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ON THE SECOND $\mathfrak{osp}(1|2)$ -RELATIVE COHOMOLOGY OF THE LIE SUPERALGEBRA OF CONTACT VECTOR FIELDS ON $\mathbb{C}^{1|1}$

ABDERRAOUF GHALLABI, NIZAR BEN FRAJ, AND SALEM FAIDI

ABSTRACT. Let $\mathcal{K}(1)$ be the Lie superalgebra of contact vector fields on the $(1,1)$ -dimensional complex superspace; it contains the Möbius superalgebra $\mathfrak{osp}(1|2)$. We classify $\mathfrak{osp}(1|2)$ -invariant superanti-symmetric binary differential operators from $\mathcal{K}(1) \wedge \mathcal{K}(1)$ to $\mathfrak{D}_{\lambda,\mu}$ vanishing on $\mathfrak{osp}(1|2)$, where $\mathfrak{D}_{\lambda,\mu}$ is the superspace of linear differential operators acting on the superspaces of weighted densities. This result allows us to compute the second differential $\mathfrak{osp}(1|2)$ -relative cohomology of $\mathcal{K}(1)$ with coefficients in $\mathfrak{D}_{\lambda,\mu}$.

1. INTRODUCTION

Let $\mathbf{vect}(1)$ be the Lie algebra of polynomial vector fields on $\mathbb{C}$. Consider the 1-parameter deformation of the $\mathbf{vect}(1)$ -action on $\mathbb{C}[x]$:

$$L_X^\lambda \frac{d}{dx}(f) = Xf' + \lambda X'f,$$

where $X, f \in \mathbb{C}[x]$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\mathbf{vect}(1)$ -module structure on $\mathbb{C}[x]$ defined by L^λ for a fixed λ . Geometrically, $\mathcal{F}_\lambda = \{f dx^\lambda \mid f \in \mathbb{C}[x]\}$ is the space of polynomial weighted densities of weight $\lambda \in \mathbb{C}$. The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1, 0$ and 1 , respectively.

Denote by $\mathcal{D}_{\lambda,\mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ the $\mathbf{vect}(1)$ -module of linear differential operators with the natural $\mathbf{vect}(1)$ -action denoted $L_X^{\lambda,\mu}(A)$. If we restrict ourselves to the Lie subalgebra of $\mathbf{vect}(1)$ generated by $\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\}$, isomorphic to $\mathfrak{sl}(2)$, we get a family of infinite dimensional $\mathfrak{sl}(2)$ -modules, still denoted $\mathcal{F}_\lambda$ and $\mathcal{D}_{\lambda,\mu}$. Bouarroudj [9] computed the space

$$H_{\text{diff}}^2(\mathbf{vect}(1), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu}),$$

where H_{diff}^i denotes the differential cohomology; that is, only cochains given by differential operators are considered. These spaces appear naturally in the problem of describing the $\mathfrak{sl}(2)$ -trivial deformations of the $\mathbf{vect}(1)$ -module $\mathcal{S}_{\mu-\lambda} = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\mu-\lambda-k}$, the space of symbols of differential operators (for example, see [3, 18]).

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The purpose of this paper is to study the simplest super analogue of the problem solved in [9], namely, we consider the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the linear supermanifold, i.e., the ringed space $\mathcal{C}^{1|1} := (\mathbb{C}, \mathbb{C}[x, \theta])$ associated with the linear superspace $\mathbb{C}^{1|1}$, where x is a coordinate on $\mathbb{C}$ and θ is a generator of the Grassmann algebra, x is supposed to be even and θ odd. We also consider the $\mathcal{K}(1)$ -module $\mathfrak{F}_\lambda$ of λ -densities on $\mathcal{C}^{1|1}$ and the $\mathcal{K}(1)$ -module of linear differential operators $\mathfrak{D}_{\lambda, \mu} := \text{Hom}_{\text{diff}}(\mathfrak{F}_\lambda, \mathfrak{F}_\mu)$, which are super analogues of the spaces $\mathcal{F}_\lambda$ and $\mathcal{D}_{\lambda, \mu}$, respectively. The Lie superalgebra $\mathfrak{osp}(1|2)$, a super analogue of $\mathfrak{sl}(2)$, can be realized as a subalgebra of $\mathcal{K}(1)$, see Eq. (2.2) below.

It was discovered in [12] that $\mathfrak{D}_{\lambda, \mu}$ has an important refinement to a $\mathcal{K}(1)$ -invariant $\mathbb{N}/2$ -filtration:

$$\mathfrak{D}_{\lambda, \mu}^0 \subset \mathfrak{D}_{\lambda, \mu}^{\frac{1}{2}} \subset \mathfrak{D}_{\lambda, \mu}^1 \subset \mathfrak{D}_{\lambda, \mu}^{\frac{3}{2}} \subset \cdots \subset \mathfrak{D}_{\lambda, \mu}^{i-\frac{1}{2}} \subset \mathfrak{D}_{\lambda, \mu}^i \cdots \quad (1.1)$$

The quotient module $\mathfrak{D}_{\lambda, \mu}^i / \mathfrak{D}_{\lambda, \mu}^{i-\frac{1}{2}}$ is isomorphic to the module of weighted densities $\Pi^{2i}(\mathfrak{F}_{\mu-\lambda-i})$ (see, e.g., [12]), where Π is the change of parity map. Thus, the graded $\mathcal{K}(1)$ -module $\text{gr}\mathfrak{D}_{\lambda, \mu}$ associated with the filtration (1.1) is a direct sum of density modules:

$$\text{gr}\mathfrak{D}_{\lambda, \mu} = \bigoplus_{i=0}^{\infty} \Pi^i \left(\mathfrak{F}_{\mu-\lambda-\frac{i}{2}} \right).$$

Note that this module depends only on the shift, $\mu - \lambda$, of the weights and not on μ and λ independently. We call this $\mathcal{K}(1)$ -module the space of symbols of differential operators and denote it $\mathfrak{S}_{\mu-\lambda}$, a super analogue of $\mathcal{S}_{\mu-\lambda}$. For generic values of λ and μ , $\mathfrak{S}_{\mu-\lambda}$ and $\mathfrak{D}_{\lambda, \mu}$ are isomorphic as $\mathfrak{osp}(1|2)$ -modules (cf. [12]).

