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COMBINATORIAL FORMULAS FOR DETERMINANT, PERMANENT, AND INVERSE OF SOME CIRCULANT MATRICES WITH THREE PARAMETERS

CRISTIAN PANELO, ANDRÉS M. ENCINAS, AND DENIS E. VIDELA

ABSTRACT. In this work, we give closed formulas for determinant, permanent, and inverse of circulant matrices with three non-zero coefficients. The techniques that we use are related to digraphs associated with these matrices.

1. INTRODUCTION

The circulant matrices appear in many applications, for example, to approximate finite difference of elliptic equations with periodic boundary conditions, or to approximate periodic functions with splines. The circulant matrices play an important role in coding theory and in statistics, the standard reference is [6].

Among of the main problems about circulant matrices are to determining invertibility conditions and computing the inverse of circulant matrices are among the main problems related to such matrices. These problems have been widely treated in the literature by using the primitive n -th root of unity and some polynomial associated with the circulant matrices, see [5, 6, 13]. There exist some classical and well-known results that enable us to solve almost everything, we would raise about the inverse of circulant matrices. Nevertheless, when we deal with specific families of circulant matrices, these classical results give us unmanageable formulas. Therefore, it is interesting to find alternative descriptions and in fact, there exist many papers devoted to this question. The direct computation for the inverse of some circulant matrices have been proposed in many works, see [4, 5, 9, 10, 12, 13, 15].

In this work, we delve into the combinatorial structure of circulant matrices with only three non-zero generators, by considering the digraphs associated with this kind of matrices. Therefore, we extend the previous work of some of the authors, see [7, 8], where only an specific class of these matrices was considered.

In the present work, we use digraphs, for all of graph-theoretic notions not explicitly defined here, the reader is referred to [1]. As is natural for circulant

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matrices, our matrix indices and permutations start at zero and so permutations in this work are bijections over $\{0, \dots, n-1\}$, and if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

then the set of indices of A is $\{0, 1\}$, in this case we have that $A_{00} = 1$ and $A_{10} = 3$. With $[n]$ we denote the set $\{0, \dots, n-1\}$ instead of $\{1, \dots, n\}$.

Given a permutation α of $[n]$, we denote by P_α the $n \times n$ matrix defined by $(P_\alpha)_{\alpha(j)j} = 1$ and 0 otherwise. The matrix P_α is known as the permutation matrix associated with α . It is well-known that $P_\alpha^{-1} = P_\alpha^T$ and $P_{\alpha\rho} = P_\alpha P_\rho$. We use the matrix associated with the permutation

$$\tau_n = (n-1 \ n-2 \ \dots \ 2 \ 1 \ 0)$$

many times along this work, so instead of P_{τ_n} we just write P_n . Notice that, for $k \in \mathbb{Z}$, we have that $P_n^0 = I_n$, $P_n^k = P_n^{(k) \bmod n} = P_{\tau_n^k}$, $\det(P_n) = (-1)^{n-1}$, and $(P_n^k)^{-1} = P_n^{n-k}$. Moreover $\tau_n^k(i) = (i-k) \bmod n$.

A matrix $C = (c_{ij})$ is called circulant with parameters $c_0, c_1, \dots, c_{n-1}$ if

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_0 \end{bmatrix}$$

in this case we denote C by $\text{Circ}(c_0, \dots, c_{n-1})$. We have that

$$\text{Circ}(c_0, \dots, c_{n-1}) = c_0 I_n + \dots + c_{n-1} P_n^{n-1}.$$

The numbers $c_0, \dots, c_{n-1}$ are called the parameters of C . Let a, b, c be non-zero complex numbers. We begin this work by studying the following kind of circulant matrices

$$\text{Circ}(a, b, c, 0, \dots, 0) = aI_n + bP_n + cP_n^2,$$

that we just call circulant matrices with three parameters.

This paper is organized as follows. In Section 2, we find explicit formulas for the determinant and the permanent of circulant matrices with three parameters. In Section 3, we give an explicit formula for the inverse of non-singular circulant matrices with three parameters. In both sections, we extend the results to a more general case where the parameters are in other positions.

