

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

RICCI-BOURGUIGNON SOLITONS ON REAL HYPERSURFACES IN THE COMPLEX PROJECTIVE SPACE

IMSOON JEONG AND YOUNG JIN SUH

ABSTRACT. In this paper, we give a complete classification of Ricci-Bourguignon soliton on real hypersurfaces in the complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1 \cdot U_n)$. Next, as an application, we give some non-existence properties for *gradient* Ricci-Bourguignon soliton on real hypersurfaces with isometric Reeb flow and contact real hypersurfaces in the complex projective space $\mathbb{C}P^n$.

1. INTRODUCTION

Among the class of Hermitian symmetric spaces with rank 1 of compact type, we have complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1 \cdot U_n)$, which is geometrically quite different from the case of rank 2. It has a Kähler structure and Fubini-Study metric g of constant holomorphic sectional curvature 4 (see Romero [26], [27], and Smyth [28]). The complex projective space $\mathbb{C}P^n$ is considered as a kind of real Grassmann manifolds of compact type with rank 1 (see Kobayashi and Nomizu [20]). In Hermitian symmetric space of rank 2, Jeong and Suh [18] gave a classification of Ricci soliton real hypersurfaces the complex two-plane Grassmannian $G_2(\mathbb{C}^{n+2})$, and the other geometric properties of $G_2(\mathbb{C}^{n+2})$ are investigated in Suh [29], [30], and [31].

Recently, Yamabe solitons and Ricci solitons on almost co-Kähler manifolds and three dimensional $N(k)$ -contact manifolds have been investigated by Chaubey-De-Suh [11] and [13]. Moreover, the study of the Yamabe flow was initiated in the work of Hamilton [17], Morgan and Tian [21] and Perelman [24] as a geometric method to construct Yamabe metrics on Riemannian manifolds.

A time dependent metric $g(t)$ on a Riemannian manifold M is said to be evolved by the Yamabe flow if the metric g satisfies

$$\frac{\partial}{\partial t}g(t) = -\gamma g(t), \quad g(0) = g_0$$

2020 *Mathematics Subject Classification.* Primary 53C40. Secondary 53C55.

Key words and phrases. Ricci-Bourguignon soliton, gradient Ricci-Bourguignon soliton, pseudo-anti commuting, pseudo-Einstein, complex projective space.

The first author was supported by grant Proj. No. NRF-2021-R1F1A-1064192, and the second by NRF-2018-R1D1A1B-05040381 from National Research Foundation of Korea.

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

on M , where γ denotes the scalar curvature on M . From such a view point, in this paper we want to give a complete classification of Yamabe solitons and gradient Yamabe solitons on Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$.

On the other hand, it is well known that there exist two focal submanifolds of real hypersurfaces in Hermitian symmetric spaces of compact type and only one focal submanifold in Hermitian symmetric spaces of non-compact type (see Berndt and Suh [1], and Helgason [16]). Since the complex projective space $\mathbb{C}P^n$ is an Hermitian symmetric space of compact type, any real hypersurface has two focal submanifolds (see Djorić and Okumura [14], Pérez [25]). Among them we consider two kinds of real hypersurfaces in $\mathbb{C}P^n$ with isometric Reeb flow or contact hypersurfaces. In $\mathbb{C}P^n$, Cecil and Ryan [9], and Okumura [22] gave a classification of real hypersurfaces with isometric Reeb flow as follows:

Theorem A. Let M be a real hypersurface of the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube of radius $0 < r < \frac{\pi}{2}$ around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^n$ for some $k \in \{0, \dots, n-1\}$ or a tube with radius $\frac{\pi}{2} - r$ over $\mathbb{C}P^\ell$, where $k + \ell = n - 1$.

When a real hypersurface M in the complex projective space $\mathbb{C}P^n$ satisfies the formula $A\phi + \phi A = k\phi$, $k \neq 0$ and constant, we say that M is a *contact* real hypersurface in $\mathbb{C}P^n$. In the papers due to Blair [2], Okumura [22] and Yano and Kon [35], they introduce the classification of contact real hypersurfaces in $\mathbb{C}P^n$ as follows:

Theorem B. Let M be a connected orientable real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is a contact real hypersurface if and only if M is congruent to an open part of a tube of radius $0 < r < \frac{\pi}{4}$ around an n -dimensional real projective space $\mathbb{R}P^n$ or a tube of radius $\frac{\pi}{4} - r$ over Q^{n-1} , where $0 < r < \frac{\pi}{4}$.

Motivated by these results, in this paper we give some characterizations of real hypersurfaces in the complex projective space $\mathbb{C}P^n$ regarding a family of geometric flows introduced by J.P. Bourguignon. We call this the *Ricci-Bourguignon flow*, which generalize the Ricci flow. It is an intrinsic geometric flow on Riemannian manifolds, whose fixed points are solitons. Indeed, we know that a solution of the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2\text{Ric}(g(t))$ is given by

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \text{Ric}(X, Y) = \Omega g(X, Y),$$

where the function Ω is constant and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Chaubey-Suh-De [12], Morgan-Tian [21], Perelman [24], Wang [33] and [34]). Then this solution (M, V, Ω, g) is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant Ω . In the complex two-plane Grassmannian $G_2(\mathbb{C}^{n+2})$, Jeong and Suh [18] gave a classification of Ricci soliton for real hypersurfaces.

Submitted: January 6, 2023

Accepted: August 15, 2023

Published (early view): August 24, 2024

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

As a generalization of the notion of Ricci flow, the Ricci-Bourguignon flow (see Bourguignon [4] and [5], Catino-Cremaschi-Djadli-Mantegazza-Mazzieri [8]) is given by

$$\frac{\partial}{\partial t}g(t) = -2(\text{Ric}(g(t)) - \theta\gamma g(t)), \quad g(0) = g_0.$$

This family of geometric flows with $\theta = 0$ reduces to the Ricci flow $\frac{\partial}{\partial t}g(t) = -2\text{Ric}(g(t))$, $g(0) = g_0$. If the constant $\theta = \frac{1}{2}$, it is said to be *Einstein flow*. The critical point of the following Einstein flow

$$\frac{\partial}{\partial t}g(t) = -2(\text{Ric}(g(t)) - \frac{1}{2}\gamma g(t)), \quad g(0) = g_0,$$

implies that the Einstein gravitational tensor $\text{Ric}(g(t)) - \frac{1}{2}\gamma g(t)$ vanishes. For a four-dimensional space time M^4 , this is equivalent to the vanishing Ricci tensor by virtue of $d\gamma = 2\text{div}(\text{Ric})$. In this case M^4 becomes vacuum. That is, $g(t) = g(0)$, the metric is constant along the time (see O'Neill [23]). For $\theta = \frac{1}{n}$, the tensor $\text{Ric} - \frac{\gamma}{n}g$ is said to be traceless Ricci tensor, and for $\theta = \frac{1}{2(n-1)}$, it is said to be the Schouten tensor.