In this paper we classify all $\mathfrak{osp}(1|2)$ -invariant superanti-symmetric binary differential operators from $\mathcal{K}(1) \wedge \mathcal{K}(1)$ to $\mathfrak{D}_{\lambda, \mu}$. We use the result to compute $H_{\text{diff}}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$. We show that nonzero cohomology $H_{\text{diff}}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$ only appear for resonant values of weights that satisfy $\mu - \lambda \in \frac{1}{2}\mathbb{N} + 3$. These spaces allow us to study the generic formal $\mathfrak{osp}(1|2)$ -trivial deformations of the natural action of $\mathcal{K}(1)$ on the superspace of symbols $\mathfrak{S}_{\mu-\lambda}$ and classify the nontrivial projectively invariant extensions of the Lie superalgebra $\mathcal{K}(1)$ by the module $\mathfrak{D}_{\lambda, \mu}$.

Recall that the same problem was considered in [4] for the case of $(1, 2)$ -dimensional real superspace and solved for values of weights that satisfy $\mu - \lambda \leq 6$ or $\mu - \lambda$ is semi-integer, and conjecturally for $\mu - \lambda \in \mathbb{N} + 7$. In the case of $(1, n)$ -dimensional complex superspace, $n \geq 3$, this problem is related to the classification of $\mathfrak{osp}(n|2)$ -invariant bilinear differential operators acting on the superspaces of weighted densities, where $\mathfrak{osp}(n|2)$ is the orthosymplectic Lie superalgebra—superization of Cohen-Rankin operators related with n -extended superstrings (see [14, 15]), and still out of reach.

2. DEFINITIONS AND NOTATIONS

Let $\mathcal{C}^{1|1} := (\mathbb{C}, \mathbb{C}[x, \theta])$ be the ringed space associated with the linear superspace $\mathbb{C}^{1|1}$, where x is a coordinate on $\mathbb{C}$ and θ is a generator of the Grassmann algebra,

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x is supposed to be even and θ odd. Here $\theta^2 = 0$, so $\mathbb{C}[x, \theta]$ has a basis $\{1, \theta\}$ over $\mathbb{C}[x]$. On the space $\mathbb{C}[x, \theta]$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|}\bar{\eta}(F) \cdot \bar{\eta}(G).$$

where the subscript $'$ stands for $\frac{\partial}{\partial x}$, $|F|$ is the parity of F and $\bar{\eta} = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial x}$. Note that the derivation $\bar{\eta}$ is the generator of 1-extended supersymmetry and generates the kernel of the contact 1-form

$$\alpha = dx + \theta d\theta$$

as a module over the ring of polynomial functions. Let $\mathbf{vect}(1|1)$ be the superspace of polynomial vector fields on $\mathcal{C}^{1|1}$:

$$\mathbf{vect}(1|1) = \{F_0\partial_x + F_1\partial_\theta \mid F_0, F_1 \in \mathbb{C}[x, \theta]\},$$

where $\partial_\theta = \frac{\partial}{\partial \theta}$ and $\partial_x = \frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(1)$ of contact polynomial vector fields on $\mathcal{C}^{1|1}$. That is, $\mathcal{K}(1)$ is the superspace of vector fields on $\mathcal{C}^{1|1}$ preserving the distribution singled out by the 1-form α :

$$\mathcal{K}(1) = \{X \in \mathbf{vect}(1|1) \mid \text{there exists } F \in \mathbb{C}[x, \theta] \text{ such that } L_X(\alpha) = F\alpha\},$$

where L_X is the Lie derivative along the vector field X . The Lie superalgebra $\mathcal{K}(1)$ is spanned by the fields of the form:

$$X_F = F\partial_x - \frac{1}{2}(-1)^{|F|}\bar{\eta}(F)\bar{\eta}, \quad \text{where } F \in \mathbb{C}[x, \theta].$$

Of course, $\mathcal{K}(1)$ is a subalgebra of $\mathbf{vect}(1|1)$, and $\mathcal{K}(1)$ acts on $\mathbb{C}[x, \theta]$ through:

$$\mathfrak{L}_{X_F}(G) = FG' - \frac{1}{2}(-1)^{|F|}\bar{\eta}(F) \cdot \bar{\eta}(G).$$

The bracket in $\mathcal{K}(1)$ can be written as: $[X_F, X_G] = X_{\{F, G\}}$.

The orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ can be realized as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta}, X_\theta). \quad (2.2)$$

The space of even elements is isomorphic to $\mathfrak{sl}(2)$, while the space of odd elements is two dimensional:

$$(\mathfrak{osp}(1|2))_{\bar{1}} = \text{Span}(X_\theta, X_{x\theta}).$$

We define the space of λ -densities as

$$\mathfrak{F}_\lambda = \{F(x, \theta)\alpha^\lambda \mid F(x, \theta) \in \mathbb{C}[x, \theta]\}$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $\mathbb{C}[x, \theta]$, but the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_F in $\mathcal{K}(1)$ is now:

$$\mathfrak{L}_{X_F}(G\alpha^\lambda) = \mathfrak{L}_{X_F}^\lambda(G)\alpha^\lambda, \quad \text{with } \mathfrak{L}_{X_F}^\lambda(G) = \mathfrak{L}_{X_F}(G) + \lambda F'G.$$

A differential operator on $\mathbb{C}^{1|1}$ is an operator on $\mathbb{C}[x, \theta]$ of the form:

$$A = \sum_{k=0}^M \sum_{\varepsilon} a_{k,\varepsilon}(x, \theta) \partial_x^k \partial_\theta^\varepsilon; \quad \varepsilon = 0, 1; \quad M \in \mathbb{N}.$$