2. DETERMINANT AND PERMANENT OF THE MATRICES $\text{Circ}(a, b, c, 0, \dots, 0)$

In order to obtain an explicit formula for the determinant of circulant matrices of the form $aI_n + bP_n + cP_n^2$, we have the following definitions.

Definition 2.1 ([3]). Let $A = [a_{ij}]$ be a matrix of order n . We associate with A a digraph $D(A)$ with n vertices. The vertices of $D(A)$ are denoted by $0, 1, 2, \dots, n-1$. If $a_{ij} \neq 0$ there is an arc from vertex i to vertex j of weight a_{ij} for each $i, j \in [n]$, we denote by a_{ij} -arc to this arc.

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Definition 2.2 ([3]). Let D be a digraph. A linear subdigraph of D is a spanning subdigraph of D in which each vertex has indegree 1 and outdegree 1, i.e. exactly one arc into each vertex and exactly one (possibly the same) out of each vertex.

The following theorem gives the determinant and permanent of a given matrix in terms of its associated digraph.

Theorem 2.3 ([3]). Let $A = [a_{ij}]$ be a square matrix of order n . Then

$$\det(A) = \sum_{L \in \mathcal{L}(D(A))} (-1)^{n-c(L)} w(L)$$

and

$$\text{perm}(A) = \sum_{L \in \mathcal{L}(D(A))} w(L)$$

where $\mathcal{L}(D(A))$ is the set of all linear subdigraphs of $D(A)$, $c(L)$ is the number of cycles contained in L (included loops), and $w(L)$ is the product of the weights of the arcs of L .

Now we study the structure of the digraph $D(aI_n + bP_n + cP_n^2)$. Since this digraph appears many times in our work, we just write $D_n(a, b, c)$ instead of $D(aI_n + bP_n + cP_n^2)$.

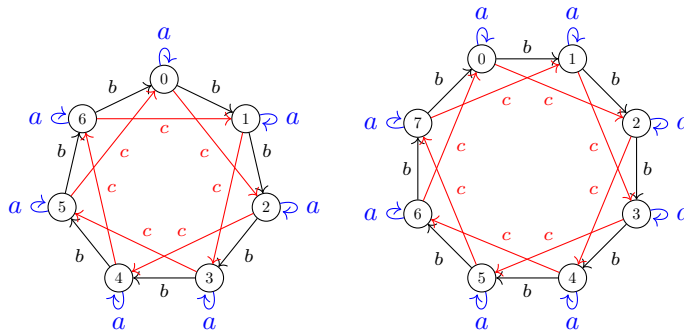


FIGURE 1. $D_7(a, b, c)$ and $D_8(a, b, c)$

Notice that the digraph $D_n(a, b, c)$ has a directed cycle C , with $V(C) = \{v_0, v_1, \dots, v_{n-1}\}$ such that, for $0 \leq i \leq n-1$, $v_i = i$ and C has a b -arc from v_i to v_{i+1} (with $v_n = v_0$). From [11], we have that the number of ways to choose k non-adjacent vertices of C , with $k \in [n]$, is

$$\frac{n}{n-k} \binom{n-k}{k}. \quad (1)$$

Theorem 2.4. Let $n \in \mathbb{Z}$ such that $n > 2$ and let $a, b, c \in \mathbb{C} - \{0\}$. Then $\det(aI_n + bP_n + cP_n^2)$ equals

$$a^n - (-b)^n + c^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-j-1} \frac{n}{n-j} \binom{n-j}{j} a^j b^{n-2j} c^j.$$

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Proof. Notice that in the digraph $D_n(a, b, c)$ we have the following linear subdigraphs

- (1) A linear subdigraph L_1 such that L_1 has all the loops (a -arcs). Note that $(-1)^{n-c(L_1)}w(L_1) = (-1)^0a^n$, see Figure 2b.
- (2) A linear subdigraph L_2 such that L_2 is a cycle with b -arcs. Note that $(-1)^{n-c(L_2)}w(L_2) = (-1)^{n-1}b^n$, see Figure 2c.
- (3) If n be even, then we have a linear subdigraph L_3 such that L_3 has two cycles of c -arcs. Note that $(-1)^{n-c(L_3)}w(L_3) = (-1)^{n-2}c^n$. If n be odd, then the linear subdigraph L_3 has one cycle of c -arcs. Note that $(-1)^{n-c(L_3)}w(L_3) = (-1)^{n-1}c^n$, see Figure 1.