Now let us introduce Ricci-Bourguignon soliton $(M, V, \Omega, \theta, \gamma, g)$ which is a solution of the Ricci-Bourguignon flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \text{Ric}(X, Y) = (\Omega + \theta\gamma)g(X, Y), \quad (1.1)$$

for any tangent vector fields X and Y on M , where Ω is a soliton constant, θ any constant and γ the scalar curvature on M , and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan-Tian [21]). Then (M, g) is said to be a *Ricci-Bourguignon soliton* with potential vector field V and Ricci-Bourguignon soliton constant Ω . In recent years, many authors studied the Ricci-Bourguignon soliton on Riemannian manifolds. In [7], Catino-Mazzieri-Mongodi classified non-compact gradient Ricci-Bourguignon soliton $(M, \nabla f, \Omega, \theta, \gamma, g)$ with bounded non-negative sectional curvature. Moreover, Blaga-Tastan [3] and Dwivedi [15] obtained some results on Ricci-Bourguignon solitons and almost Ricci-Bourguignon solitons on Riemannian manifolds.

On the other hand, when the Reeb vector field ξ satisfies $A\xi = \alpha\xi$ for the shape operator A on a real hypersurface M in the complex projective space $\mathbb{C}P^n$, M is said to be *Hopf*. By using this notion, in section 4 it can be easily seen that a Hopf Ricci-Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ in the complex projective space $\mathbb{C}P^n$ also satisfies the generalized pseudo-anti commuting property mentioned above.

Now, we want to introduce some backgrounds for the study of real hypersurfaces in the complex projective space $\mathbb{C}P^n$, which will be used a main key to give our theorems.

If the Ricci operator Ric of a real hypersurface M in $\mathbb{C}P^n$ satisfies

$$\text{Ric}(X) = aX + b\eta(X)\xi \quad (1.2)$$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

for smooth functions a, b on M , then M is said to be *pseudo-Einstein*. Then by virtue of the classification for pseudo-Einstein Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$ due to Cecil and Ryan [9] and Propositions 3.3 and 3.4 in section 3, we introduce the following:

Theorem C. Let M be a pseudo-Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is locally congruent to one of the following:

- (i) a geodesic hypersphere, $a = 2n + 2(n-1)\cot^2(r)$, and $b = -2n$,
- (ii) a tube of radius r around a totally geodesic CP^k , $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$, $\cot^2 r = \frac{k}{n-k-1}$, $a = 2n$, and $b = -2$,
- (iii) a tube of radius r around a complex quadric Q^{n-1} where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = n-2$, $a = 2n$, and $b = -2n-1$.

Now let us consider an Einstein hypersurface in the complex quadric $\mathbb{C}P^n$. Then the Ricci tensor of M becomes $\text{Ric} = \lambda g$. In case (i) in above Theorem C, there do not exist any Einstein hypersurfaces in $\mathbb{C}P^n$, because the case (i) has two distinct constant principal curvatures, the function $b = -2n$ in Theorem C is non-vanishing. Moreover, the tubes in (ii) and (iii) are known to be 3 distinct constant principal curvatures. So it can be easily checked that the smooth functions $b = -2$ for (ii), and $b = -2(n-1)$ for (iii) in Theorem C are not vanishing (see Cecil-Ryan [9] and Takagi [32]). From such a view point we can conclude the following

Theorem D. There does not exist an Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$.

In section 4, by using the notion of generalized pseudo-anti commuting property, we will show that a Hopf Ricci-Bourguignon soliton real hypersurface in the complex projective space $\mathbb{C}P^n$ satisfies generalized pseudo-anti commuting property. Then by virtue of the classification due to Ki and Suh [19], we introduce a classification given in Proposition 4.1. After doing this, we should check whether a geodesic hypersphere of type A_1 , a pseudo-Einstein real hypersurface in type A_2 or of type B could admit Ricci-Bourguignon soliton or not. Then we can assert the following:

Theorem 1. There does not exist a Hopf Ricci-Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$ in the complex projective space $\mathbb{C}P^n$, $n \geq 3$.

By virtue of non-existence theorem of a Hopf Ricci-Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$ in the complex projective space $\mathbb{C}P^n$, we can assert the following

Corollary 1.1. *There does not exist a Hopf Ricci soliton (M, ξ, Ω, g) in the complex projective space $\mathbb{C}P^n$, $n \geq 3$.*

Now let us denote by Df the gradient vector field of the function f on M defined by $g(Df, X) = g(\text{grad} f, X) = X(f)$ for any tangent vector field X on M .

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

We consider the *gradient Ricci-Bourguignon soliton* $(M, Df, \Omega, \theta, \gamma, g)$ (see Catino-Mazzieri [6], Cernea-Guan [10]) defined by

$$\text{Hess}(f) + \text{Ric} = (\Omega + \theta\gamma)g,$$

where $\text{Hess}(f)$ is defined by $\text{Hess}(f) = \nabla Df$ for any tangent vector fields X and Y on M in such a way that

$$\text{Hess}(f)(X, Y) = g(\nabla_X Df, Y).$$

Then the gradient Ricci-Bourguignon soliton can be given by

$$\nabla_X Df + \text{Ric}(X) = (\Omega + \theta\gamma)X \quad (1.3)$$

for any vector field X tangent to M in $\mathbb{C}P^n$. Then first by virtue of Theorem A we can give a non-existence theorem for gradient Ricci-Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ as follows:

Theorem 2. There does not exist a real hypersurface with isomeric Reeb flow in the complex projective space $\mathbb{C}P^n$, $n \geq 3$, admitting the gradient Ricci-Bourguignon soliton.

Next by Theorem B for a contact real hypersurface in the complex projective space $\mathbb{C}P^n$, we can assert the following for gradient Ricci-Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$.

Theorem 3. There does not exist a contact real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$, admitting the gradient Ricci-Bourguignon soliton.

2. THE COMPLEX PROJECTIVE SPACE

This section is due to Berndt and Suh [1]. Let $(\bar{M}, g, J)$ be a Kähler manifold and $\bar{R}$ the Riemannian curvature tensor of $(\bar{M}, g)$. Since $\bar{\nabla}J = 0$, we immediately see that

$$\bar{R}(X, Y)JZ = J\bar{R}(X, Y)Z$$

holds for all $X, Y, Z \in T_x(\bar{M})$, $x \in \bar{M}$. From the curvature identities in Kobayashi and Nomizu [20] we also get

$$g(\bar{R}(X, Y)Z, W) = g(\bar{R}(JX, JY)Z, W) = g(\bar{R}(X, Y)JZ, JW).$$

Let $G_2^J(T\bar{M})$ be the Grassmann bundle over $\bar{M}$ consisting of all 2-dimensional J -invariant linear subspaces V of $T_p\bar{M}$, $p \in \bar{M}$. Thus every $V \in G_2^J(T\bar{M})$ is a complex line in the corresponding tangent space of $\bar{M}$. The restriction of the section curvature function K to $G_2^J(T\bar{M})$ is called the *holomorphic sectional curvature function* on $\bar{M}$ and $K(V)$ is called the *holomorphic sectional curvature* of $\bar{M}$ with respect to $V \in G_2^J(T\bar{M})$.

A Kähler manifold M is said to have *constant holomorphic sectional curvature* if the holomorphic sectional curvature function is constant. Now we want to introduce the following.

