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AN IMPROVED LOPSIDED SHIFT-SPLITTING PRECONDITIONER FOR THREE-BY-THREE BLOCK SADDLE POINT PROBLEMS

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Abstract. In this paper, an improved lopsided shift-splitting (ILSS) preconditioner is considered to solve the three-by-three block saddle point problems. This method is an improvement of the work of Zhang et al. [Comput. Appl. Math. (2022),41:261]. We proved that the iteration method produced by the ILSS preconditioner is unconditionally convergent. In addition, we proved that all eigenvalues of the ILSS preconditioned matrix are real and non-unit eigenvalues are located in a positive interval. Numerical experiments show that the ILSS preconditioner is effective.

1. Introduction

In this paper, we focus on the following three-by-three block system of linear equations

$$\hat{\mathcal{A}}\mathrm{u} = \begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \equiv \hat{b}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{p\times m}$ are of full row rank, $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$ and $h \in \mathbb{R}^p$ are given vectors. With the above conditions, the coefficient matrix of linear system (1.1) is non-singular [11, 12, 13]. The linear system (1.1) arises in many scientific computing and engineering applications such as solving quadratic programs [7], least squares problems [9], the Picard iteration method for a class of mixed finite element scheme for stationary magnetohydrodynamics models [8], the finite element method to solve the time-dependent Maxwell equations having discontinuous coefficients in polyhedral domains with Lipschitz boundary [6, 14] and so on.

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It is obvious that (1.1) can equivalently be rewritten in the following form:

$$\mathcal{A}\mathrm{u} = \begin{pmatrix} A & B^T & 0 \\ -B & 0 & -C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ -g \\ h \end{pmatrix} \equiv b, \tag{1.2}$$

which facilitates the possibility to apply classical iteration solution methods. To solve (1.2), many numerical methods have been considered. Huang et al. [12] proposed an exact block diagonal preconditioner, which has the following structure

$$P_{BD} = \begin{pmatrix} A & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & CS^{-1}C^T \end{pmatrix}, \tag{1.3}$$

where $S = BA^{-1}B^T$. And the inexact versions of the block preconditioner were also studied. Then, Cao [5] established the shift splitting (SS) and the relaxed shift splitting (RSS) preconditioner:

$$P_{SS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B^T & 0 \\ -B & \alpha I & -C^T \\ 0 & C & \alpha I \end{pmatrix}, \quad P_{RSS} = \frac{1}{2}\begin{pmatrix} A & B^T & 0 \\ -B & \alpha I & -C^T \\ 0 & C & \alpha I \end{pmatrix}. \tag{1.4}$$

Besides, a generalized shift-splitting (GSS) preconditioner was considered in [17] on the basis of the SS preconditioner. Furthermore, Zhang et al. [20] proposed a lopsided shift-splitting (LSS) preconditioner for saddle point problems (1.2):

$$P_{LSS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B^T & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & \beta I \end{pmatrix}, \tag{1.5}$$

they also proved that the LSS prconditioner is superior to SS, RSS and GSS preconditioners by numerical results.

To further enhance the preconditioning effect of the block diagonal preconditioner (1.3), Xie et al. [19] considered three efficient block preconditioners:

$$P_1 = \begin{pmatrix} A & 0 & 0 \\ B & -S & C^T \\ 0 & 0 & -CS^{-1}C^T \end{pmatrix}, \quad P_2 = \begin{pmatrix} A & 0 & 0 \\ B & -S & C^T \\ 0 & 0 & CS^{-1}C^T \end{pmatrix},$$
$$P_3 = \begin{pmatrix} A & B^T & 0 \\ B & -S & 0 \\ 0 & 0 & -CS^{-1}C^T \end{pmatrix} \tag{1.6}$$

for the linear system (1.1). They showed that the above three preconditioners can significantly improve the convergence speed of GMRES method, and preconditioner P_3 outperforms other two preconditioners in terms of both required CPU time and number of iterations for the convergence. After, Aslani et al. [2] presented a new matrix splitting and deduced a preconditioner for solving saddle point system (1.2). Abdolmaleki et al. [1] proposed a block three-by-three diagonal preconditioner for (1.2). Wang et al. [18] considered an exact parameterized block SPD preconditioner and its inexact version for saddle point problems (1.2). The alternating positive semi-definite splitting iteration methods and corresponding preconditioners were also proposed for solving three-by-three block saddle point problems [3, 16].