The other linear subdigraphs has k loops, with $0 < k < n$. Notice that there is no exist a linear subdigraphs L with two loops at successive vertices, since if L has loops in j and $j + 1$, then L has no arcs that out from the vertex $j - 1$. Thus, if we choose the loop at vertex j , then we cannot choose the loops at vertices $j - 1$ and $j + 1$. By (1), we have $\frac{n}{n-k} \binom{n-k}{k}$ choices to take k loops, each of these options gives us a linear subdigraph L_k with k loops such that $(-1)^{n-c(L_k)}w(L_k) = (-1)^{n-k-1}a^kb^{n-2k}c^k$, since for each loop at a vertex j chosen, we lose two b -arcs and we have an c -arc from $j - 1$ to $j + 1$, see Figure 2d. Moreover, if we choose k loops, then $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, since if we take $k > \lfloor \frac{n}{2} \rfloor$ loops, then the linear subdigraphs is repeated. ■

The following corollary follows from Theorems 2.3 and 2.4.

Corollary 2.5. *Let n be a positive integer such that $n > 2$ and let a, b, c non-zero complex numbers. Then,*

$$\text{perm}(aI_n + bP_n + cP_n^2) = a^n + b^n + c^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} a^j b^{n-2j} c^j.$$

Example 2.6. Let us consider $A = aI_7 + bP_7 + cP_7^2$ and $B = aI_8 + bP_8 + cP_8^2$, then

$$\begin{aligned} \det(A) &= a^7 + b^7 + c^7 - 7ab^5c + 14a^2b^3c^2 - 7a^3bc^3, \\ \text{perm}(A) &= a^7 + b^7 + c^7 + 7ab^5c + 14a^2b^3c^2 + 7a^3bc^3, \end{aligned}$$

and

$$\begin{aligned} \det(B) &= a^8 - b^8 + c^8 + 8ab^6c - 20a^2b^4c^2 + 16a^3b^2c^3 - 2a^4c^4, \\ \text{perm}(B) &= a^8 + b^8 + c^8 + 8ab^6c + 20a^2b^4c^2 + 16a^3b^2c^3 + 2a^4c^4. \end{aligned}$$

With $\otimes$ we denote the usual Kronecker product between matrices, see [14]. Follow the notation in [8], given n and s two non-zero integers we have the number $n \setminus s := \frac{n}{\gcd(n, s)}$.

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Theorem 2.7 ([8]). *Let $n, s \in \mathbb{Z}$ such that $0 < s < n$ and let $a_k \in \mathbb{C} - \{0\}$, for $k \in [n \setminus s]$. Then, there exists a permutation $\sigma_{n,s}$ of $[n]$ such that*

$$P_{\sigma_{n,s}}^T \left(\sum_{k=0}^{(n \setminus s)-1} a_k P_n^{sk} \right) P_{\sigma_{n,s}} = I_{\gcd(n,s)} \otimes \left(\sum_{k=0}^{(n \setminus s)-1} a_k P_{n \setminus s}^k \right).$$

By Theorem 2.7 we have the following corollary.

Corollary 2.8. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \leq s_1 < s_2 < s_3 \leq n$, and $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2(s_2 - s_1)$, then*

$$aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3} = P_n^{s_1} P_{\sigma_{n,s}} \left(I_{\gcd(n,s)} \otimes \left(aI_{n \setminus s} + bP_{n \setminus s} + cP_{n \setminus s}^2 \right) \right) P_{\sigma_{n,s}}^T,$$

where $s = s_2 - s_1$.

By Corollary 2.8 we have the following corollary.