Submitted: January 6, 2023

Accepted: August 15, 2023

Published (early view): August 24, 2024

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Theorem 2.1. *A Kähler manifold $(\bar{M}, g, J)$ has constant holomorphic sectional curvature $c \in \mathbb{R}$ if and only if its Riemannian curvature tensor $\bar{R}$ is of the form*

$$\begin{aligned} \bar{R}(X, Y)Z = & \frac{c}{4} \left\{ g(Y, Z)X - g(X, Z)Y \right. \\ & \left. + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \right\} \end{aligned}$$

for any vector fields X, Y and Z on $\bar{M}$.

The complex vector space $\mathbb{C}^n$ ($n \in \mathbb{N}$) is in a canonical way an n -dimensional complex manifold. For $p \in \mathbb{C}^n$ denote by $\pi_p : T_p \mathbb{C}^n \rightarrow \mathbb{C}^n$ the canonical isomorphism. We define a Riemannian metric g on $\mathbb{C}^n$ by

$$g_p(u, v) = \langle \pi_p(u), \pi_p(v) \rangle$$

for all $u, v \in T_p \mathbb{C}^n$ and $p \in \mathbb{C}^n$, where $\langle \cdot, \cdot \rangle$ is the real part of the standard Hermitian inner product on $\mathbb{C}^n$, that is,

$$\langle a, b \rangle = \operatorname{Re} \left(\sum_{\nu=1}^n a_\nu \bar{b}_\nu \right) \quad (a, b \in \mathbb{C}^n).$$

The metric g is called the *canonical Riemannian metric on $\mathbb{C}^n$* . The complex structure J on $\mathbb{C}^n$ is given by the equation $\pi_p(Ju) = i\pi_p(u)$. It is easy to verify that $(\mathbb{C}^n, g, J)$ is a Kähler manifold. In fact, $(\mathbb{C}^n, g, J)$ a complex Euclidean space with vanishing constant holomorphic sectional curvature. The Kähler manifold $(\mathbb{C}^n, g, J)$ is known to be the *n -dimensional complex Euclidean space*.

We define an equivalence relation $\sim$ on $\mathbb{C}^{n+1} \setminus \{0\}$ by $z_1 \sim z_2$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ so that $z_2 = z_1 \lambda$. We denote the quotient space $\mathbb{C}^{n+1} \setminus \{0\} / \sim$ by $\mathbb{C}P^n$. By construction, the points in $\mathbb{C}P^n$ are in one-to-one correspondence with the complex lines through $0 \in \mathbb{C}^{n+1}$. We equip $\mathbb{C}P^n$ with the quotient topology with respect to the canonical projection $\tau : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}P^n$. Then $\mathbb{C}P^n$ is a compact Hausdorff space and τ is a continuous map. There exists a unique complex manifold structure on $\mathbb{C}P^n$ so that τ is a holomorphic submersion. In this way $\mathbb{C}P^n$ becomes an n -dimensional complex manifold $(\mathbb{C}P^n, J)$. For $z \in \mathbb{C}^{n+1} \setminus \{0\}$ we also write $[z] = \tau(z) \in \mathbb{C}P^n$.

Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ and denote by π the restriction of τ to S^{2n+1} . We consider S^{2n+1} with the Riemannian metric induced from $\mathbb{C}^{n+1}$, which is the standard metric on S^{2n+1} turning it into a real space form with constant sectional curvature 1. The map $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$ is a surjective submersion whose fibers are 1-dimensional circles. There exists a unique Riemannian metric g on $\mathbb{C}P^n$ so that π becomes a Riemannian submersion. In such a way, the map $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$ is known as the *Hopf map* from S^{2n+1} onto $\mathbb{C}P^n$ and the Riemannian metric g is known as the *Fubini-Study metric* on $\mathbb{C}P^n$. The manifold $(\mathbb{C}P^n, J, g)$ is a Kähler manifold and called the *n -dimensional complex projective space*. The complex projective space $(\mathbb{C}P^n, J, g)$ is a complex space form with constant holomorphic sectional curvature 4.

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

By virtue of Theorem 2.1, the Riemannian curvature tensor $\bar{R}$ of $\mathbb{C}P^n$ can be given for any vector fields X, Y and Z in $T_z(\mathbb{C}P^n)$, $z \in \mathbb{C}P^n$ as follows:

$$\bar{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ.$$

3. SOME GENERAL EQUATIONS

Let M be a real hypersurface in the complex projective space $\mathbb{C}P^n$ and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M . Then the vector field ξ is said to be the *Reeb* vector field on M in $\mathbb{C}P^n$. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM . The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

In another way, the complex projective space $\mathbb{C}P^n$ is defined by using the fibration

$$\tilde{\pi} : S^{2n+1}(1) \rightarrow \mathbb{C}P^n, \quad z \rightarrow [z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for a real hypersurface in the complex projective space $\mathbb{C}P^n$ as follows:

$$\begin{array}{ccc} M' = \tilde{\pi}^{-1}(M) & \xrightarrow{\tilde{i}} & S^{2n+1}(1) \subset \mathbb{C}^{n+1} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{i} & \mathbb{C}P^n \end{array}$$

We now assume that M is a Hopf hypersurface. Then we have

$$A\xi = \alpha\xi,$$

where A denotes the shape operator of M in $\mathbb{C}P^n$ and the smooth function α is defined by $\alpha = g(A\xi, \xi)$ on M . When we consider the transformed vector field JX by the Kähler structure J on $\mathbb{C}P^n$ for any vector field X on M in $\mathbb{C}P^n$, we may write

$$JX = \phi X + \eta(X)N.$$

Then by using Kähler structure $\nabla J = 0$, we get the following

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \text{ and } \nabla_X \xi = \phi AX.$$

Now we consider the equation of Codazzi

$$g((\nabla_X A)Y - (\nabla_Y A)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y).$$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in $\mathbb{C}P^n$ induced from the curvature tensor $\bar{R}$ of $\mathbb{C}P^n$ can be described in terms of the almost contact structure tensor ϕ and the shape operator A of M in $\mathbb{C}P^n$ as follows:

$$\begin{aligned} R(X, Y)Z = & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \quad (3.1)$$

for any vector fields $X, Y, Z \in T_z M$, $z \in M$. From this, contracting Y and Z on M in $\mathbb{C}P^n$, we get the Ricci tensor of a real hypersurface M in $\mathbb{C}P^n$ as follows:

$$\text{Ric}(X) = (2n + 1)X - 3\eta(X)\xi + (\text{Tr}A)AX - A^2X. \quad (3.2)$$

Then by contracting the Ricci operator in (3.2) the scalar curvature γ of M in $\mathbb{C}P^n$ is given by

$$\gamma = \sum_{i=1}^{2n-1} g(\text{Ric}(e_i), e_i) = 4(n^2 - 1) + h^2 - \text{Tr}A^2, \quad (3.3)$$

where the function h denotes the trace of the shape operator A of M in $\mathbb{C}P^n$.