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In this paper, in order to improve the preconditioning effect of the LSS preconditioner, we propose an improved LSS (ILSS) preconditioner for solving the linear system (1.2). In the solving process of the residual equations, the coefficient matrices of the linear subsystems corresponding to ILSS preconditioner are more simpler than that of the linear subsystems corresponding to LSS preconditioner, so it has an advantage in solving time. Theoretically, we show the unconditionally convergence of the iteration method produced by the ILSS preconditioner, and the eigenvalue distribution of the ILSS preconditioned matrix will also be discussed. Numerical experiments reveal that the ILSS preconditioner is easy to implement and takes less time compared with some existing preconditioners.

2. The improved lopsided shift-splitting preconditioner

The main purpose of this section is to establish new preconditioner and discuss the convergence of iteration method produced by preconditioner.

Firstly, let us recall the lopsided shift-splitting (LSS) preconditioner. Zhang et al. [20] proposed the LSS preconditioner (1.5) for solving the problems (1.2) in 2022. When it is used to accelerate Krylov subspace methods (such as GMRES method), the following generalized residual equation needs to be solved:

$$P_{LSS} z = r, \tag{2.1}$$

where $r = (r_1; r_2; r_3)$ and $z = (z_1; z_2; z_3)$ are given residual vector and the current vector, respectively, and $r_1,\ z_1 \in \mathbb{R}^n$, $r_2,\ z_2 \in \mathbb{R}^m$, $r_3,\ z_3 \in \mathbb{R}^p$. Thus we have the following algorithmic implementation to solve the linear system (2.1) in actual computation.

LSS algorithm For a given residual vector $r = [r_1^T, r_2^T, r_3^T]^T$, the current vector $z = [z_1^T, z_2^T, z_3^T]^T$. (2.1) can be computed according to the following procedures:
(1). Solve $(\beta I + \frac{1}{\alpha} CC^T) z_3 = 2r_3 - \frac{2}{\alpha} C r_2$;
(2). Solve $z_2 = \frac{1}{\alpha}(C^T z_3 + 2r_2)$;
(3). Solve $(\alpha I + A) z_1 = 2r_1 - B^T z_2$.

We see from above algorithm that only two subsystems need to be solved. In order to further improve the preconditioning effect of the LSS preconditioner, we can simplify the coefficient matrices $\beta I + \frac{1}{\alpha} CC^T$ and $\alpha I + A$ in above algorithm as CC^T and A, respectively, and an extra preconditioner is produced:

$$P = \frac{1}{2}\begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & 0 \end{pmatrix}.$$

Because the per-factor $\frac{1}{2}$ has no effect on the preconditioned linear system in actual computations, it will be neglected and an improved LSS (ILSS) preconditioner is established:

$$P_{ILSS} = \begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & 0 \end{pmatrix}. \tag{2.2}$$

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Therefore, for the preconditioner P_{ILSS}, we can describe the implementing process as in following algorithm.
ILSS algorithm For a given residual vector $r = [r_1^T, r_2^T, r_3^T]^T$, the current vector $z = [z_1^T, z_2^T, z_3^T]^T$, residual equation corresponding to P_{ILSS} can be computed according to the following procedures:
(1). Solve $Az_1 = r_1$;
(2). Solve $CC^T z_3 = \alpha r_3 - Cr_2$;
(3). Solve $z_2 = \frac{1}{\alpha}(C^T z_3 + r_2)$.

By comparing LSS and ILSS algorithms, we see that the preconditioner P_{ILSS} consumes less computational cost than the preconditioner P_{LSS}. Due to the symmetric positive definiteness of the coefficient matrices in linear subsystems, we can choose the Cholesky decomposition or the PCG method to solve linear subsystems at each step for the ILSS algorithm.

In fact, $\mathcal{A}$ admits the following splitting

$$\mathcal{A} = P_{ILSS} - Q_{ILSS} = \begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & 0 \end{pmatrix} - \begin{pmatrix} 0 & -B^T & 0 \\ B & \alpha I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

which results in a matrix splitting iteration method, called the ILSS iteration method, for solving the linear system (1.2).
ILSS iteration method. Let $\alpha > 0$ be a given parameter and $(x_0^T, y_0^T, z_0^T)^T$ be the initial guess vector. For $k = 0, 1, 2, \ldots$ until certain stopping criterion is satisfied, compute

$$\begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & -B^T & 0 \\ B & \alpha I & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} + \begin{pmatrix} f \\ -g \\ h \end{pmatrix}. \tag{2.4}$$

It is easy to rewrite the ILSS iteration method in the following fixed-point form:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix} = \Gamma_\alpha \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} + \Upsilon, \tag{2.5}$$

where

$$\Gamma_\alpha = \begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -B^T & 0 \\ B & \alpha I & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.6}$$

is the iteration matrix and

$$\Upsilon = \begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & 0 \end{pmatrix}^{-1} \begin{pmatrix} f \\ -g \\ h \end{pmatrix}.$$

It is well known that the fixed-point iteration method (2.5) converges to the exact solution $x_* = \mathcal{A}^{-1}b$ for arbitrary initial guess if the spectral radius $\rho(\Gamma_\alpha) < 1$.