Corollary 2.9. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \leq s_1 < s_2 < s_3 \leq n$, and $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2(s_2 - s_1)$, then $\det(aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3})$ equals*

$$(-1)^{s_1(n-1)} \left(\det \left(aI_{n \setminus (s_2-s_1)} + bP_{n \setminus (s_2-s_1)} + cP_{n \setminus (s_2-s_1)}^2 \right) \right)^{\gcd(n, s_2-s_1)}.$$

Proof. Let $s = s_2 - s_1$ and $A = aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3}$. By Corollary 2.8

$$\begin{aligned} \det(A) &= \det \left(P_n^{s_1} P_{\sigma_{n,s}} \left(I_{\gcd(n,s)} \otimes \left(aI_{n \setminus s} + bP_{n \setminus s} + cP_{n \setminus s}^2 \right) \right) P_{\sigma_{n,s}}^T \right) \\ &= \det(P_n^{s_1}) \det \left(I_{\gcd(n,s)} \otimes \left(aI_{n \setminus s} + bP_{n \setminus s} + cP_{n \setminus s}^2 \right) \right). \end{aligned}$$

Let C and D be matrices of order n and m respectively, then $\det(C \otimes D) = (\det(C))^m (\det(D))^n$, see [14]. On other hand $\det(P_n^{s_1}) = (-1)^{s_1(n-1)}$, see [8]. Therefore,

$$\det(A) = (-1)^{s_1(n-1)} \left(\det \left(aI_{n \setminus s} + bP_{n \setminus s} + cP_{n \setminus s}^2 \right) \right)^{\gcd(n,s)}. \quad \blacksquare$$

Given a square matrix A of order n . We have that $\text{perm}(PA) = \text{perm}(A)$, where P is a permutation matrix of order n , and $\text{perm}(I_m \otimes A) = (\text{perm}(A))^m$, see [2]. Thus, we have the following corollary.

Corollary 2.10. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \leq s_1 < s_2 < s_3 \leq n-1$, and $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2(s_2 - s_1)$, then $\text{perm}(aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3})$ equals*

$$\left(\text{perm} \left(aI_{n \setminus (s_2-s_1)} + bP_{n \setminus (s_2-s_1)} + cP_{n \setminus (s_2-s_1)}^2 \right) \right)^{\gcd(n, s_2-s_1)}.$$

3. INVERSE OF THE MATRICES $\text{Circ}(a, b, c, 0, \dots, 0)$

For the rest of the work we use the following **convention**: let $j, k, t \in \mathbb{Z}$, and let $x_j \in \mathbb{C} - \{0\}$. We set $\sum_{j=t}^k x_j = 0$ and $\binom{k}{t} = 0$, when $k < t$. Also, we set $y^\ell = z^\ell = 0$, when $\ell < 0$, for $y, z \in \mathbb{C}$ and $\ell \in \mathbb{R}$.

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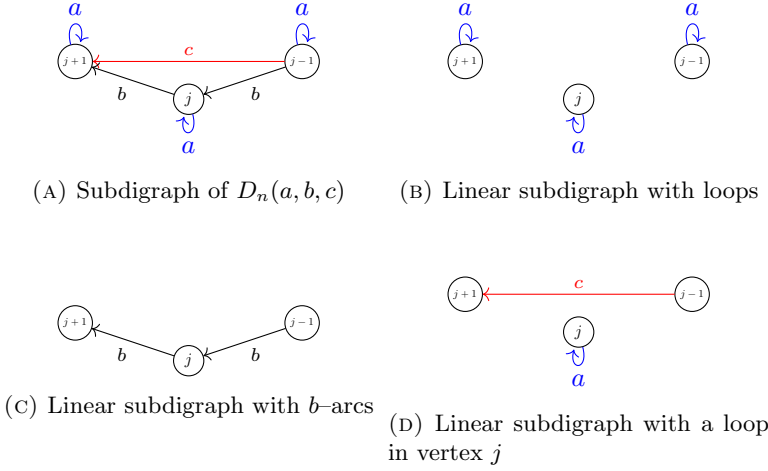