Putting $Z = \xi$ in the Codazzi equation, we get

$$g((\nabla_X A)Y - (\nabla_Y A)X, \xi) = -2g(\phi X, Y).$$

Since we have assumed that M is Hopf in $\mathbb{C}P^n$, differentiating $A\xi = \alpha\xi$ gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.$$

From this, the left side of the above equation becomes

$$\begin{aligned} & g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ = & g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ = & (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

Putting $X = \xi$ in above two equations and using the almost contact structure of (M, g) , we have

$$Y\alpha = (\xi\alpha)\eta(Y).$$

Inserting this formula into two previous equation implies

$$0 = 2g(A\phi AX, Y) - \alpha g((\phi A + A\phi)X, Y) - 2g(\phi X, Y).$$

By virtue of this equation, we can assert the following

Lemma 3.1. *Let M be a Hopf real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then we obtain*

$$2A\phi AX = \alpha(A\phi + \phi A)X + 2\phi X$$

for any tangent vector field X on M .

In the proof of our Theorems 1 and 2, we want to give more information about Hopf real hypersurfaces in the complex projective space. By using the formulas given in section 3 we want to introduce an important lemma due to Okumura [22] and Yano and Kon [35] as follows:

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Lemma 3.2. *Let M be a Hopf real hypersurface in $\mathbb{C}P^n$. Then the Reeb function α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ , then $2\lambda \neq \alpha$ and ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$ on M .*

Now by using (3.2) and (3.3), we introduce an important proposition due to Cecil-Ryan [9], Djorić and Okumura [14] as follows:

Proposition 3.3. *Let M be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^k$, $k \in \{1, \dots, n-2\}$ in $\mathbb{C}P^n$. Then the following statements hold:*

- (1) M is a Hopf hypersurface.
- (2) The principal curvatures and corresponding principal curvature spaces of M are given by

principal curvature	eigenspace	multiplicity
$\lambda = \cot(r)$	T_λ	2ℓ
$\mu = -\tan(r)$	T_μ	$2k$
$\alpha = 2 \cot(2r)$	$T_\alpha = \mathbb{R}JN$	1

where $\ell = n - k - 1$.

- (3) The shape operator A commutes with the structure tensor field ϕ as

$$A\phi = \phi A.$$

- (4) The trace h of the shape operator A and its square h^2 becomes the following respectively

$$h = (2\ell + 1)\cot(r) - (2k + 1)\tan(r),$$

$$h^2 = (2\ell + 1)^2\cot^2(r) + (2k + 1)^2\tan^2(r) - 2(2\ell + 1)(2k + 1).$$

- (5) The trace of the matrix A^2 is given by

$$\text{Tr} A^2 = (2\ell + 1)\cot^2(r) + (2k + 1)\tan^2(r) - 2.$$

- (6) The scalar curvature γ of the tube M is given by

$$\gamma = 4(n - 1)n - 8k\ell + 2(2\ell + 1)\ell\cot^2(r) + 2(2k + 1)k\tan^2(r).$$

- (7) M is pseudo-Einstein, that is,

$$\text{Ric}(X) = 2nX - 2\eta(X)\xi$$

$$\text{for } \cot^2(r) = \frac{k}{n-k-1}.$$

The tube of radius r around totally geodesic and totally real projective space $\mathbb{R}P^n$ has therefore three distinct constant principal curvatures $2\tan(2r)$, $-\cot(r)$, and $\tan(r)$. It also can be regarded as a tube of radius $\frac{\pi}{4} - r$ over a totally geodesic complex quadric Q^{n-1} . Then by (3.2) and (3.3), we want to give an important proposition due to Berndt and Suh [1], Cecil and Ryan [9], and Takagi [32] as follows:

Submitted: January 6, 2023

Accepted: August 15, 2023

Published (early view): August 24, 2024

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Proposition 3.4. *Let M be the tube of radius $0 < r < \frac{\pi}{4}$, around the complex quadric Q^{n-1} which is a complex hypersurface in $\mathbb{C}P^n$. Then the following statements hold:*

- (1) M is a Hopf hypersurface.
- (2) The principal curvatures and corresponding principal curvature spaces of M are

principal curvature	eigenspace	multiplicity
$\lambda = -\cot(\frac{\pi}{4} - r)$	T_λ	$n - 1$
$\mu = \tan(\frac{\pi}{4} - r)$	T_μ	$n - 1$
$\alpha = 2 \cot(2r)$	$\mathbb{R}JN$	1

- (3) The shape operator A and the structure tensor field ϕ satisfy

$$A\phi + \phi A = k\phi, \quad k \neq 0 : \text{const.}$$

- (4) The trace h of the shape operator A and its square h^2 becomes the following respectively

$$\begin{aligned} h &= \text{Tr}A = 2\cot(2r) - 2(n-1)\tan(2r), \\ h^2 &= 4\cot^2(2r) + 4(n-1)^2\tan^2(2r) - 8(n-1). \end{aligned}$$

- (5) The trace of the matrix A^2 is given by

$$\text{Tr}A^2 = 4\cot^2(2r) + 4(n-1)\tan^2(2r).$$

- (6) The scalar curvature γ of the tube M is given by

$$\gamma = 4(n-1)^2 + 4(n-1)(n-2)\tan^2(2r).$$

- (7) M is pseudo-Einstein, that is,

$$\text{Ric}(X) = 2nX - 2(n-1)\eta(X)\xi$$

$$\text{for } \cot^2(2r) = n - 2.$$

4. HOPF RICCI-BOURGUIGNON SOLITON REAL HYPERSURFACES

Now let us introduce Ricci-Bourguignon soliton $(M, V, \Omega, \theta, \gamma, g)$ which is a solution of the Ricci-Bourguignon flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \text{Ric}(X, Y) = (\Omega + \theta\gamma)g(X, Y),$$

for any tangent vector fields X and Y on M , where Ω is a Ricci-Bourguignon soliton constant, θ any constant and γ the scalar curvature on M , and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan-Tian [21]). Then let us consider the Reeb vector field ξ as the Ricci-Bourguignon soliton vector field V as follows:

$$\frac{1}{2}(\mathcal{L}_\xi g)(X, Y) + \text{Ric}(X, Y) = (\Omega + \theta\gamma)g(X, Y), \quad (4.1)$$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Then by virtue of the Lie derivative $(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$, the formula (4.1) can be given by

$$\text{Ric}(X) = \frac{1}{2}(A\phi - \phi A)X + (\Omega + \theta\gamma)X. \quad (4.2)$$

From this, by applying the structure tensor ϕ to both sides, we get the following two formulas

$$\text{Ric}(\phi X) = \frac{1}{2}(A\phi^2 - \phi A\phi)X + (\Omega + \theta\gamma)\phi X,$$

and

$$\phi \text{Ric}(X) = \frac{1}{2}(\phi A\phi - \phi^2 A)X + (\Omega + \theta\gamma)\phi X.$$

By using the almost contact structure (ϕ, ξ, η, g) in the right side above, we know that the *generalized pseudo-anti commuting property* holds as follows:

$$\text{Ric}(\phi X) + \phi \text{Ric}(X) = 2(\Omega + \theta\gamma)\phi X. \quad (4.3)$$

Now we want to introduce an important proposition due to Ki and Suh [19], and Yano and Kon [35], which will be used in the proof of our Theorem 1 as follows:

Proposition 4.1. *Let M be a connected complete Hopf real hypersurface in the complex projective space $\mathbb{C}P^n$. If M satisfies the generalized pseudo-anti commuting property, then M is locally congruent to a geodesic hypersphere in type A_1 , a pseudo-Einstein hypersurface in type A_2 , or M is locally congruent to of type B .*

Among real hypersurfaces of type A_2 satisfying the generalized pseudo-anti commuting property (4.2) is only pseudo-Einstein. Then it is exactly the second case in Theorem C. That is M is locally congruent to a tube of radius r around a totally geodesic CP^k , $0 < k < n - 1$, where $0 < r < \frac{\pi}{2}$ and $\cot^2(r) = \frac{k}{n-k-1}$.