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Of course any differential operator defines a linear mapping $F\alpha^\lambda \mapsto (AF)\alpha^\mu$ from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any $\lambda, \mu \in \mathbb{C}$, thus the space of differential operators becomes a family of $\mathcal{K}(1)$ -modules $\mathfrak{D}_{\lambda,\mu}$ for the natural action:

$$\mathfrak{L}_{X_F}^{\lambda,\mu}(A) = \mathfrak{L}_{X_F}^\mu \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}_{X_F}^\lambda.$$

2.1. Lie superalgebra cohomology. Let us first recall some fundamental concepts from cohomology theory (see, e.g., [10, 11]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a superspace $V = V_0 \oplus V_1$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{h}$ is omitted it assumed to be $\{0\}$.) The space of $\mathfrak{h}$ -relative n -cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$ -module

$$C^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}); V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g}, \mathfrak{h}; V) \longrightarrow C^{n+1}(\mathfrak{g}, \mathfrak{h}; V)$ is an even map satisfying $\delta_n \circ \delta_{n-1} = 0$ (see for instance, [16]): for $\phi \in C^n(\mathfrak{g}, \mathfrak{h}; V)$,

$$\begin{aligned} (\delta_n \phi)(g_0, \dots, g_n) &= \sum_{i=0}^n (-1)^i (-1)^{|g_i|(|\phi|+|g_0|+\dots+|g_{i-1}|)} g_i \phi(g_0, \dots, \hat{g}_i, \dots, g_n) + \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} (-1)^{|g_i|(|g_0|+\dots+|g_{i-1}|)} (-1)^{|g_j|(|g_0|+\dots+\hat{g}_i+\dots+|g_{j-1}|)} \phi([g_i, g_j], g_0, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_n). \end{aligned}$$

The kernel of δ_n , denoted $Z^n(\mathfrak{g}, \mathfrak{h}; V)$, is the space of $\mathfrak{h}$ -relative n -cocycles, among them, the elements in the range of δ_{n-1} are called $\mathfrak{h}$ -relative n -coboundaries. We denote $B^n(\mathfrak{g}, \mathfrak{h}; V)$ the space of n -coboundaries.

By definition, the n^{th} $\mathfrak{h}$ -relative cohomology space is the quotient space

$$H^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V) / B^n(\mathfrak{g}, \mathfrak{h}; V).$$

We can also define an $\mathfrak{g}$ -action π on $C^n(\mathfrak{g}, V)$ by setting, for any $g \in \mathfrak{g}$,

$$(\pi(g)\phi)(g_1, \dots, g_n) = g\phi(g_1, \dots, g_n) - \sum_{i=1}^n (-1)^{|g|(|\phi|+|g_1|+\dots+|g_{i-1}|)} \phi(g_1, \dots, [g, g_i], \dots, g_n),$$

and a contraction operator $\iota(g)$ from C^n to C^{n-1} by

$$(\iota(g)\phi)(g_1, \dots, g_{n-1}) = (-1)^{|g||\phi|} \phi(g, g_1, \dots, g_{n-1}).$$

A direct computation gives the classical formula

$$\pi(g)\phi = (\delta_{n-1} \circ \iota(g) + \iota(g) \circ \delta_n)\phi,$$

and thus $\delta_n(\pi(g)\phi) = \pi(g)(\delta_n\phi)$; that is δ_n is a $\mathfrak{g}$ -map. Note that $C^n(\mathfrak{g}, \mathfrak{h}; V)$ may be viewed as the subspace of $C^n(\mathfrak{g}, V)$ annihilated by both $\iota(\mathfrak{h})$ and $\pi(\mathfrak{h})$. We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0, 1 and 2: for $v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{h}}$, $\delta v(g) := (-1)^{|g||v|} g \cdot v$, where

$$V^{\mathfrak{h}} = \{v \in V \mid h \cdot v = 0 \text{ for all } h \in \mathfrak{h}\}.$$

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3. THE $\mathfrak{osp}(1|2)$ -RELATIVE COHOMOLOGY OF $\mathcal{K}(1)$ ACTING ON $\mathfrak{D}_{\lambda,\mu}$

3.1. $\mathfrak{osp}(1|2)$ -invariant binary differential operators. The following steps to compute the relative cohomology has intensively been used in [4, 5, 6, 7, 8, 9, 17]. First, we classify $\mathfrak{osp}(1|2)$ -invariant differential operators, then we isolate among them those that are 2-cocycles. To do that, we need the following Lemma.

Lemma 3.1. *Any 2-cocycle vanishing on the subalgebra $\mathfrak{osp}(1|2)$ of $\mathcal{K}(1)$ is $\mathfrak{osp}(1|2)$ -invariant.*

Proof. The 2-cocycle condition reads as follows

$$(-1)^{|X||c|} \mathfrak{L}_X^{\lambda,\mu} c(Y, Z) - (-1)^{|Y|(|X|+|c|)} \mathfrak{L}_Y^{\lambda,\mu} c(X, Z) + (-1)^{|Z|(|X|+|Y|+|c|)} \mathfrak{L}_Z^{\lambda,\mu} c(X, Y) - c([X, Y], Z) + (-1)^{|Y||Z|} c([X, Z], Y) + c(X, [Y, Z]) = 0$$

for every $X, Y, Z \in \mathcal{K}(1)$. Now, if $X \in \mathfrak{osp}(1|2)$, then the equation above becomes

$$c([X, Y], Z) - (-1)^{|Y||Z|} c([X, Z], Y) = (-1)^{|X||c|} \mathfrak{L}_X^{\lambda,\mu} c(Y, Z).$$

This relation is nothing but the $\mathfrak{osp}(1|2)$ -invariance property of the bilinear map c . $\square$

As our 2-cocycles vanish on $\mathfrak{osp}(1|2)$, we will investigate $\mathfrak{osp}(1|2)$ -invariant superanti-symmetric binary differential operators that vanish on $\mathfrak{osp}(1|2)$. The following Proposition is known and it is a corollary of the results by Gieres and Theisen, see [14].