Next, we will discuss the convergence of the ILSS iteration method.

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Theorem 2.1. *Assume that $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (SPD) matrix, $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{p\times m}$ are of full row rank. α is a given positive constant. Then the ILSS iteration method is unconditionally convergent.*

Proof. Let λ be an eigenvalue of the iteration matrix Γ_α and $[u; v; w]$ be the corresponding eigenvector. If $\lambda = 0$, the ILSS iteration method is convergent. Next, we only consider the case of $\lambda \neq 0$. It follows according to the relationships between eigenvalues and eigenvectors that

$$\begin{pmatrix} 0 & -B^T & 0 \\ B & \alpha I & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \lambda \begin{pmatrix} A & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{2.7}$$

Eq.(2.7) can equivalently be reformulated into

$$\begin{cases} \lambda A u + B^T v = 0, \\ Bu + \alpha(1-\lambda)v + \lambda C^T w = 0, \\ Cv = 0. \end{cases} \tag{2.8}$$

It is obvious that both u and v are not zero vectors, otherwise, $(u; v; w) = (0; 0; 0)$, which is a contradiction. For simplicity, we can assume that $\|v\| = 1$.

If $\lambda = 1$,

$$\begin{cases} Au + B^T v = 0, \\ Bu + C^T w = 0, \\ Cv = 0. \end{cases} \tag{2.9}$$

Multiplying the second equation in (2.9) from left by v^*, combining with the first and third equation in (2.9) gives $u^* A u = 0$, i.e., $u = 0$, furthermore, it follows that $v = 0$ and $w = 0$, contradiction happens. Thus $\lambda \neq 1$.

Similarly, multiplying the second equation in (2.8) from left by v^*, combining with the first and third equation in (2.8), we have

$$-\bar{\lambda} u^* A u + \alpha(1-\lambda) = 0,$$

i.e.,

$$\alpha\lambda + u^* A u \bar{\lambda} = \alpha,$$

where $\bar{\lambda}$ denotes the conjugate number of λ. Because the matrix A is symmetric positive definite and α is a positive constant, $\bar{\lambda} = \lambda$. Thus it holds that

$$\lambda = \frac{\alpha}{\alpha + u^* A u} < 1.$$

In this case, the ILSS iteration method is convergent.

In conclusion, ILSS iteration method is unconditionally convergent. □

3. Spectral properties of the preconditioned matrix

In this section, we derive the eigenvalue distribution of the preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$.

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Theorem 3.1. *Assume that the conditions in Theorem 2.1 are satisfied. Then the all eigenvalues of preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$ are real numbers and*

$$sp\ (P_{ILSS}^{-1}\mathcal{A}) \subseteq \left(0, \frac{a}{\alpha + a}\right] \cup \{1\},$$

where $sp\ (\cdot)$ represents the spectrum of the corresponding matrix, $\lambda_{min}(A) \leq a \leq \lambda_{max}(A)$.

Proof. Let θ be the eigenvalue of preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$, By $P_{ILSS}^{-1}\mathcal{A} = I - P_{ILSS}^{-1}Q_{ILSS}$, it holds from Theorem 2.1 that

$$\theta = 1 - \lambda = 1 - \frac{\alpha}{\alpha + u^*Au} = \frac{u^*Au}{\alpha + u^*Au}. \tag{3.1}$$

If $\lambda = 0$, the eigenvalue of $P_{ILSS}^{-1}\mathcal{A}$ is 1. It is obvious from (3.1) that θ is a real number if $\lambda \neq 0$. Let $a = u^*Au$, we have

$$0 < \theta \leq \frac{a}{\alpha + a}.$$

This completes the proof. □

Corollary 3.2. *Assume that the conditions in Theorem 2.1 hold. Then the eigenvalues of the ILSS preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$ are either clustered at 1 if the parameter α is close to 0_+ or clustered at points 0 and 1 if $\alpha \to +\infty$.*

Proof. The conclusion is obvious by the expression of (3.1). □

As a matter of fact, the convergence rate of the preconditioned Krylov subspace method is determined not only by the eigenvalue distribution of the preconditioned matrix, but also depends on the eigenvector distribution of the preconditioned matrix. The eigenvector distribution of the preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$ is presented in the following theorem.