Now geodesic hyperspheres and pseudo-Einstein real hypersurfaces are included in the class of type A_1 and A_2 respectively. So by Theorem A, they are characterized by the commuting shape operator. That is, $A\phi = \phi A$. Accordingly, from the notion of Ricci-Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$ of M , (4.1) becomes

$$\text{Ric} = (\Omega + \theta\gamma)g.$$

This mean that those hypersurfaces are Einstein. But by Theorem D there does not exist such hypersurfaces in complex projective space $\mathbb{C}P^n$.

Next, in the remained case let us check real hypersurfaces of type B in $\mathbb{C}P^n$. It is characterized by $A\phi + \phi A = k\phi$, where $k \neq 0$: constant. Moreover, by Proposition 3.4, the principal curvature are given by $\lambda = -\cot(\frac{\pi}{4} - r)$, $\mu = \tan(\frac{\pi}{4} - r)$ and $\alpha = 2\cot(2r)$. So $k = \lambda + \mu = -\frac{4}{\alpha}$. For any $X \in T_\lambda$ the vector field $\phi X \in T_\mu$. So the Ricci-Bourguignon soliton (4.1) gives the following

$$(-\mu - \frac{2}{\alpha})\phi X + \text{Ric}X = (\Omega + \theta\gamma)X$$

for any $X \in T_\lambda$. Since the first term is skew-symmetric, by taking symmetric part the first term vanishes. So it becomes $\text{Ric}(X) = (\Omega + \theta\gamma)X$ for any $X \in T_\lambda$.

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Similarly, for any $Y \in T_\mu$ we get $\text{Ric}(Y) = (\Omega + \theta\gamma)Y$. Of course, if we put $X = \xi$ in (4.2), we get $\text{Ric}(\xi) = (\Omega + \theta\gamma)\xi$. Consequently, $\text{Ric} = \lambda g$. That is, it is Einstein. But by Theorem D there does not exist such a type B on a hypersurface in complex projective space $\mathbb{C}P^n$. From this fact we can assert our Main Theorem 1 in the introduction.

5. GRADIENT RICCI-BOURGUIGNON SOLITON ON ISOMETRIC REEB FLOW IN $\mathbb{C}P^n$

In this section let M be a tube of radius r , $0 < r < \frac{\pi}{2}$, over a totally geodesic $\mathbb{C}P^k$, $k \in \{0, 1, \dots, n-2, n-1\}$ in $\mathbb{C}P^n$, which is said to be of type A_1 and of type A_2 . In Theorem A, we have mentioned that the Reeb flow on M in $\mathbb{C}P^n$ is isometric if and only if M is locally congruent to a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for $k \in \{0, 1, \dots, n-1\}$. Then for $k = 0$ or $k = n-1$ we say that M is a geodesic hypersphere which is said to be of type A_1 and it has with two distinct principal curvatures. For $k \in \{1, \dots, n-2\}$, M is locally congruent to a tube over $\mathbb{C}P^k$ in $\mathbb{C}P^n$. Moreover, it is said to be of type A_2 and has with three distinct constant principal curvatures.

Then the shape operator of M in the complex projective space $\mathbb{C}P^n$ with isometric Reeb flow can be expressed as

$$A = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cot(r) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cot(r) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\tan(r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan(r) \end{bmatrix}$$

for three constant principal curvatures $\alpha = 2\cot(2r)$, $\cot(r)$ and $-\tan(r)$ with multiplicities 1, 2ℓ and $2k$ respectively, where $\ell = n - k - 1$.

Then, by putting $X = \xi$ in (3.2), and using $A\xi = \alpha\xi$, we have the following

$$\begin{aligned} \text{Ric}(\xi) &= (2n+1)\xi - 3\xi + hA\xi - A^2\xi \\ &= 2(n-1)\xi + (h\alpha - \alpha^2)\xi \\ &= \kappa\xi, \end{aligned}$$

where we have put $\kappa = 2(n-1) + h\alpha - \alpha^2$. So by Proposition 3.3, the constant κ is given by

$$\begin{aligned} \kappa &= 2(n-1) + (h\alpha - \alpha^2) \\ &= 2(n-1) + \{(2\ell+1)\cot(r) - (2k+1)\tan(r)\}2\cot(2r) - (2\cot(2r))^2 \\ &= 2(n-1) + 2\{\ell\cot^2(r) + k\tan^2(r) - (k+\ell)\} \\ &= 2\ell\cot^2(r) + 2k\tan^2(r). \end{aligned}$$

Then by taking the covariant derivative we get the following two formulas

$$(\nabla_X \text{Ric})\xi = \kappa \phi AX - \text{Ric}(\phi AX),$$

and

$$(\nabla_\xi \text{Ric})X = h(\nabla_\xi A)X - (\nabla_\xi A^2)X.$$

Since M admits a gradient Ricci-Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$, we could consider the soliton vector field W as $W = Df$ for any smooth function on M . In the introduction we have noted that $\text{Hess}(f)$ is defined by $\text{Hess}(f) = \nabla Df$ for any tangent vector fields X and Y on M in such a way that

$$\text{Hess}(f)(X, Y) = g(\nabla_X Df, Y).$$

Then the gradient Ricci-Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ can be given by

$$\nabla_X Df + \text{Ric}(X) = (\Omega + \theta\gamma)X.$$

for any tangent vector field X on M . Then by covariant differentiating and using that the scalar curvature γ is constant for isometric Reeb flow, it gives

$$\begin{aligned} \nabla_X \nabla_Y Df + (\nabla_X \text{Ric})(Y) + \text{Ric}(\nabla_X Y) \\ = (\Omega + \theta\gamma)\nabla_X Y \end{aligned}$$

for any vector field X and Y tangent to M in $\mathbb{C}P^n$. From this, together with the above two formulas, it follows that

$$\begin{aligned} R(\xi, Y)Df &= \nabla_\xi \nabla_Y Df - \nabla_Y \nabla_\xi Df - \nabla_{[\xi, Y]} Df \\ &= (\nabla_Y \text{Ric})\xi - (\nabla_\xi \text{Ric})Y \\ &= \kappa \phi AY - \text{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y. \end{aligned} \tag{5.1}$$