Proposition 3.2. *The space of superanti-symmetric bilinear differential operators $\mathcal{K}(1) \wedge \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\mu}$, which are $\mathfrak{osp}(1|2)$ -invariant and vanish on $\mathfrak{osp}(1|2)$, is purely even if $\mu - \lambda$ is integer and it is purely odd if $\mu - \lambda$ is semi-integer; moreover it is as follows:*

- (i) *It is $(2p - 5)$ -dimensional if $(\mu - \lambda) = 2p - 2$ and $p \geq 3$.*
- (ii) *It is $(2p - 3)$ -dimensional if $(\mu - \lambda) = 2p - 1$ or $(\mu - \lambda) = 2p - \frac{1}{2}$ and $p \geq 2$.*
- (iii) *It is $(2p - 4)$ -dimensional if $(\mu - \lambda) = 2p - \frac{3}{2}$ and $p \geq 3$.*
- (iv) *It is 0-dimensional, otherwise.*

Proof. The general form of any such a differential operator is:

$$c(X_F, X_G) = \sum_{\substack{\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \\ 0 \leq k_1, k_2, k_3 \leq M}} c_\varepsilon^{k_1, k_2, k_3}(x, \theta, |F|, |G|) \bar{\eta}^{\varepsilon_1}(F^{(k_1)}) \bar{\eta}^{\varepsilon_2}(G^{(k_2)}) \bar{\eta}^{\varepsilon_3} \partial_x^{k_3},$$

where $\varepsilon_i = 0$ or 1 , $M \in \mathbb{N}$, $\varepsilon_1 + k_1 \geq 3$ and $\varepsilon_2 + k_2 \geq 3$. The invariance property of c with respect to X_1 and X_x implies that

$$\frac{d}{dx} c_\varepsilon^{k_1, k_2, k_3} = 0, \quad \bar{\eta}(c_\varepsilon^{k_1, k_2, k_3}) = 0 \quad \text{and} \quad \sum_{i=1}^3 (\varepsilon_i + 2k_i) = 2(\mu - \lambda) + 4.$$

Therefore, the coefficients $c_\varepsilon^{k_1, k_2, k_3}$ are functions of $|F|$ and $|G|$ and the parameters λ and μ must satisfy $2(\mu - \lambda) + 4 = n$, where $n \in \mathbb{N}$. The corresponding operator

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can be expressed as:

$$c(X_F, X_G) = \sum_{\varepsilon, k_1, k_2} c_{\varepsilon}^{k_1, k_2, n}(|F|, |G|) \bar{\eta}^{\varepsilon_1}(F^{(k_1)}) \bar{\eta}^{\varepsilon_2}(G^{(k_2)}) \bar{\eta}^{\varepsilon_3} \partial_x^{\frac{1}{2}(n - \sum_{i=1}^3 \varepsilon_i) - k_1 - k_2}. \quad (3.3)$$

The superanti-symmetric condition: $c(X_F, X_G) = -(-1)^{|F||G|}c(X_G, X_F)$ leads to the following relation

$$c_{\varepsilon_1, \varepsilon_2, \varepsilon_3}^{k_1, k_2, n}(|F|, |G|) = -(-1)^{\varepsilon_2 \cdot |F| + \varepsilon_1 \cdot |G| + \varepsilon_1 \cdot \varepsilon_2} c_{\varepsilon_2, \varepsilon_1, \varepsilon_3}^{k_2, k_1, n}(|G|, |F|). \quad (3.4)$$

Now, for the sake of completeness, we consider the invariance property with respect to $X_{x\theta}$. According to the parity of n , we distinguish two cases.

The case where n is even.

In this case, the invariance property of c with respect to $X_{x\theta}$ leads to the following relations

$$c_{\varepsilon_1, \varepsilon_2, \varepsilon_3}^{k_1, k_2, n}(|F|, |G|) = (-1)^{\varepsilon_1} c_{\varepsilon_1, \varepsilon_2, \varepsilon_3}^{k_1, k_2, n}(|F| + 1, |G|) = (-1)^{\varepsilon_1 + \varepsilon_2} c_{\varepsilon_1, \varepsilon_2, \varepsilon_3}^{k_1, k_2, n}(|F|, |G| + 1). \quad (3.5)$$

and

$$\begin{aligned} \Lambda_{k_1, k_2}^n c_{0,0,0}^{k_1, k_2, n} - (k_1 - 2) c_{1,0,1}^{k_1, k_2, n} - (-1)^{|F|} (k_2 - 2) c_{0,1,1}^{k_1, k_2, n} &= 0, & k_1 + k_2 \leq \frac{n-2}{2} \text{ and } k_1 > k_2, \\ \Lambda_{k_1, k_2}^n c_{1,1,0}^{k_1, k_2, n} + (k_1 + 1) c_{0,1,1}^{k_1+1, k_2, n} - (-1)^{|F|} (k_2 + 1) c_{1,0,1}^{k_1, k_2+1, n} &= 0, & k_1 + k_2 \leq \frac{n-4}{2} \text{ and } k_1 \geq k_2 \\ \Lambda_{k_1, k_2-2\lambda}^n c_{1,0,1}^{k_1, k_2, n} + (k_1 + 1) c_{0,0,0}^{k_1+1, k_2, n} + (-1)^{|F|} (k_2 - 2) c_{1,1,0}^{k_1, k_2, n} &= 0, & k_1 + k_2 \leq \frac{n-2}{2}, \end{aligned} \quad (3.6)$$

where $\Lambda_{k_1, k_2}^n = (-1)^{|F|+|G|}(\frac{n}{2} - k_1 - k_2)$. According to formulae (3.4) and (3.5), we deduce that $c_{0,0,0}^{k_1, k_1, n} = 0$. Now, we can see, with the help of the maple, that the system (3.6) is linearly independent. Further, by (3.4), we can see that all the coefficients $c_{\varepsilon}^{k_1, k_2, n}$ can be expressed in terms of

$$c_{1,0,1}^{k_1, k_2, n} \text{ with } k_1 \geq 2 \text{ and } k_2 \geq 3, \quad c_{0,0,0}^{k_1, k_2, n} \text{ with } k_1 > k_2 \geq 3 \text{ and } c_{1,1,0}^{k_1, k_2, n} \text{ with } k_1 \geq k_2 \geq 2. \quad (3.7)$$