Theorem 3.3. *Assume that the conditions of Theorem 2.1 hold. Then the preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$ has $n+i(0 \leq i \leq m+p)$ linearly independent eigenvectors. There are*
(1). n eigenvectors of the form $[u_\ell; 0; w_\ell]$ $(\ell = 1, 2, \ldots, n)$ that correspond to the eigenvalue 1, where w_ℓ $(\ell = 1, 2, \ldots, n)$ are arbitrary linearly independent vectors, $u_\ell \in N(B)$, $N(\cdot)$ denotes the null space of the corresponding matrix.
(2). i $(0 \leq i \leq m + p)$ eigenvectors of the form $[u_\ell; v_\ell; w_\ell]$ $(1 \leq \ell \leq i)$ that correspond to eigenvalues $\theta_\ell \neq 1$, where $0 \neq v_\ell \in N(C)$, $u_\ell = \frac{1}{\theta - 1}A^{-1}B^Tv_\ell$ and $w_\ell = \frac{1}{(\theta-1)^2}(CC^T)^{-1}CBA^{-1}B^Tv_\ell$.

The proof process of above theorem is similar to [10, Theorem 2], we omit it. And we also give an accurate elaboration about the degree of the minimal polynomial of the ILSS preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$.

Theorem 3.4. *Assume that the conditions of Theorem 2.1 hold. Then the dimension of the Krylov subspace $\mathcal{K}(P_{ILSS}^{-1}\mathcal{A}, b)$ is at most $m + p + 1$.*

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4. Numerical experiments

In this section, we give two examples which demonstrates the effectiveness of the proposed preconditioner. In our implementations, the initial guess is chosen to be the zero vector, the exact solution is the vector with all its element being ones and $b = \mathcal{A} \cdot \mathbf{1}$. The iteration is terminated if the current iteration satisfies $\mathrm{RES} = \|b - \mathcal{A}u^{(k)}\|_2 / \|b\|_2 \le 10^{-6}$ or if the prescribed iteration number $k_{\max} = 1500$ is exceeded. Besides, the relative errors of the computed approximations, denoted as ERR, i.e.,

$$\mathrm{ERR} = \frac{\|u_k - u_*\|_2}{\|u_*\|_2}$$

are reported with u_* the initially known exact solution. All experiments are performed in MATLAB 2017(a) on an Intel Core(8G RAM) Windows 10 system.

To compared with ILSS preconditioner, we also test BD preconditioner (1.3), SS preconditioner (1.4), LSS preconditioner (1.5), block preconditioner P_3 (1.6) for solving three-by-three block saddle point problems (1.2). When preconditioners are applied to GMRES method, the coefficient matrices of linear subsystems corresponding to each preconditioner are SPD, we can use sparse Cholesky decomposition to solve them. All of them are used as the left preconditioned GMRES iteration method to solve the three-by-three block saddle point problems.

Example 4.1. [5, 16] Consider the three-by-three block saddle point problems (1.2), in which

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2},$$

$$B = \begin{pmatrix} I \otimes F & F \otimes I \end{pmatrix} \in \mathbb{R}^{p^2 \times 2p^2} \quad \text{and} \quad C = E \otimes F \in \mathbb{R}^{p^2 \times p^2},$$

where

$$T = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{p \times p}, \ F = \frac{1}{h} tridiag(0, 1, -1) \in \mathbb{R}^{p \times p},$$

and $E = \mathrm{diag}(1, p+1, \ldots, p^2 - p + 1)$, $\otimes$ means the Kronecker product symbol and $h = \frac{1}{p+1}$.

Table 1. The CPU times of the ILSS preconditioned GMRES method with various parameter α in different dimensions for Example 4.1.

$p \setminus \alpha$	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
16	0.0195	0.0119	0.0034	0.0019	**0.0015**	0.0022
32	0.0284	0.0178	0.0085	0.0068	**0.0061**	50.7946
48	0.0526	0.0481	0.0318	**0.0302**	0.0376	94.4258
56	0.0372	0.0344	0.0308	**0.0293**	123.5037	-

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Table 2. Numerical results of preconditioned GMRES iteration methods for Example 4.1.