FIGURE 2. Subgraphs of $D_n(a, b, c)$

Definition 3.1. Let $i, n \in \mathbb{Z}$ such that $n > 0$, and let $a, b, c \in \mathbb{C} - \{0\}$. We define

$$\begin{aligned}
 L_n(i) &= (-1)^t a^{n-1-t} b^t + (-1)^{n-2-t} b^{n-2-t} c^{t+1} \\
 &+ \sum_{j=1}^{\lfloor \frac{n-2-t}{2} \rfloor} (-1)^{n-t-j} \binom{n-2-t-j}{j} a^j b^{n-2-t-2j} c^{t+j+1} \\
 &+ \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} (-1)^{t-j} \binom{t-j}{j} a^{n-1-t+j} b^{t-2j} c^j,
 \end{aligned}$$

where $t = (i) \bmod n$

Example 3.2.

i	$L_7(i)$	i	$L_8(i)$
0	$a^6 - b^5 c + 4ab^3 c^2 - 3a^2 b c^3$	0	$a^7 + b^6 c - 5ab^4 c^2 + 6a^2 b^2 c^3 - a^3 c^4$
1	$-a^5 b + b^4 c^2 - 3ab^2 c^3 + a^2 c^4$	1	$-a^6 b - b^5 c^2 + 4ab^3 c^3 - 3a^2 b c^4$
2	$a^4 b^2 - b^3 c^3 + 2abc^4 - a^5 c$	2	$a^5 b^2 + b^4 c^3 - 3ab^2 c^4 + a^2 c^5 - a^6 c$
3	$-a^3 b^3 + b^2 c^4 - ac^5 + 2a^4 b c$	3	$-a^4 b^3 - b^3 c^4 + 2abc^5 + 2a^5 b c$
4	$a^2 b^4 - bc^5 - 3a^3 b^2 c + a^4 c^2$	4	$a^3 b^4 + b^2 c^5 - ac^6 - 3a^4 b^2 c + a^5 c^2$
5	$-ab^5 + c^6 + 4a^2 b^3 c - 3a^3 b c^2$	5	$-a^2 b^5 - bc^6 + 4a^3 b^3 c - 3a^4 b c^2$
6	$b^6 - 5ab^4 c + 6a^2 b^2 c^2 - a^3 c^3$	6	$ab^6 + c^7 - 5a^2 b^4 c + 3a^3 b^2 c^2 - a^4 c^3$
		7	$-b^7 + 6ab^5 c - 10a^2 b^3 c^2 + 4a^3 b c^3$

Notice that when $i, n, t \in \mathbb{Z}$ such that $n > 2$, $1 \leq i \leq n-1$ and $1 \leq t \leq \lfloor \frac{n-i}{2} \rfloor$, we have the following useful properties that it can be proved by simple calculations:

$$\binom{n-1-i-t}{t-1} + \binom{n-1-i-t}{t} = \binom{n-i-t}{t} \quad \text{and} \quad \binom{i-1-t}{t-1} + \binom{i-1-t}{t} = \binom{i-t}{t}. \quad (2)$$

$$2 \binom{n-1-t}{t-1} + \binom{n-1-t}{t} = \frac{n}{n-t} \binom{n-t}{t}, \quad (3)$$

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$$\left\lfloor \frac{n-i}{2} \right\rfloor = \left\lfloor \frac{n-2-i}{2} \right\rfloor + 1 \quad (4)$$

and

$$\left\lfloor \frac{n-i}{2} \right\rfloor = \begin{cases} \left\lfloor \frac{n-1-i}{2} \right\rfloor & \text{if } n \not\equiv (i) \pmod{2}, \\ \left\lfloor \frac{n-1-i}{2} \right\rfloor + 1 & \text{if } n \equiv (i) \pmod{2}. \end{cases} \quad (5)$$

In order to obtain an explicit formula for the inverse of a non-singular matrices of the form $aI_n + bP_n + cP_n^2$ we define

$$\rho_n(a, b, c, i) := a L_n(i) + b L_n(i-1) + c L_n(i-2).$$

Lemma 3.3. *Let $i, n \in \mathbb{Z}$ such that $n > 2$, and let $a, b, c \in \mathbb{C} - \{0\}$. If $1 \leq i \leq n-1$, then $\rho_n(a, b, c, i) = 0$.*