So, we deduce that the dimension of the space of solution is equal to

$$\text{number (of coefficients } c_{\varepsilon}^{k_1, k_2, n} \text{ given by (3.7))} - \text{number (of equations given by (3.6))}.$$

A straightforward computation shows that that the space of $\mathfrak{osp}(1|2)$ -invariant operators has the following structure:

(i) For $n = 4p$, it is $(\frac{n}{2} - 5)$ -dimensional and can be spanned by

$$c_{1,0,1}^{2,3,n}, c_{1,0,1}^{2,4,n}, \dots, c_{1,0,1}^{2, \frac{n}{2}-3, n}.$$

(ii) For $n = 4p + 2$, it is $(\frac{n}{2} - 4)$ -dimensional and can be spanned by $c_{0,0,0}^{\frac{n}{2}-3, 3, n}$, $c_{1,1,0}^{2,2,n}$ and

$$c_{1,0,1}^{2,4,n}, c_{1,0,1}^{2,5,n}, \dots, c_{1,0,1}^{2, \frac{n}{2}-3, n}.$$

The case where n is odd.

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In this case, the invariance property of c with respect to $X_{x\theta}$ leads to the following relations

$$c_{\varepsilon_1, \varepsilon_2, \varepsilon_3}^{k_1, k_2, n}(|F|, |G|) = (-1)^{\varepsilon_2 + \varepsilon_3} c_{\varepsilon_1, \varepsilon_2, \varepsilon_3}^{k_1, k_2, n}(|F| + 1, |G|) = (-1)^{\varepsilon_3} c_{\varepsilon_1, \varepsilon_2, \varepsilon_3}^{k_1, k_2, n}(|F|, |G| + 1). \quad (3.8)$$

and

$$\begin{aligned} \Lambda_{k_1, k_2 + \frac{1}{2} - 2\lambda}^n c_{0,0,1}^{k_1, k_2, n} + (k_1 - 2) c_{1,0,0}^{k_1, k_2, n} + (-1)^{|F|} (k_2 - 2) c_{0,1,0}^{k_1, k_2, n} &= 0, \quad k_1 + k_2 \leq \frac{n-1}{2} \text{ and } k_1 > k_2, \\ \Lambda_{k_1, k_2 + \frac{3}{2} - 2\lambda}^n c_{1,1,1}^{k_1, k_2, n} - (k_1 + 1) c_{0,1,0}^{k_1+1, k_2, n} + (-1)^{|F|} (k_2 + 1) c_{1,0,0}^{k_1, k_2+1, n} &= 0, \quad k_1 + k_2 \leq \frac{n-3}{2} \text{ and } k_1 \geq k_2, \\ \Lambda_{k_1, k_2 + \frac{1}{2}}^n c_{1,0,0}^{k_1, k_2, n} - (k_1 + 1) c_{0,0,1}^{k_1+1, k_2, n} - (-1)^{|F|} (k_2 - 2) c_{1,1,1}^{k_1, k_2, n} &= 0, \quad k_1 + k_2 \leq \frac{n-3}{2}, \end{aligned}$$

According to formulae (3.4) and (3.8), we deduce that $c_{0,0,1}^{k_1, k_1, n} = 0$. Now, the same arguments as in the previous case, show that the space of $\mathfrak{osp}(1|2)$ -invariant operators has the following structure:

- (i) For $n = 4p + 1$, it is $(\frac{n-9}{2})$ -dimensional and can be spanned by $c_{1,0,0}^{2, \frac{n-5}{2}, n}$ and

$$c_{1,1,1}^{2,2,n}, c_{1,1,1}^{3,2,n}, \dots, c_{1,1,1}^{\frac{n-9}{2}, 2, n}.$$

- (ii) For $n = 4p + 3$, it is $(\frac{n-9}{2})$ -dimensional and can be spanned by $c_{0,0,1}^{\frac{n-7}{2}, 3, n}$ and

$$c_{1,1,1}^{2,2,n}, c_{1,1,1}^{3,2,n}, \dots, c_{1,1,1}^{\frac{n-9}{2}, 2, n}.$$

Proposition 3.2 is proved. $\square$

3.2. The $\mathfrak{osp}(1|2)$ -relative cohomology of $\mathcal{K}(1)$. In this subsection, we will compute the second differential $\mathfrak{osp}(1|2)$ -relative cohomology spaces $H_{\text{diff}}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$. Our main result is the following:

Theorem 3.3. *The space $H_{\text{diff}}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$ has the following structure:*

$$H_{\text{diff}}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}) \simeq \begin{cases} \mathbb{R} & \text{if } \left\{ \begin{aligned} &(\lambda, \mu) \in \left\{ \begin{aligned} &(0, 3), (-\frac{5}{2}, \frac{1}{2}), (-\frac{3}{2}, 2), (-2, \frac{5}{2}) \\ &(\frac{-7+\sqrt{33}}{4}, \frac{9+\sqrt{33}}{4}), (\frac{-7-\sqrt{33}}{4}, \frac{9-\sqrt{33}}{4}) \end{aligned} \right\} \\ &\mu - \lambda = 5 \text{ for all } \lambda, \\ &\mu - \lambda = \frac{11}{2} \text{ and } \lambda \neq -\frac{5}{2}, \\ &\mu - \lambda = p - \frac{3}{2} \text{ and } \lambda = 1 - \frac{p}{2}, \quad p \in \mathbb{N} + 8, \\ &\mu - \lambda = p - 2 \text{ and } \lambda = -\frac{2p-5 \pm \sqrt{8p-15}}{4}, \quad p \in \mathbb{N} + 8, \end{aligned} \right. \\ \mathbb{R}^2 & \text{if } \mu - \lambda = \frac{11}{2} \text{ and } \lambda = -\frac{5}{2} \\ 0 & \text{otherwise,} \end{cases}$$

We can easily check that all the results of Theorem 3.3 are invariant under passage to adjoint values. We refer to [2] for adjoint differential operator modules.