Pre.	I	P_{BD}	P_3	P_{SS}	P_{LSS}	P_{ILSS}
$p=16$						
(α,β)	-	-	-	$(10^{-2},-)$	$(10^{-3},10^{-6})$	$(10^{-4},-)$
IT	865	4	3	2	3	3
RES	8.7e-07	1.5e-10	1.3e-10	7.6e-07	2.3e-09	2.0e-08
ERR	2.3e-06	4.1e-11	3.5e-11	4.9e-06	3.5e-10	1.9e-09
CPU	3.1177	0.0575	0.0213	0.0336	0.0129	0.0103
$p=32$						
(α,β)	-	-	-	$(10^{-2},-)$	$(10^{-3},10^{-6})$	$(10^{-4},-)$
IT	-	4	3	2	2	3
RES	-	1.2e-08	1.4e-08	3.4e-07	9.6e-07	2.4e-07
ERR	-	3.0e-09	3.6e-09	3.5e-06	1.1e-05	1.5e-08
CPU	-	0.5097	0.3952	0.1618	0.0204	0.0186
$p=48$						
(α,β)	-	-	-	$(10^{-2},-)$	$(10^{-3},10^{-6})$	$(10^{-3},-)$
IT	-	4	3	2	2	3
RES	-	7.7e-07	1.3e-09	2.1e-07	6.5e-07	1.0e-07
ERR	-	1.8e-07	3.1e-10	2.8e-06	9.7e-06	5.1e-09
CPU	-	4.7037	3.6303	0.8774	0.0315	0.0296
$p=56$						
(α,β)	-	-	-	$(10^{-2},-)$	$(10^{-3},10^{-6})$	$(10^{-3},-)$
IT	-	4	3	2	2	3
RES	-	9.9e-08	3.2e-07	1.7e-07	5.8e-07	1.7e-07
ERR	-	2.3e-08	7.5e-08	2.6e-06	9.1e-06	8.3e-09
CPU	-	11.1635	9.5971	1.7265	0.0403	0.0379
$p=64$						
(α,β)	-	-		$(10^{-2},-)$	$(10^{-3},10^{-6})$	$(10^{-2},-)$
IT	-	5	3	2	2	3
RES	-	4.9e-07	6.6e-07	1.5e-07	5.4e-07	2.4e-08
ERR	-	1.1e-07	1.5e-07	2.4e-06	8.6e-06	1.1e-09
CPU	-	28.4426	25.4994	3.1639	0.0582	0.0432
$p=80$						
(α,β)	-	-	-	$(10^{-2},-)$	$(10^{-3},10^{-6})$	$(10^{-2},-)$
IT	-	6	5	2	2	3
RES	-	2.5e-07	5.7e-10	1.3e-07	6.9e-07	5.6e-08
ERR	-	5.8e-08	1.1e-10	2.2e-06	7.9e-06	2.2e-09
CPU	-	54.2340	48.7028	9.4477	0.1443	0.1192

Example 4.2. [5, 16] We consider the three-by-three saddle point problems (1.2), in which

$$A = \mathrm{diag}(2W^TW + D_1, D_2, D_3) \in \mathbb{R}^{n\times n}$$

is a block diagonal matrix,

$$B = [E, -I_{2\tilde{p}}, I_{2\tilde{p}}] \in \mathbb{R}^{m\times n},\ C = E^T \in \mathbb{R}^{l\times m}$$

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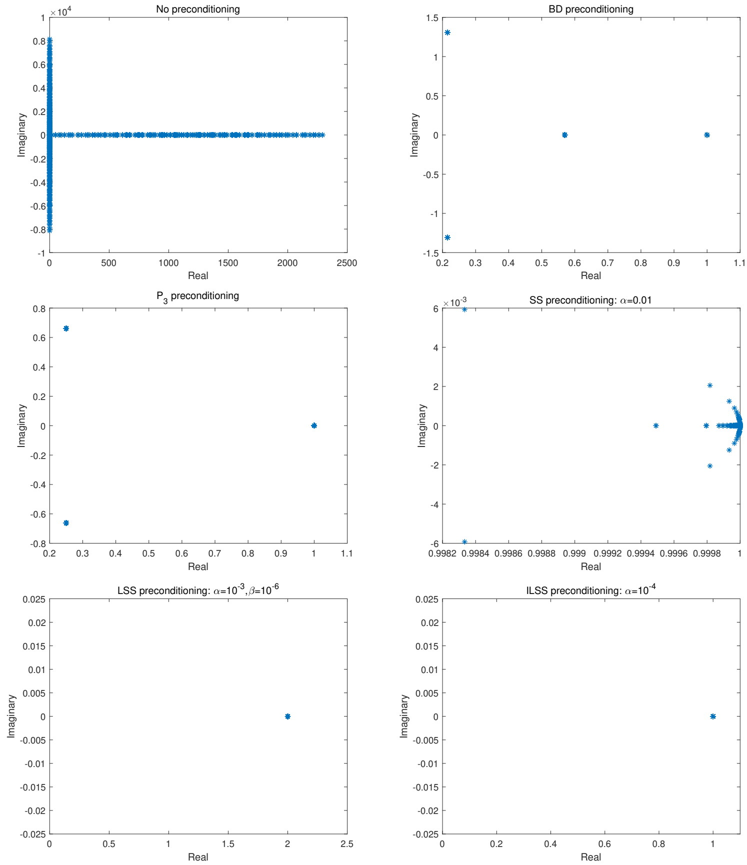