Proof. Let $i > 0$, let us consider

$$x_{a,i} = (-1)^{n-2-i} ab^{n-2-i} c^{i+1} + \sum_{j=1}^{\left\lfloor \frac{n-2-i}{2} \right\rfloor} \alpha_{i,j} + \sum_{j=1}^{\left\lfloor \frac{n-1-i}{2} \right\rfloor} \beta_{i,j} + \sum_{j=1}^{\left\lfloor \frac{n-i}{2} \right\rfloor} \gamma_{i,j}$$

and

$$x_{c,i} = (-1)^{i-2} a^{n+1-i} b^{i-2} c + \sum_{j=1}^{\left\lfloor \frac{i-2}{2} \right\rfloor} \alpha'_{i,j} + \sum_{j=1}^{\left\lfloor \frac{i}{2} \right\rfloor} \beta'_{i,j} + \sum_{j=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \gamma'_{i,j},$$

where

$$\begin{aligned} \alpha_{i,j} &= (-1)^{n-i-j} \binom{n-2-i-j}{j} a^{j+1} b^{n-2-i-2j} c^{i+j+1}; \\ \beta_{i,j} &= (-1)^{n+1-i-j} \binom{n-1-i-j}{j} a^j b^{n-i-2j} c^{i+j}; \\ \gamma_{i,j} &= (-1)^{n+2-i-j} \binom{n-i-j}{j} a^j b^{n-i-2j} c^{i+j}, \end{aligned}$$

and

$$\begin{aligned} \alpha'_{i,j} &= (-1)^{i-2-j} \binom{i-2-j}{j} a^{n+1-i+j} b^{i-2-2j} c^{j+1}; \\ \beta'_{i,j} &= (-1)^{i-j} \binom{i-j}{j} a^{n-i+j} b^{i-2j} c^j; \\ \gamma'_{i,j} &= (-1)^{i-1-j} \binom{i-1-j}{j} a^{n-i+j} b^{i-2j} c^j. \end{aligned}$$

Claim: $x_{a,i} = x_{c,i} = 0$.

Note that if we consider $x_{a,i}$ as a polynomial on a, b, c , we have $x_{a,i}$ only has monomials of the form $a^j b^{n-i-2j} c^{i+j}$, with $1 \leq j \leq \left\lfloor \frac{n-i}{2} \right\rfloor$, i.e.

$$x_{a,i} = \sum_{j=1}^{\left\lfloor \frac{n-i}{2} \right\rfloor} A_j a^j b^{n-i-2j} c^{i+j},$$

where, by (5), we have that

$$A_1 = (-1)^{n-2-i} ab^{n-2-i} c^{i+1} + \beta_{i,1} + \gamma_{i,1}$$

and, if $i \not\equiv (n) \pmod{2}$, then for $2 \leq j \leq \left\lfloor \frac{n-i}{2} \right\rfloor$

$$A_j = \alpha_{i,j-1} + \beta_{i,j} + \gamma_{i,j},$$

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if $i = (n) \bmod 2$, then

$$A_j = \begin{cases} \alpha_{i,j-1} + \beta_{i,j} + \gamma_{i,j}, & \text{if } 2 \leq j \leq \left\lfloor \frac{n-i}{2} \right\rfloor - 1, \\ \alpha_{i,j-1} + \gamma_{i,j}, & \text{if } j = \left\lfloor \frac{n-i}{2} \right\rfloor. \end{cases}$$

By (2) we have that $A_j = 0$ for all j . Therefore, $x_{a,i} = 0$ as desired.

A similar argument shows that $x_{c,i} = 0$. Therefore, $x_{a,i} = x_{c,i} = 0$ as we claimed.