4. COMPUTING THE COHOMOLOGY

The proof of Theorem 3.3 will be the subject of subsection 4.2. In fact, we need first the description of $\mathfrak{osp}(1|2)$ -invariant bilinear operators, from $\mathfrak{F}_{-1} \otimes \mathfrak{F}_{\lambda}$ to $\mathfrak{F}_{\lambda+k-1}$, called supertransvectants.

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4.1. Supertransvectants: an explicit formula. Gieres and Theisen [14] listed the supertransvectants and they expressed them in terms of supercovariant derivatives then Gargoubi and Ovsienko [13] gave an interpretation of these operators, later Ben Fraj et al. [1] gave another description of the space of $\mathfrak{osp}(1|2)$ -invariant bilinear differential operators

$$\begin{aligned}\mathfrak{J}_k^\lambda : \mathfrak{F}_{-1} \otimes \mathfrak{F}_\lambda &\longrightarrow \mathfrak{F}_{\lambda+k-1} \\ (F\alpha^{-1}, G\alpha^\lambda) &\mapsto \mathfrak{J}_k^\lambda(F, G)\alpha^{\lambda+k-1}\end{aligned}$$

vanishing on $\mathfrak{osp}(1|2)$, where $k \in \frac{1}{2}\mathbb{N}$. They showed that these spaces are one dimensional for $k > 2$. The operators $\mathfrak{J}_k^\lambda$, where $k \in \mathbb{N}$, are even and they are given by:

$$\mathfrak{J}_k^\lambda(F, G) = \sum_{\substack{i+j=k-1 \\ i \geq 2}} \Gamma_{i-2, j, \frac{2k-5}{2}}^{\frac{3}{2}, \lambda} \left((2l+k-1-i)F^{(i+1)}G^{(j)} - (-1)^{|F|}(k-j)\bar{\eta}(F^{(i)})\bar{\eta}(G^{(j)}) \right). \quad (4.9)$$

The operators $\mathfrak{J}_k^\lambda$ labeled by semi-integer k are odd; set $\mathfrak{J}_{\frac{\lambda}{2}}^\lambda(F, G) = \bar{\eta}(F'')G$ and

$$\mathfrak{J}_k^\lambda(F, G) = \sum_{\substack{i+j=k-\frac{1}{2} \\ i \geq 3}} (-1)^{|F|} \Gamma_{i-3, j, \frac{2k-7}{2}}^{\frac{3}{2}, \lambda} F^{(i)}\bar{\eta}(G^{(j)}) - \sum_{\substack{i+j=k-\frac{1}{2} \\ i \geq 2}} \Gamma_{i-2, j, \frac{2k-7}{2}}^{\frac{3}{2}, \lambda} \bar{\eta}(F^{(i)})G^{(j)}, \quad (4.10)$$

with $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$ and $[k]$ denotes the integer part of k , $k > 0$, and

$$\Gamma_{i, j, k}^{\tau, \lambda} = (-1)^j \binom{2\tau + [k]}{j} \binom{2\lambda + [k]}{i}.$$

Remark 4.1. *The supertransvectants appear in many contexts, especially in the computation of cohomology (cf. [1, 12]). We refer to [14] for their history.*

In order to prove Theorem 3.3, we will study properties of the coboundaries.

Lemma 4.2. *If $\delta(B)$ belongs to $B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$, where $B : \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda, \mu}$ is an operator vanishing on $\mathfrak{osp}(1|2)$, then B is a supertransvectant.*

Proof. For all $X, Y \in \mathcal{K}(1)$ we have

$$\delta(B)(X, Y) := (-1)^{|X||B|} \mathfrak{L}_X^{\lambda, \mu} B(Y) - (-1)^{|Y|(|X|+|B|)} \mathfrak{L}_Y^{\lambda, \mu} B(X) - B([X, Y]).$$

Since $B(X) = 0$ for all $X \in \mathfrak{osp}(1|2)$ we deduce that

$$(-1)^{|X||B|} \mathfrak{L}_X^{\lambda, \mu} B(Y) - B([X, Y]) = 0.$$

Thus, the operator B is $\mathfrak{osp}(1|2)$ -invariant; therefore it coincides with the supertransvectants.

Now, clearly the coboundary $\delta(\mathfrak{J}_k^\lambda)$ has the following form:

$$\delta(\mathfrak{J}_k^\lambda)(X_F, X_G) = \sum_{\varepsilon, k_1, k_2} \beta_\varepsilon^{k_1, k_2, 2k+2} (|F|, |G|) \bar{\eta}^{\varepsilon_1}(F^{(k_1)}) \bar{\eta}^{\varepsilon_2}(G^{(k_2)}) \bar{\eta}^{\varepsilon_3} \partial_x^{k+1-k_1-k_2-\frac{1}{2} \sum_{i=1}^3 \varepsilon_i}, \quad (4.11)$$

where the coefficients $\beta_\varepsilon^{k_1, k_2, 2k+2}$ are well-determined values of the coefficients $c_\varepsilon^{k_1, k_2, 2k+2}$ given by (3.3). The following Lemma will be useful in the proof of Theorem 3.3.