Then from (e31) we have the following for a real hypersurface M in $\mathbb{C}P^n$ with isometric Reeb flow

$$\begin{aligned} R(\xi, Y)Df &= g(Y, Df)\xi - g(\xi, Df)Y \\ &\quad + g(AY, Df)A\xi - g(A\xi, Df)AY. \end{aligned} \tag{5.2}$$

From this, let us take a vector field $Y \in T_\lambda$, $\lambda = \cot(r)$. Moreover, we can decompose the tangent space $T\mathbb{C}P^n$ as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\mu \oplus T_\alpha \oplus \mathbb{R}N,$$

where $\lambda = \cot(r)$, $\mu = -\tan(r)$ and $\alpha = 2\cot(2r)$. If M is of type A_1 , that is, a geodesic hypersphere in $\mathbb{C}P^n$, it can be decomposed as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\alpha \oplus \mathbb{R}N,$$

or otherwise

$$T\mathbb{C}P^n = T_\mu \oplus T_\alpha \oplus \mathbb{R}N.$$

Then for $Y \in T_\lambda$ (5.2) gives

$$\begin{aligned} R(\xi, Y)Df &= g(Y, Df)\xi - g(\xi, Df)Y + \alpha \lambda g(Y, Df)\xi - \alpha \lambda g(\xi, Df)Y \\ &= (1 + \alpha \lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}. \end{aligned} \tag{5.3}$$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Then by taking the inner product of (5.3) with the Reeb vector field ξ and using (5.1), it follows that $(1 + \alpha\lambda)g(Y, Df) = \cot^2(r)g(Y, Df) = 0$. But $\cot^2(r) \neq 0$ for the radius $0 < r < \frac{\pi}{2}$ of isometric Reeb flow M in $\mathbb{CP}^n$. It means the following for any $Y \in T_\lambda$

$$g(Y, Df) = 0. \quad (5.4)$$

Now let us check (5.2) for $Y \in T_\mu$, $\mu = -\tan(r)$. Then (5.2) gives

$$\begin{aligned} R(\xi, Y)Df \\ = g(Y, Df)\xi - g(\xi, Df)Y + \alpha\mu g(Y, Df)\xi - \alpha\mu g(\xi, Df)Y. \end{aligned} \quad (5.5)$$

Then by taking the inner product (5.5) with the Reeb vector field ξ and $Y \in T_\mu$ respectively and using (5.1), we get

$$(1 + \alpha\mu)g(Y, Df) = 0 \quad \text{and} \quad (1 + \alpha\mu)g(\xi, Df) = 0, \quad (5.6)$$

where $g(R(\xi, Y)Df, \xi) = 0$ and the left side $g(R(\xi, Y)Df, Y) = 0$ is given by virtue of the following formulas

$$g(\phi AY, Y) = \mu g(\phi Y, Y) = 0,$$

$$\text{Ric}(\phi AY) = \mu\{(2n + 1) + \mu h - \mu^2\}\phi Y,$$

and

$$g((\nabla_\xi A)Y, Y) = -\mu g(\nabla_\xi Y, Y) = 0.$$

Since $1 + \alpha\mu = 1 + (\cot(r) - \tan(r))(-\tan(r)) = \tan^2(r) \neq 0$ for $0 < r < \frac{\pi}{2}$ for isometric Reeb flow M in $\mathbb{CP}^n$, (5.6) implies that

$$g(Y, Df) = 0 \quad \text{and} \quad g(\xi, Df) = 0 \quad (5.7)$$

for any $Y \in T_\mu$, $\mu = -\tan r$. For a geodesic hypersphere of type A_1 in $\mathbb{CP}^n$ it holds either $g(Y, Df) = 0$ for $Y \in T_\lambda = \mathcal{C}$ or for $Y \in T_\mu = \mathcal{C}$ from the above decomposition, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ in the tangent space TM of M in $\mathbb{CP}^n$. Of course, it also holds $g(\xi, Df) = 0$ for a geodesic hypersphere in $\mathbb{CP}^n$.

Summing up (5.3), (5.7) and the above documents, the gradient of the smooth function f is identically vanishing, that is, $g(Y, Df) = 0$ for any tangent vector field $Y \in T_z M$, $z \in M$. Consequently, we can conclude that the gradient Ricci Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ is trivial. So it becomes Einstein. That is, $\text{Ric}(X) = \lambda X$. Then by Theorem D, we get a complete proof of our Main Theorem 2 in the Introduction.

6. GRADIENT RICCI-BOURGUIGNON SOLITON ON CONTACT REAL HYPERSURFACES IN $\mathbb{CP}^n$

In this section, we want to give a property for gradient Ricci-Bourguignon soliton on a contact real hypersurface M in the complex projective space $\mathbb{CP}^n$. Of course, by Theorem B we know that the scalar curvature γ is constant. The $(M, Df, \Omega, \theta, \gamma, g)$ gives the following for any tangent vector field X on M in $\mathbb{CP}^n$

$$\nabla_X Df + \text{Ric}(X) = (\Omega + \theta\gamma)X. \quad (6.1)$$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Then by differentiating (6.1), the curvature tensor of $\text{grad} f$ is given by the following

$$\begin{aligned} R(X, Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X, Y]} Df \\ &= -(\nabla_X \text{Ric})Y - \text{Ric}(\nabla_X Y) + (\Omega + \theta\gamma)\nabla_X Y \\ &\quad + (\nabla_Y \text{Ric})X + \text{Ric}(\nabla_Y X) - (\Omega + \theta\gamma)\nabla_Y X \\ &\quad + \text{Ric}([X, Y]) - (\Omega + \theta\gamma)[X, Y] \\ &= (\nabla_Y \text{Ric})X - (\nabla_X \text{Ric})Y, \end{aligned} \quad (6.2)$$

where we have used the Ricci soliton constant ν and gradient Ricci-Bourguignon soliton constant ρ , and the scalar curvature γ is constant on a contact real hypersurface M in $\mathbb{C}P^n$ in Proposition 3.4.

Now let us assume that M is a contact real hypersurface in $\mathbb{C}P^n$, which is characterized by

$$A\phi + \phi A = \kappa\phi, \text{ where } \kappa \neq 0 : \text{const.}$$

Then it is Hopf and the Ricci operator is given by

$$\text{Ric}(X) = (2n+1)X - 3\eta(X)\xi + hAX - A^2X$$

for any tangent vector field X on M . From this, let us put $X = \xi$. Then M being Hopf and $A\xi = \alpha\xi$ implies

$$\text{Ric}(\xi) = \ell\xi,$$

where $\ell = 2(n-1) + h\alpha - \alpha^2$ is constant, and the mean curvature $h = \text{Tr} A$ is constant for a contact hypersurface M in $\mathbb{C}P^n$. Then by taking covariant derivative to the Ricci operator, we have

$$(\nabla_X \text{Ric})\xi = \nabla_X (\text{Ric}(\xi)) - \text{Ric}(\nabla_X \xi) = \ell\phi AX - \text{Ric}(\phi AX),$$

and

$$\begin{aligned} (\nabla_\xi \text{Ric})X &= \nabla_\xi (\text{Ric}X) - \text{Ric}(\nabla_\xi X) \\ &= h(\nabla_\xi A)X - (\nabla_\xi A^2)X. \end{aligned}$$