FIGURE 1. The eigenvalues distribution of the preconditioned matrices with $p = 16$ for Example 4.1.

are both full row rank matrices, where $\tilde{p} = p^2$, $\hat{p} = p(p+1)$; $W = (w_{ij}) \in \mathbb{R}^{\hat{p}\times\hat{p}}$ with $w_{ij} = e^{-2((i/3)^2+(j/3)^2)}$; $D_1 = I_{\hat{p}}$ is an identity matrix; $D_i = \mathrm{diag}(d_j^{(i)}) \in \mathbb{R}^{2\tilde{p}\times 2\tilde{p}}$, $i = 2, 3$, are diagonal matrices, with

$$d_j^2 = \begin{cases} 1, \ for\ 1 \le j \le \tilde{p}, \\ 10^{-5}(j-\tilde{p})^2, \ for\ \tilde{p}+1 \le j \le 2\tilde{p}, \end{cases}$$

$$d_j^3 = 10^{-5}(j+\tilde{p})^2, \ for\ 1 \le j \le 2\tilde{p};$$

and

$$E = \begin{pmatrix} \hat{E} \otimes I_p \\ I_p \otimes \hat{E} \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} 2 & -1 & & & \\ & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & & 2 & -1 \end{pmatrix} \in \mathbb{R}^{p\times(p+1)}.$$

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TABLE 3. Numerical results of preconditioned GMRES iteration methods for Example 4.2.

Pre.	I	P_{BD}	P_3	P_{SS}	P_{LSS}	P_{ILSS}
$p=16$						
(α,β)	-	-	-	$(10^{-2},-)$	$(0.6,10^{-2})$	$(10^{7},-)$
IT	1207	6	3	4	22	40
RES	9.5e-07	7.1e-12	1.5e-11	7.4e-08	8.7e-07	8.7e-07
ERR	1.1e-05	1.9e-11	4.1e-11	1.4e-07	8.8e-06	2.3e-06
CPU	1.0345	0.4359	0.3725	0.2705	0.2587	0.2451
$p=32$						
(α,β)	-	-	-	$(10^{-2},-)$	$(0.5,0.1)$	$(10^{8},-)$
IT	1452	6	3	3	16	22
RES	9.8e-07	2.1e-14	4.5e-14	4.9e-07	9.2e-07	9.5e-07
ERR	4.5e-05	5.5e-13	1.2e-12	1.2e-06	2.9e-05	1.8e-05
CPU	23.6692	10.8788	8.6491	5.5974	3.9426	3.4267
$p=48$						
(α,β)	-	-	-	$(10^{-2},-)$	$(0.5,0.1)$	$(10^{8},-)$
IT	-	6	3	3	17	16
RES	-	1.5e-14	9.1e-15	5.9e-07	1.5e-07	7.5e-07
ERR	-	1.8e-12	1.2e-12	1.5e-06	4.5e-05	7.3e-05
CPU	-	119.0142	88.0379	55.1203	24.5532	11.0027
$p=56$						
(α,β)	-	-	-	$(10^{-2},-)$	$(0.5,0.1)$	$(10^{8},-)$
IT	-	6	3	3	17	16
RES	-	5.6e-15	3.2e-15	5.8e-07	3.2e-07	2.9e-07
ERR	-	1.3e-12	7.4e-13	1.6e-06	5.5e-05	5.3e-05
CPU	-	289.4107	215.7352	159.5430	61.4487	23.5894

The experiment problems are formed by setting different values of p. For existing preconditioners, in order to make the preconditioners closer to the coefficient matrix $\mathcal{A}$, we use the parameter selection method in [5, 20]. For ILSS preconditioner, we know by the corollary 3.2 that the parameter α should be as small or big as possible to make the spectrum of preconditioned matrix more clustered, and then accelerate the convergence speed of GMRES method. Therefore, we list the CPU time of preconditioned GMRES method with various α in different dimensions in table 1. We find by table 1 that when α is from 1 to 10^{-5}, the CPU times of the corresponding ILSS preconditioned GMRES method first decreases and then increases. Based on this phenomenon, we choose that $\alpha = 10^{-4}$ when $p = 16,\ 32$; if $p = 48, 56$, $\alpha = 10^{-3}$ and $\alpha = 10^{-2}$ as $p = 64,\ 80$ for Example 4.1. For Example 4.2, we give the tested optimal parameters α for the ILSS preconditioned GMRES method.

