathcal{D}$ and sometimes I_h to denote the dyadic support of $h \in \mathcal{H}$. From the basic character of $\mathcal{H}$ in $L^2(\mathbb{R}^+)$ we have that, given $f \in L^2(\mathbb{R}^+)$,

$$f = \sum_{h \in \mathcal{H}} \langle f, h \rangle h,$$

in the $L^2(\mathbb{R}^+)$ -sense, with $\langle f, h \rangle = \int_{\mathbb{R}^+} f(x)h(x)dx$. The sequence of coefficients $\{\langle f, h \rangle : h \in \mathcal{H}\}$ is well defined even for functions in $L_{\text{loc}}^1(\mathbb{R}^+)$, since the Haar functions are bounded and have bounded support.

3. PROOF OF THEOREM 1.1

The easy part of Theorem 1.1 follows as usual from the vanishing of the mean of the Haar functions. Let us state and prove it.

Proposition 3.1. *Let $f \in \text{Lip}_{\delta}(\alpha)$, $\alpha > 0$. Set $[f]_{\text{Lip}_{\delta}(\alpha)}$ to denote the infimum of the constants $C > 0$ such that $|f(x) - f(y)| \leq C\delta^{\alpha}(x, y)$, $x, y \in \mathbb{R}^+$. Then $|\langle f, h_I \rangle| \leq [f]_{\text{Lip}_{\delta}(\alpha)} |I|^{\alpha + \frac{1}{2}}$ for every $I \in \mathcal{D}$.*

Proof. For $I = [a_I, b_I) \in \mathcal{D}$ we have $\int_{\mathbb{R}^+} h_I(x)dx = 0$, hence

$$\begin{aligned} |\langle f, h_I \rangle| &= \left| \int_{\mathbb{R}^+} f(x)h_I(x)dx \right| \\ &= \left| \int_{\mathbb{R}^+} (f(x) - f(a_I))h_I(x)dx \right| \\ &\leq \int_I |f(x) - f(a_I)| |h_I(x)| dx \\ &\leq [f]_{\text{Lip}_{\delta}(\alpha)} \int_I \delta^{\alpha}(x, a_I) |I|^{-\frac{1}{2}} dx \\ &\leq [f]_{\text{Lip}_{\delta}(\alpha)} |I|^{\alpha - \frac{1}{2}} \int_I dx \\ &= [f]_{\text{Lip}_{\delta}(\alpha)} |I|^{\alpha - \frac{1}{2} + 1} \end{aligned}$$

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$$= [f]_{\text{Lip}_\delta(\alpha)} |I|^{\alpha+\frac{1}{2}}.$$

□

In order to prove that the size of the coefficients guarantee the regularity of f we start by stating and proving a lemma. Given an interval $I \in \mathcal{D}$ we denote with I^- and I^+ its left and right halves respectively. Notice that when $I \in \mathcal{D}^j$ then I^- and I^+ both belong to $\mathcal{D}^{j+1}$. Given a locally integrable function f we write $m_I(f)$ to denote the mean value of f on $I \in \mathcal{D}$. In other words $m_I(f) = \frac{1}{|I|} \int_I f(x) dx$.

Lemma 3.2. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^+)$. Then, for every $I \in \mathcal{D}$ we have*

$$|m_{I^-}(f) - m_{I^+}(f)| = 2 |I|^{-\frac{1}{2}} |\langle f, h_I \rangle|.$$

Proof. Let $I \in \mathcal{D}$ be given, then

$$\begin{aligned} |m_{I^-}(f) - m_{I^+}(f)| &= \left| \frac{2}{|I|} \int_{I^-} f(x) dx - \frac{2}{|I|} \int_{I^+} f(x) dx \right| \\ &= 2 |I|^{-\frac{1}{2}} \left| \int_I |I|^{-\frac{1}{2}} (\mathcal{X}_{I^-}(x) - \mathcal{X}_{I^+}(x)) f(x) dx \right| \\ &= 2 |I|^{-\frac{1}{2}} \left(\int_{\mathbb{R}^+} h_I(x) f(x) dx \right) \\ &= 2 |I|^{-\frac{1}{2}} |\langle f, h_I \rangle|. \end{aligned}$$