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Lemma 4.3. *i) We have*

$$\begin{aligned}\beta_{1,1,1}^{2,2,13} &= -(-1)^{|G|}30(\lambda+2), \\ \beta_{0,0,0}^{4,3,14} &= \frac{20}{3}\lambda(2\lambda+3)(\lambda+2)(\lambda+4), \\ \beta_{1,0,1}^{2,4,14} &= (-1)^{|F|+|G|}30\left(\lambda+\frac{9+\sqrt{41}}{4}\right)\left(\lambda+\frac{9-\sqrt{41}}{4}\right), \\ \beta_{0,0,1}^{4,3,15} &= (-1)^{|F|+|G|}\frac{5}{6}(2\lambda+1)(2\lambda+5)(2\lambda+9), \\ \beta_{1,1,1}^{2,2,15} &= (-1)^{|G|}45(2\lambda+5).\end{aligned}$$

ii) For $\lambda \in \mathbb{R}$ and $p \in \mathbb{N} + 8$, we have

$$\begin{aligned}\beta_{1,1,1}^{2,2,2p+1} &= -(-1)^{|G|+p}\binom{p-1}{p-5}(6\lambda+3p-6), \\ \beta_{1,0,1}^{2,4,2p} &= -\frac{(-1)^{|F|+|G|+p}}{8}\binom{p-5}{2}\binom{p-1}{p-5}(4\lambda+2p-5+\sqrt{8p-15}) \times \\ &\quad \times (4\lambda+2p-5-\sqrt{8p-15}).\end{aligned}$$

Proof. i) • To compute the coefficient $\beta_{1,1,1}^{2,2,13}$, we must compute the coboundary $\delta(\mathfrak{J}_{\frac{1}{2}}^\lambda)(X_F, X_G)$, where

$$\begin{aligned}\mathfrak{J}_{\frac{1}{2}}^\lambda(X_G) &= (-1)^{|G|}\left(\Gamma_{0,2,2}^{\frac{3}{2},\lambda}G^{(3)}\bar{\eta}\partial_x^2 + \Gamma_{1,1,2}^{\frac{3}{2},\lambda}G^{(4)}\bar{\eta}\partial_x + \Gamma_{2,0,2}^{\frac{3}{2},\lambda}G^{(5)}\bar{\eta}\right) - \\ &\quad - \left(\Gamma_{0,3,2}^{\frac{3}{2},\lambda}\bar{\eta}(G'')\partial_x^3 + \Gamma_{1,2,2}^{\frac{3}{2},\lambda}\bar{\eta}(G^{(3)})\partial_x^4 + \Gamma_{2,1,2}^{\frac{3}{2},\lambda}\bar{\eta}(G^{(4)})\partial_x + \Gamma_{3,0,2}^{\frac{3}{2},\lambda}\bar{\eta}(G^{(5)})\right).\end{aligned}$$

Collecting the terms in $\bar{\eta}(F'')\bar{\eta}(G'')\bar{\eta}\partial_x$, we get

$$\beta_{1,1,1}^{2,2,13} = 3(-1)^{|G|}\left(\Gamma_{0,3,2}^{\frac{3}{2},\lambda} + \Gamma_{1,1,2}^{\frac{3}{2},\lambda}\right) = -30(-1)^{|G|}(\lambda+2).$$

• To compute the coefficient $\beta_{1,0,1}^{2,4,14}$, we must compute the coboundary $\delta(\mathfrak{J}_6^\lambda)(X_F, X_G)$, where

$$\begin{aligned}\mathfrak{J}_6^\lambda(X_G) &= \Gamma_{0,3,3}^{\frac{3}{2},\lambda}\left((2\lambda+3)G^{(3)}\partial_x^3 - 3(-1)^{|G|}\bar{\eta}(G'')\bar{\eta}\partial_x^3\right) + \\ &\quad \Gamma_{1,2,3}^{\frac{3}{2},\lambda}\left((2\lambda+2)G^{(4)}\partial_x^2 - 4(-1)^{|G|}\bar{\eta}(G^{(3)})\bar{\eta}\partial_x^2\right) + \\ &\quad \Gamma_{2,1,3}^{\frac{3}{2},\lambda}\left((2\lambda+1)G^{(5)}\partial_x - 5(-1)^{|G|}\bar{\eta}(G^{(4)})\bar{\eta}\partial_x\right) + \\ &\quad \Gamma_{3,0,3}^{\frac{3}{2},\lambda}\left((2\lambda)G^{(6)} - 6(-1)^{|G|}\bar{\eta}(G^{(5)})\bar{\eta}\right).\end{aligned}$$

Collecting the terms in $\bar{\eta}(F'')G^{(4)}\bar{\eta}$, we get

$$\begin{aligned}\beta_{1,0,1}^{2,4,14} &= (-1)^{|F|+|G|}\left((\lambda+1)\Gamma_{1,2,3}^{\frac{3}{2},\lambda} - (3\lambda+\frac{3}{2})\Gamma_{0,3,3}^{\frac{3}{2},\lambda}\right) \\ &= (-1)^{|F|+|G|}(15(2\lambda+3)(\lambda+1) + 20(3\lambda+\frac{3}{2})) \\ &= (-1)^{|F|+|G|}15(2\lambda^2+9\lambda+5) \\ &= (-1)^{|F|+|G|}30\left(\lambda+\frac{9+\sqrt{41}}{4}\right)\left(\lambda+\frac{9-\sqrt{41}}{4}\right).\end{aligned}$$

We point out that, under the coboundary operator δ , only the terms $-3(-1)^{|G|}\Gamma_{0,3,3}^{\frac{3}{2},\lambda}\bar{\eta}(G'')\bar{\eta}\partial_x^3$, $\Gamma_{1,2,3}^{\frac{3}{2},\lambda}(2\lambda+2)G^{(4)}\partial_x^2$ and $\Theta(X_G) = -6(-1)^{|G|}\Gamma_{3,0,3}^{\frac{3}{2},\lambda}\bar{\eta}(G^{(5)})\bar{\eta}$ in the expression of $\mathfrak{J}_6^\lambda$ allow us to compute the coefficient $\beta_{1,0,1}^{2,4,14}$. Note that the coefficient of $\bar{\eta}(F'')G^{(4)}\bar{\eta}$ in $\delta(\Theta)(X_F, X_G)$ vanishes.