Now, notice that

$$\begin{aligned} a L_n(i) &= (-1)^i a^{n-i} b^i + (-1)^{n-2-i} a b^{n-2-i} c^{i+1} \\ &\quad + \sum_{j=1}^{\left\lfloor \frac{n-2-i}{2} \right\rfloor} (-1)^{n-i-j} \binom{n-2-i-j}{j} a^{j+1} b^{n-2-i-2j} c^{i+j+1} \\ &\quad + \sum_{j=1}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^{i-j} \binom{i-j}{j} a^{n-i+j} b^{i-2j} c^j, \end{aligned}$$

$$\begin{aligned} b L_n(i-1) &= (-1)^{i-1} a^{n-i} b^i + (-1)^{n-1-i} b^{n-i} c^i \\ &\quad + \sum_{j=1}^{\left\lfloor \frac{n-1-i}{2} \right\rfloor} (-1)^{n+1-i-j} \binom{n-1-i-j}{j} a^j b^{n-i-2j} c^{i+j} \\ &\quad + \sum_{j=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} (-1)^{i-1-j} \binom{i-1-j}{j} a^{n-i+j} b^{i-2j} c^j \end{aligned}$$

and

$$\begin{aligned} c L_n(i-2) &= (-1)^{i-2} a^{n+1-i} b^{i-2} c + (-1)^{n-i} b^{n-i} c^i \\ &\quad + \sum_{j=1}^{\left\lfloor \frac{n-i}{2} \right\rfloor} (-1)^{n+2-i-j} \binom{n-i-j}{j} a^j b^{n-i-2j} c^{i+j} \\ &\quad + \sum_{j=1}^{\left\lfloor \frac{i-2}{2} \right\rfloor} (-1)^{i-2-j} \binom{i-2-j}{j} a^{n+1-i+j} b^{i-2-2j} c^{j+1}. \end{aligned}$$

For $1 \leq i \leq n-1$, we have that

$$a L_n(i) + b L_n(i-1) + c L_n(i-2) = x_{a,i} + x_{c,i}.$$

Since $x_{a,i} = x_{c,i} = 0$, we obtain that $\rho_n(a, b, c, i) = 0$, as asserted. ■

Lemma 3.4. *Let $n \in \mathbb{Z}$ such that $n > 2$, and let $a, b, c \in \mathbb{C} - \{0\}$. Then*

$$\rho_n(a, b, c, 0) = \det(aI_n + bP_n + cP_n^2).$$

Proof. We have that

$$\begin{aligned} \rho_n(a, b, c, 0) &= a L_n(0) + b L_n(-1) + c L_n(-2) \\ &= a L_n(0) + b L_n(n-1) + c L_n(n-2) \end{aligned}$$

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and

$$\begin{aligned}
 L_n(0) &= a^{n-1} + (-1)^{n-2} b^{n-2} c \\
 &\quad + \sum_{j=1}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^{n-j} \binom{n-2-j}{j} a^j b^{n-2-2j} c^{j+1} \\
 &\quad + \sum_{j=1}^0 (-1)^{-j} \binom{-j}{j} a^{n-1+j} b^{-2j} c^j, \\
 L_n(n-1) &= (-1)^{n-1} b^{n-1} \\
 &\quad + \sum_{j=1}^{-1} (-1)^{1-j} \binom{-1-j}{j} a^j b^{-1-2j} c^{n+j} \\
 &\quad + \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-1-j} \binom{n-1-j}{j} a^j b^{n-1-2j} c^j \\
 L_n(n-2) &= (-1)^{n-2} a b^{n-2} + c^{n-1} \\
 &\quad + \sum_{j=1}^0 (-1)^{2-j} \binom{-j}{j} a^j b^{-2j} c^{n-1+j} \\
 &\quad + \sum_{j=1}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^{n-2-j} \binom{n-2-j}{j} a^{1+j} b^{n-2-2j} c^j.
 \end{aligned}$$

Then $a L_n(0) + b L_n(n-1) + c L_n(n-2)$ is equal to

$$a^n - (-b)^n + c^n + (-1)^{n-2} 2 a b^{n-2} c + \sum_{j=1}^{\lfloor \frac{n-2}{2} \rfloor} \alpha_{0,j} + \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \beta_{0,j},$$

where

$$\begin{aligned}
 \alpha_{0,j} &= (-1)^{n-j} 2 \binom{n-2-j}{j} a^{j+1} b^{n-2-2j} c^{j+1}, \quad \text{and} \\
 \beta_{0,j} &= (-1)^{n-1-j} \binom{n-1-j}{j} a^j b^{n-2j} c^j.
 \end{aligned}$$