From (6.2), together with above formula, by putting $X = \xi$ we have the following for a contact hypersurface M in $\mathbb{C}P^n$

$$\begin{aligned} R(\xi, Y)Df &= (\nabla_Y \text{Ric})\xi - (\nabla_\xi \text{Ric})Y \\ &= \ell\phi AY - \text{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y. \end{aligned} \quad (6.3)$$

Then the diagonalization of the shape operator A of the contact real hypersurface in complex hyperbolic quadric $\mathbb{C}P^n$ is given by

$$A = \begin{bmatrix} 2\cot(2r) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -\cot(\frac{\pi}{4} - r) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & -\cot(\frac{\pi}{4} - r) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \tan(\frac{\pi}{4} - r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \tan(\frac{\pi}{4} - r) \end{bmatrix}.$$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Here by Proposition 3.4 the principal curvatures are given by $\alpha = 2\cot(2r)$, $\lambda = -\cot(\frac{\pi}{4} - r)$ and $\mu = \tan(\frac{\pi}{4} - r)$ with multiplicities 1, $n - 1$ and $n - 1$ respectively. All of these principal curvatures satisfy

$$\kappa = \lambda + \mu = -\cot(\frac{\pi}{4} - r) + \tan(\frac{\pi}{4} - r) = -2\tan(2r) = -\frac{4}{\alpha}.$$

On the other hand, the curvature tensor $R(X, Y)Z$ of M induced from the curvature tensor $\bar{R}(X, Y)Z$ of the complex projective space $\mathbb{C}P^n$ gives

$$\begin{aligned} R(\xi, Y)Df &= g(Y, Df)\xi - g(\xi, Df)Y \\ &\quad + g(AY, Df)A\xi - g(A\xi, Df)AY \\ &= (1 + \alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}. \end{aligned} \quad (6.4)$$

for any $Y \in T_\lambda$, $\lambda = -\cot(\frac{\pi}{4} - r)$ for a contact real hypersurface M in the complex projective space $\mathbb{C}P^n$. Consequently, (6.3) and (6.4) give

$$\ell\phi AY - \text{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1 + \alpha\lambda)\{g(Y, Df)\xi - g(\xi, Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ , we have

$$(1 + \alpha\lambda)g(Y, Df) = 0. \quad (6.5)$$

Then for any $Y \in T_\lambda$ in (6.5) it follows that

$$g(Y, Df) = 0, \quad (6.6)$$

where we have noted that $1 + \alpha\lambda = 1 + 2\cot(2r)(-\cot(\frac{\pi}{4} - r)) \neq 0$. Because if we assume that $2\cot(2r)\cot(\frac{\pi}{4} - r) = 1$, then $\tan(2r) = 2\cot(\frac{\pi}{4} - r)$. Then it follows that

$$(\cos(r) - \sin(r))\sin(r)\cos(r) = (\cos(r) + \sin(r))^2(\cos(r) - \sin(r)),$$

which gives $\sin(r)\cos(r) = -1$. This implies a contradiction for $0 < r < \frac{\pi}{4}$. Accordingly, the gradient vector field Df is orthogonal to the eigenspace T_λ , that is, $g(Y, Df) = 0$ for any $Y \in T_\lambda$.

Next, we consider for $Y \in T_\mu$, $\mu = \tan(\frac{\pi}{4} - r)$ in Proposition 3.4. Then using these properties in (6.3) and (6.4) implies the following

$$\ell\phi AY - \text{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1 + \alpha\mu)\{g(Y, Df)\xi - g(\xi, Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ , we get

$$g(Y, Df) = 0 \quad \text{for any } Y \in T_\mu, \quad (6.7)$$

where $1 + \alpha\mu \neq 0$. If we assume that $1 + \alpha\mu = 0$, then by Proposition 3.4, we can write $1 + 2\cot(2r)\tan(\frac{\pi}{4} - r) = 0$. Then it gives $-\tan(2r) = 2\frac{\cos(r) - \sin(r)}{\cos(r) + \sin(r)}$. Since $\tan(2r) = \frac{\sin(2r)}{\cos(2r)}$, we get the following

$$\begin{aligned} (\cos(r) + \sin(r))\sin(r)\cos(r) &= -(\cos(r) - \sin(r))(\cos^2(r) - \sin^2(r)) \\ &= -(\cos(r) - \sin(r))^2(\cos(r) + \sin(r)). \end{aligned}$$

Since $\cos(r) + \sin(r) \neq 0$ for $0 < r < \frac{\pi}{4}$, we get $\sin(r)\cos(r) = 1$, which gives also a contradiction.

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

Finally, let us take the inner product the above formula with $Y \in T_\mu$, and use $AY = \mu Y$, $A\phi Y = \lambda\phi Y$ for a contact hypersurface in $\mathbb{C}P^n$, we have

$$\begin{aligned} -(1 + \alpha\mu)g(\xi, Df) &= \ell g(\phi AY, Y) - g(\text{Ric}(\phi AY), Y) \\ &\quad - hg((\nabla_\xi A)Y, Y) + g((\nabla_\xi A^2)Y, Y) \\ &= 0, \end{aligned} \quad (6.8)$$

where in the second equality we have used the following formulas

$$\begin{aligned} \text{Ric}(\phi AY) &= (2n + 1)\phi AY + hA\phi AY - A^2\phi AY \\ &= \mu\{(2n + 1) + \lambda h - \lambda^2\}\phi Y, \\ g((\nabla_\xi A)Y, Y) &= g(\nabla_\xi(AY) - A\nabla_\xi Y, Y) \\ &= g(\mu\nabla_\xi Y - A\nabla_\xi Y, Y) = 0, \end{aligned}$$

and

$$\begin{aligned} g((\nabla_\xi A^2)Y, Y) &= g(\nabla_\xi(A^2Y) - A^2\nabla_\xi Y, Y) \\ &= g(\mu^2\nabla_\xi Y - A^2\nabla_\xi Y, Y) \\ &= \mu^2g(\nabla_\xi Y, Y) - \mu^2g(\nabla_\xi Y, Y) = 0. \end{aligned}$$

From this, together with (6.8), it follows that

$$g(\xi, Df) = 0. \quad (6.9)$$

Consequently, from (6.6), (6.7) and (6.9) it follows that the gradient vector field Df is identically vanishing. Then $Df = 0$ in (6.1) means that M is Einstein. But Theorem D in the introduction gives that there does not exist an Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$. From this, we give a complete proof of our Main Theorem 3 in the introduction.

Remark 6.1. The metric g of a Riemannian manifold M of dimension $n \geq 3$ is said to be a *gradient Einstein soliton* [6] if there exists a smooth function f on M such that

$$\text{Ric} + \nabla^2 f = (\Omega + \frac{1}{2}\gamma)g,$$

where γ denotes the scalar curvature of M and $\theta = \frac{1}{2}$ a constant on M . Here $\nabla^2 f$ denotes the Hessian operator of g and f the Einstein potential function of the gradient Einstein soliton. So this soliton is an example of gradient Ricci-Bourguignon soliton.