In tables 2 and 3, we list the number of iteration steps 'IT', the residuals 'RES', the relative errors 'ERR' and the corresponding CPU times of the preconditioned

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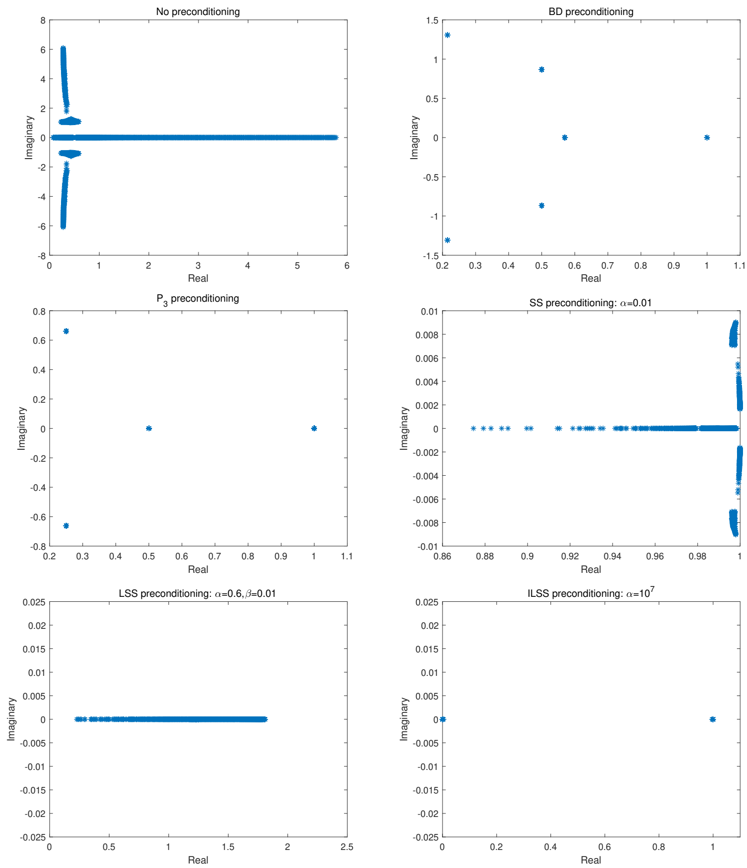

FIGURE 2. The eigenvalues distribution of the preconditioned matrices with $p = 16$ for Example 4.2.

GMRES iteration methods with I (i.e., no preconditioning), P_{BD}, P_3, P_{SS}, P_{LSS} and P_{ILSS} with different p for Examples 4.1 and 4.2.

From these results in table 2, we find that the unpreconditioned GMRES method converges very slowly and it is not convergent if $p \geq 32$. Although P_{BD} and P_3 preconditioned GMRES method has fewer iteration steps, its CPU time increases greatly with the increase of problem sizes. Besides, comparing SS, LSS and ILSS preconditioners, it is easy to find that the iteration steps of the SS and LSS preconditioned GMRES methods are equal, but the CPU time of SS preconditioner is greatly increasing with dimension augmenting. Although the iteration steps of the ILSS preconditioned GMRES method are more than those of the SS and LSS preconditioned GMRES method, ILSS method is the least time-consuming, which also proves that our comparison conclusion from the algorithm is correct. In addition, the approximate solution obtained by GMRES method preconditioned by P_{ILSS} is

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closer to the exact solution than that obtained by GMRES method preconditioned by P_{SS} and P_{LSS}. This fact can be confirmed by the value of 'ERR' in table 2. From table 3, it is easy to find that, although the P_{BD}, P_3 and P_{SS} preconditioned GMRES methods have the least iteration steps, their CPU times increase greatly with the increase of p. On the contrary, the LSS and ILSS preconditioned GMRES methods have more iteration steps, but they take less time than the other three preconditioners. Besides, comparing LSS preconditioner with ILSS preconditioner, we find that the iteration steps of ILSS preconditioned GMRES method are more than that of LSS preconditioned GMRES method when $p \leq 32$, if $p > 32$ it's the other way around; in each case, the CPU time of ILSS method is always less than that of LSS method, and the level of approximating exact solution of iteration solution obtained by two methods is basically the same. The above facts mean that the parameter we select is effective and the ILSS preconditioner is optimal preconditioner for solving three-by-three block saddle point problems (1.2).