Note that if we consider $\rho_n(a, b, c, 0) - a^n + (-b)^n - c^n$ as a polynomial on a, b, c we have that only has monomials of the form $a^j b^{n-j} c^j$, i.e.

$$\rho_n(a, b, c, 0) - a^n + (-b)^n - c^n = \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} B_j a^j b^{n-2j} c^j$$

where, by (5), we have that

$$B_1 = (-1)^{n-2} 2 a b^{n-2} c + \beta_{0,1}$$

and, if n is odd, then for $2 \leq j \leq \lfloor \frac{n}{2} \rfloor$

$$B_j = \alpha_{0,j-1} + \beta_{0,j},$$

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if n is even, then

$$B_j = \begin{cases} \alpha_{0,j-1} + \beta_{0,j}, & \text{if } 2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ \alpha_{0,j-1}, & \text{if } j = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

By (3) we have that $B_j = (-1)^{n-j-1} \frac{n}{n-j} \binom{n-j}{j} a^j b^{n-2j} c^j$, for all j . Therefore, $\rho_n(a, b, c, 0) = \det(aI_n + bP_n + cP_n^2)$, as desired. ■

Theorem 3.5. *Let $n \in \mathbb{Z}$ such that $n > 2$, and let $a, b, c \in \mathbb{C} - \{0\}$. Then*

$$(aI_n + bP_n + cP_n^2)^{-1} = \frac{1}{\det(aI_n + bP_n + cP_n^2)} \sum_{i=0}^{n-1} L_n(i) P_n^i.$$

Proof. Follows directly from Lemmas 3.3 and 3.4, since

$$(aI_n + bP_n + cP_n^2) \left(\sum_{i=0}^{n-1} \frac{L_n(i)}{m} P_n^i \right) = \sum_{i=0}^{n-1} \frac{\rho_n(i)}{m} P_n^i. \quad \blacksquare$$

where $m = \det(aI_n + bP_n + cP_n^2)$.

Finally, by Corollary 2.8 and Theorem 3.5 we have the following theorem.

Theorem 3.6. *Let $n, s_1, s_2, s_3 \in \mathbb{Z}$ such that $0 \leq s_1 < s_2 < s_3 \leq n-1$, and let $a, b, c \in \mathbb{C} - \{0\}$. If $s_3 - s_1 = 2s$, where $s = s_2 - s_1$, and the matrix $aI_{n \setminus s} + bP_{n \setminus s} + cP_{n \setminus s}^2$ are non-singular, then $aP_n^{s_1} + bP_n^{s_2} + cP_n^{s_3}$ is non-singular and its inverse matrix is*

$$P_{\sigma_{n,s}} \left(I_r \otimes \left(\frac{1}{\det(aI_t + bP_t + cP_t^2)} \sum_{i=0}^{t-1} L_t(i) P_t^i \right) \right) P_{\sigma_{n,s}}^T P_n^{n-s_1},$$

where $r = \gcd(n, s)$ and $t = n \setminus s$.

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DEPARTAMENTO DE MATEMÁTICAS. FACULTAD DE CIENCIAS FÍSICO-MATEMÁTICAS Y NATURALES. UNIVERSIDAD NACIONAL DE SAN LUIS. SAN LUIS, ARGENTINA
Email address: `crpanelo@unsl.edu.ar`

DEPARTAMENT DE MATEMÀTIQUES. UNIVERSITAT POLITÈCNICA DE CATALUNYA. BARCELONA, SPAIN
Email address: `andres.marcos.encinas@upc.edu`

FAMAF - CIEM (CONICET), UNIVERSIDAD NACIONAL DE CÓRDOBA, ARGENTINA
Email address: `devidela@famaf.unc.edu.ar`

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