REFERENCES

- [1] J. Berndt and Y. J. Suh, Real hypersurfaces in Hermitian symmetric spaces, *Advances in Analysis and Geometry*, 5, *De Gruyter, Berlin*, [2022], ©2022. MR4500021.
- [2] D. E. Blair, Contact manifolds in Riemannian Geometry, *Lecture Notes in Mathematics*, Vol. 509, *Springer-Verlag, Berlin-New York*, 1976. MR0467588.
- [3] A. M. Blaga and H. M. Tastan, Some results on almost η -Ricci-Bourguignon soliton, *J. Geom. Phys.* **168** (2021), Paper No. 104316, 9 pp. MR4281917.
- [4] J. P. Bourguignon, Une stratification de l'espace des structures riemanniennes, *Compositio Math.* **30** (1975), 1–41. MR0418147.

Submitted: January 6, 2023

Accepted: August 15, 2023

Published (early view): August 24, 2024

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

- [5] J. P. Bourguignon, Ricci curvature and Einstein metrics, *Global differential geometry and global analysis (Berlin, 1979)*, pp. 42–63, Lecture Notes in Math., 838, Springer, Berlin, 1981. MR0636265.
- [6] G. Catino and L. Mazziere, Gradient Einstein solitons, *Nonlinear Anal.* **132** (2016), 66–94. MR3433954.
- [7] G. Catino, L. Mazziere and S. Mongodi, Rigidity of gradient Einstein shrinkers, *Commun. Contemp. Math.* **17**(2015), no. 6, 1550046, 18 pp. MR3485875.
- [8] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza and L. Mazziere, The Ricci-Bourguignon flow, *Pacific J. Math.* **287**(2017), no. 2, 337–370. MR3632891
- [9] T.E. Cecil and P.J. Ryan, Focal sets and real hypersurfaces in complex projective space, *Trans. Amer. Math. Soc.* **269** (1982), no. 2, 481–499. MR0637703.
- [10] P. Cerna and D. Guan, Killing fields generated by multiple solutions to the Fischer-Marsden equation, *Internat. J. Math.* **26** (2015), no. 4, 1540006, 18 pp. MR3338070.
- [11] S.K. Chaubey, Y.J. Suh and U.C. De, Characterizations of the Lorentzian manifolds admitting a type of semi-symmetric metric connection, *Anal. Math. Phys.* **10** (2020), Paper No. 61, 15 pp. MR4167217.
- [12] S. K. Chaubey, U. C. De and Y. J. Suh, Kenmotsu manifolds satisfying the Fischer-Marsden equation, *J. Korean Math. Soc.* **58** (2021), no. 3, 597–607. MR4252852.
- [13] S.K. Chaubey, U.C. De and Y.J. Suh, Gradient Yamabe and gradient m -quasi Einstein metrics on three-dimensional cosymplectic manifolds, *Mediterr. J. Math.* **18** (2021), no. 3, Paper No. 80, 14 pp. MR4232874.
- [14] M. Djorić and M. Okumura, CR submanifolds of complex projective space, *Developments in Mathematics*, 19, Springer, New York, 2010. MR2566776.
- [15] S. Dwivedi, Some results on Ricci-Bourguignon solitons and almost solitons, *Canad. Math. Bull.* (2021), no. 3, 591–604. MR4313552.
- [16] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Corrected reprint of the 1978 original, Graduate Studies in Mathematics, 34, American Mathematical Society, Providence, RI, 2001. MR1834454.
- [17] R. Hamilton, The Ricci flow on surfaces, *Mathematics and general relativity (Santa Cruz, CA, 1986)*, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988. MR0954419.
- [18] I. Jeong and Y.J. Suh, Pseudo anti-commuting and Ricci soliton real hypersurfaces in complex two-plane Grassmannians, *J. Geom. Phys.* **86** (2014), 258–272. MR3282328.
- [19] U-H. Ki and Y.J. Suh, On real hypersurfaces of a complex space form, *Math. J. Okayama Univ.* **32** (1990), 207–221. MR1112028.
- [20] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. II, Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996. MR1393941.
- [21] J. Morgan and G. Tian, Ricci flow and Poincaré Conjecture, Clay Mathematics Monographs, 3. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. MR2334563
- [22] M. Okumura, On some real hypersurfaces of a complex projective space, *Trans. Amer. Math. Soc.* **212** (1975), 355–364. MR0377787.
- [23] B. O'Neill, Semi-Riemannian geometry, With applications to relativity, Pure and Applied Mathematics, 103, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. MR0719023.
- [24] G. Perelman, Ricci flow with surgery on three-manifolds, math.DG/0303109, 2003.
- [25] J.D. Pérez, Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space, *Ann. Mat. Pura Appl. (4)* **194**(2015), no. 6, 1781–1794. MR3419917.
- [26] A. Romero, Some examples of indefinite complete complex Einstein hypersurfaces not locally symmetric, *Proc. Amer. Math. Soc.* **98** (1986), no. 2, 283–286. MR0854034.

Submitted: January 6, 2023

Accepted: August 15, 2023

Published (early view): August 24, 2024

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.3849>.

- [27] A. Romero, On a certain class of complex Einstein hypersurfaces in indefinite complex space forms, *Math. Z.* **192** (1986), no. 4, 627–635. MR0847011.
- [28] B. Smyth, Differential geometry of complex hypersurfaces, *Ann. of Math. (2)* **85** (1967), 246–266. MR0206881.
- [29] Y.J. Suh, Real hypersurfaces of type B in complex two-plane Grassmannians, *Monatsh. Math.* **147** (2006), no. 4, 337–355. MR2215841.
- [30] Y.J. Suh, Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians, *Adv. in Appl. Math.* **50** (2013), no. 5, 645–659. MR3032310.
- [31] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature, *J. Math. Pures Appl.* **100** (2013), no. 1, 16–33. MR3057300.
- [32] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.* **10** (1973), 495–506. MR0336660.
- [33] Y. Wang, Ricci solitons on almost Kenmotsu 3-manifolds, *Open Math.* **15** (2017), no. 1, 1236–1243. MR3709073.
- [34] Y. Wang, Ricci solitons on almost co-Kähler manifolds, *Canad. Math. Bull.* **62** (2019), no. 4, 912–922. MR4028497.
- [35] K. Yano and M. Kon, CR submanifolds in Kaehlerian and Sasakian manifolds, Progress in Mathematics, 30. *Birkhäuser, Boston, MA*, 1983. MR0688816.

IMSOON JEONG

DEPARTMENT OF MATHEMATICS EDUCATION,
CHEONGJU UNIVERSITY,
CHUNGCHONGBUK-DO 28503, REPUBLIC OF KOREA
Email address: isjeong@cju.ac.kr

YOUNG JIN SUH

DEPARTMENT OF MATHEMATICS AND RIRCM,
KYUNGPOOK NATIONAL UNIVERSITY,
DAEGU 41566, REPUBLIC OF KOREA
Email address: yjsuh@knu.ac.kr

Submitted: January 6, 2023

Accepted: August 15, 2023

Published (early view): August 24, 2024