□

Proposition 3.3. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^+)$ be such that for some constant $A > 0$ we have*

$$|\langle f, h_I \rangle| \leq A |I|^{\alpha+\frac{1}{2}}$$

for every $I \in \mathcal{D}$, then $f \in \text{Lip}_\delta(\alpha)$ and $[f]_{\text{Lip}_\delta(\alpha)} \leq C_\alpha A$ with $C_\alpha = \sup\{2, \frac{1}{2\alpha-1}\}$.

Proof. Let $x < y$ be two points in $\mathbb{R}^+$. Let $I \in \mathcal{D}$ be the smallest dyadic interval containing x and y . In other words $|I| = \delta(x, y)$. Since $x < y$, necessarily $x \in I^-$ and $y \in I^+$. Set $I_1^x = I^-$ and $I_1^y = I^+$. Now let I_2^x be the half of I_1^x to which x belongs, and I_2^y the half of I_1^y with $y \in I_2^y$. In general, once I_l^x and I_l^y are defined, we select I_{l+1}^x as the only half of I_l^x with $x \in I_{l+1}^x$ and I_{l+1}^y as the only half of I_l^y with $y \in I_{l+1}^y$. In this way for a fixed positive integer k we have

$$I_k^x \subset I_{k-1}^x \subset \cdots \subset I_2^x \subset I_1^x \subset I,$$

and

$$I_k^y \subset I_{k-1}^y \subset \cdots \subset I_2^y \subset I_1^y \subset I.$$

Hence

$$\begin{aligned} f(x) - f(y) &= (f(x) - m_{I_k^x}(f)) + \\ &\quad + (m_{I_k^x}(f) - m_{I_{k-1}^x}(f)) + \cdots + (m_{I_2^x}(f) - m_{I_1^x}(f)) + \\ &\quad + (m_{I_1^x}(f) - m_{I_1^y}(f)) + \\ &\quad + (m_{I_1^y}(f) - m_{I_k^y}(f)) + \\ &\quad + (m_{I_k^y}(f) - f(y)). \end{aligned}$$

$$+ \left(m_{I_1^y}(f) - m_{I_2^y}(f) \right) + \cdots + \left(m_{I_{k-1}^y}(f) - m_{I_k^y}(f) \right) \\ + \left(m_{I_k^y}(f) - f(y) \right).$$

Then

$$|f(x) - f(y)| \leq |f(x) - m_{I_k^x}(f)| + \\ + \sum_{l=2}^k \left| m_{I_l^x}(f) - m_{I_{l-1}^x}(f) \right| + \\ + \left| m_{I_1^x}(f) - m_{I_1^y}(f) \right| + \\ + \sum_{l=1}^{k-1} \left| m_{I_l^y}(f) - m_{I_{l+1}^y}(f) \right| + \\ + \left| m_{I_k^y}(f) - f(x) \right| \\ = I + II + III + IV + V.$$

Let us start by bounding the central term III . Notice that $I_1^x = I^-$ and $I_1^y = I^+$, with $|I| = \delta(x, y)$. Then by Lemma 3.2,

$$III = \left| m_{I_1^x}(f) - m_{I_1^y}(f) \right| \\ = |m_{I^-}(f) - m_{I^+}(f)| \\ = 2 |I|^{-\frac{1}{2}} |\langle f, h_I \rangle| \\ \leq 2A |I|^{-\frac{1}{2}} |I|^{\alpha+\frac{1}{2}} \\ = 2A |I|^\alpha \\ = 2A \delta^\alpha(x, y),$$

which has the desired form. The terms II and IV can be handled in the same way, let us deal with II . Take a generic term of the sum II , and use again Lemma 3.2.