In Figs. 1 and 2, we plot the eigenvalues distribution of the original coefficient matrix and the P_{BD}, P_3, P_{SS}, P_{LSS} and P_{ILSS} preconditioned matrices with $p = 16$ for Examples 4.1 and 4.2. It is easy to observe from these figures that the all preconditioners improve the eigenvalues distribution of the original coefficient matrix greatly. In particular, the spectrum of the ILSS preconditioned matrix clustered at point 1 as $\alpha = 10^{-4}$ in fig. 1, however, the spectrum of the ILSS preconditioned matrix clustered at points 0 and 1 as $\alpha = 10^7$ in fig. 2, which not only supports the validity of the theoretical result, but also leads to fast convergence rate of GMRES iteration method. See tables 2 and 3 for detail results.

5. Conclusion and remarks

In this paper, we establish a new improved shift-splitting (ILSS) preconditioner for the three-by-three block saddle point problems (1.2). We show the unconditionally convergence of the ILSS iteration method. And all eigenvalues of the preconditioned matrix $P_{ILSS}^{-1}\mathcal{A}$ are located in a positive real interval. Numerical experiments reveal that this preconditioner has great superiority in suitable parameters compared with other tested preconditioners.

In fact, how to choose the optimal parameters for ILSS preconditioner is the work that needs to be considered, and ILSS preconditioned GMRES iteration method with inexact algorithms is also a key point for solving the largest linear system (1.2), which are very practical and interesting problem that needs further research.

References

[1] M. Abdolmaleki, S. Karimi, D.K. Salkuyeh, A new block-diagonal preconditioner for a class of 3×3 block saddle point problems, Mediterr. J. Math. 19 (2022) 1–15.
[2] H. Aslani, D.K. Salkuyeh, F.P.A. Beik, On the preconditioning of three-by-three block saddle point problems, Filomat 15 (2021) 5181–5194.
[3] H. Aslani, D. K. Salkuyeh, Semi-convergence of the APSS method for a class of nonsymmetric three-by-three singular saddle point problems, (2022) arXiv:2208.00814.
[4] M. Benzi, G.H. Golub, J. Liesen, Numercial solution of saddle point problems, Acta Numer. 14 (2005) 1-137.

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[5] Y. Cao, Shift-splitting preconditioners for a class of block three-by-three saddle point problems, Appl. Math. Lett. 96 (2019) 40–46.

[6] P. Degond, P.A. Raviart, An analysis of the Darwin model of approximation to Maxwell's equations, Forum Math. 4 (1992) 13-44.

[7] D. Han, X. Yuan, Local linear convergence of the alternating direction method of multipliers for quadratic programs, SIAM J. Numer. Anal. 51 (2013) 3446-3457.

[8] K.B. Hu, J.C. Xu, Structure-preserving finite element methods for stationary MHD models, (2017) arXiv:1503.06160.

[9] Q. Liu , C. Chen , Q. Zhang, Perturbation analysis for total least squares problems with linear equality constraint. Appl Numer Math 161 (2021) 69–81.

[10] L.S. Meng, J. Li, S.-X. Miao, A variant of relaxed alternating positive semi-definite splitting preconditioner for double saddle point problems, Japan J. indust. Appl. Math. 38 (2021) 979–998.

[11] N. Huang, Y.-H. Dai, Q. Hu, Uzawa methods for a class of block three-by-three saddle-point problems, Numer Linear Algebra Appl. 26 (2019), e2265.

[12] N. Huang, C.-F. Ma, Spectral analysis of the preconditioned system for the 3×3 block saddle point problem, Numer. Algor. 81 (2019) 421–444.

[13] N. Huang, Variable parameter Uzawa method for solving a class of block three-by-three saddle point problems, Numer. Algor. 85 (2020) 1233-1254.

[14] P. Monk, Analysis of a finite element method for Maxwell's equations, SIAM J. Numer. Anal. 29 (1992) 714–729.

[15] Y. Saad, Iterative Methods for Sparse Linear Systems, 2nd ed. SIAM, Philadephia, 2003.

[16] D.K. Salkuyeh, H. Aslani, Z.Z. Liang, An alternating positive semidefinite splitting preconditioner for the three-by-three block saddle point problems, Math. Commun. 26 (2021) 177–195.

[17] L. Wang, K. Zhang, Generalized shift-splitting preconditioner for saddle point problems with block three-by-three structure, Open Access Library. 06 (2019) 1–14.

[18] N.N. Wang, J.C. Li, On parameterized block symmetric positive definite preconditioners for a class of block three-by-three saddle point problems, J. Comput. Appl. Math. 405 (2022) 113–959.

[19] X. Xie, H.-B. Li, A note on preconditioning for the 3×3 block saddle point problem, Comput. Math. Appl. 79 (2020) 3289–3296.

[20] N. Zhang, R.-X. Li, J. Li, Lopsided shift-splitting preconditioner for saddle point problems with three-by-three structure, Comput. Appl. Math. 41 (2022) 261.

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