$$\left| m_{I_l^x}(f) - m_{I_{l-1}^x}(f) \right| = \left| \frac{1}{|I_l^x|} \int_{I_l^x} f - \frac{1}{|I_{l-1}^x|} \left(\int_{I_l^x} f + \int_{I_{l-1}^x \setminus I_l^x} f \right) \right| \\ = \left| \frac{1}{2} \frac{1}{|I_l^x|} \int_{I_l^x} f - \frac{1}{2} \frac{1}{|I_{l-1}^x \setminus I_l^x|} \int_{I_{l-1}^x \setminus I_l^x} f \right| \\ = \frac{1}{2} \left| m_{I_l^x}(f) - m_{I_{l-1}^x \setminus I_l^x}(f) \right| \\ = \frac{1}{2} 2 |I_{l-1}^x|^{-\frac{1}{2}} \left| \langle f, h_{I_{l-1}^x} \rangle \right| \\ \leq A |I_{l-1}^x|^{-\frac{1}{2}} |I_{l-1}^x|^{\alpha+\frac{1}{2}} \\ = A |I_{l-1}^x|^\alpha$$

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$$= A \frac{2^\alpha}{2^{\alpha l}} |I|^\alpha.$$

Then

$$\begin{aligned} II &= \sum_{l=2}^k \left| m_{I_l^x}(f) - m_{I_{l-1}^x}(f) \right| \\ &\leq A 2^\alpha |I|^\alpha \sum_{l \geq 2} \frac{1}{2^{\alpha l}} \\ &= \frac{A}{2^\alpha - 1} \delta^\alpha(x, y). \end{aligned}$$

Also

$$IV \leq \frac{A}{2^\alpha - 1} \delta^\alpha(x, y).$$

Let $C_\alpha = \sup\{2, \frac{1}{2^\alpha - 1}\}$, then

$$|f(x) - f(y)| \leq |f(x) - m_{I_k^x}(f)| + AC_\alpha \delta^\alpha(x, y) + |f(y) - m_{I_k^y}(f)|$$

uniformly in k . Now, from the differentiation theorem, we have that for almost all x and almost all y ,

$$m_{I_k^x}(f) \longrightarrow f(x); \quad k \rightarrow \infty$$

and

$$m_{I_k^y}(f) \longrightarrow f(y); \quad k \rightarrow \infty.$$

Hence, for those values of x and y in $\mathbb{R}^+$ we get the result

$$|f(x) - f(y)| \leq AC_\alpha \delta^\alpha(x, y).$$

□

Propositions 3.1 and 3.3 prove Theorem 1.1.

REFERENCES

- [1] Hugo Aimar and Ana Bernardis, *Fourier versus wavelets: a simple approach to Lipschitz regularity*, Rev. Un. Mat. Argentina **40** (1996), no. 1-2, 219–224. MR 1450839
- [2] Ingrid Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. MR 1162107
- [3] Matthias Holschneider and Philippe Tchamitchian, *Régularité locale de la fonction “non-différentiable” de Riemann*, Les ondelettes en 1989 (Orsay, 1989), Lecture Notes in Math., vol. 1438, Springer, Berlin, 1990, pp. 102–124, 209–210. MR 1083586
- [4] M. Holschneider and Ph. Tchamitchian, *Pointwise analysis of Riemann’s “nondifferentiable” function*, Invent. Math. **105** (1991), no. 1, 157–175. MR 1109624

Email address: ivanagomez@santafe-conicet.gov.ar

INSTITUTO DE MATEMÁTICA APLICADA DEL LITORAL “DRA. ELEONOR HARBOURE”, CONICET, UNL. IMAL, CCT CONICET SANTA FE, PREDIO “DR. ALBERTO CASSANO”, COLECTORA RUTA NAC. 168 KM 0, PARAJE EL POZO, S3007ABA SANTA FE, ARGENTINA

Submitted: September 14, 2022

Accepted: March 7, 2023

Published (early view): August 22, 2024