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• For the coefficients $\beta_{0,0,0}^{4,3,14}$, $\beta_{0,0,1}^{4,3,15}$ and $\beta_{1,1,1}^{2,2,15}$, the proof is the same as in the previous case.

ii) To compute the coefficient $\beta_{1,1,1}^{2,2,2p+1}$, we must compute the coboundary $\delta(\mathfrak{J}_{p-\frac{1}{2}}^\lambda)(X_F, X_G)$, where $\mathfrak{J}_{p-\frac{1}{2}}^\lambda$ is as (4.10). Collecting the terms in $\bar{\eta}(F'')\bar{\eta}(G'')\bar{\eta}\partial_x^{p-5}$, we get

$$\begin{aligned}\beta_{1,1,1}^{2,2,2p+1} &= (-1)^{|G|} \left(\binom{p-3}{2} \Gamma_{0,p-3,p-4}^{\frac{3}{2},\lambda} + \frac{1}{2} \binom{p-4}{2} \Gamma_{1,p-5,p-4}^{\frac{3}{2},\lambda} \right) \\ &= -(-1)^{|G|+p} \binom{p-1}{p-5} (6\lambda + 3p - 6).\end{aligned}$$

We point out that, under the coboundary operator δ , only the terms $-\Gamma_{0,p-3,p-4}^{\frac{3}{2},\lambda}\bar{\eta}(G'')\partial_x^{p-3}$ and $(-1)^{|G|}\Gamma_{1,p-5,p-4}^{\frac{3}{2},\lambda}G^{(4)}\partial_x^{p-3}$ in the expression of $\mathfrak{J}_{p-\frac{1}{2}}^\lambda$ allow us to compute the coefficient $\beta_{1,1,1}^{2,2,2p+1}$.

Finally, to compute the coefficient $\beta_{1,0,1}^{2,4,2p}$, we must compute the coboundary $\delta(\mathfrak{J}_{p-1}^\lambda)(X_F, X_G)$, where $\mathfrak{J}_{p-1}^\lambda$ is as (4.9). Collecting the terms in $\bar{\eta}(F'')G^{(4)}\bar{\eta}\partial_x^{p-7}$, we get

$$\begin{aligned}\beta_{1,0,1}^{2,4,2p} &= (-1)^{|F|+|G|} \left(\frac{2\lambda+p-5}{2} \binom{p-5}{2} \Gamma_{1,p-5,\frac{2p-7}{2}}^{\frac{3}{2},\lambda} - 3\Gamma_{0,p-4,\frac{2p-7}{2}}^{\frac{3}{2},\lambda} \left(\binom{p-4}{4} + (\lambda + \frac{1}{2}) \binom{p-4}{3} \right) \right) \\ &= (-1)^{|F|+|G|} \binom{p-5}{2} \left(\frac{2\lambda+p-5}{2} \Gamma_{1,p-5,\frac{2p-7}{2}}^{\frac{3}{2},\lambda} - (p-4) \Gamma_{0,p-4,\frac{2p-7}{2}}^{\frac{3}{2},\lambda} \left(\frac{p-7}{4} + (\lambda + \frac{1}{2}) \right) \right) \\ &= -(-1)^{|F|+|G|+p} \binom{p-5}{2} \binom{p-1}{p-5} \left(\frac{(2\lambda+p-5)(2\lambda+p-4)}{2} + 4\lambda + p - 5 \right) \\ &= \frac{-(-1)^{|F|+|G|+p}}{2} \binom{p-5}{2} \binom{p-1}{p-5} (4\lambda^2 + 2\lambda(2p-5) + (p-5)(p-2)) \\ &= \frac{-(-1)^{|F|+|G|+p}}{8} \binom{p-5}{2} \binom{p-1}{p-5} (4\lambda + 2p - 5 + \sqrt{8p-15}) \times \\ &\quad \times (4\lambda + 2p - 5 - \sqrt{8p-15})\end{aligned}$$

We point out that, under the coboundary operator δ , only the terms

$$\begin{aligned}&\frac{2\lambda+p-5}{2} \Gamma_{1,p-5,\frac{2p-7}{2}}^{\frac{3}{2},\lambda} G^{(4)} \partial_x^{p-5}, \quad 3(-1)^{|G|+1} \Gamma_{0,p-4,\frac{2p-7}{2}}^{\frac{3}{2},\lambda} \bar{\eta}(G'') \bar{\eta} \partial_x^{p-4}, \quad \text{and} \\ &\tilde{\Theta}(X_G) = 6(-1)^{|G|+1} \Gamma_{3,p-7,\frac{2p-7}{2}}^{\frac{3}{2},\lambda} \bar{\eta}(G^{(5)}) \bar{\eta} \partial_x^{p-7}\end{aligned}$$

in the expression of $\mathfrak{J}_{p-1}^\lambda$ allow us to compute the coefficient $\beta_{1,0,1}^{2,4,2p}$. Note that the coefficient of $\bar{\eta}(F'')G^{(4)}\bar{\eta}\partial_x^{p-7}$ in $\tilde{\Theta}(X_F, X_G)$ vanishes. □

4.2. Proof of Theorem 3.3. According to Lemma 3.1, any 2-cocycle of $\mathcal{K}(1)$ with coefficients in $\mathfrak{D}_{\lambda,\mu}$ vanishing on $\mathfrak{osp}(1|2)$ is $\mathfrak{osp}(1|2)$ -invariant. So, by Theorem 3.2, it is identically zero if $\mu - \lambda < 3$ and expressed as in (3.3) for $\mu - \lambda \in \frac{1}{2}\mathbb{N} + 3$.

For $\mu - \lambda \in \frac{1}{2}\mathbb{N} + 3$, the proof of Theorem 3.3 consists in two steps. First, we investigate operators that belong to $Z^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$. The 2-cocycle condition imposes conditions on the coefficients $c_\varepsilon^{k_1,k_2,n}$: we get a linear system for $c_\varepsilon^{k_1,k_2,n}$. Second, taking into account these conditions, we eliminate all coefficients underlying coboundaries. Gluing these bits of information together we deduce that $\dim H^2$ is equal to the number of independent coefficients $c_\varepsilon^{k_1,k_2,n}$ remaining in the expression of the 2-cocycle (3.3).

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4.2.1. *The case where $\mu - \lambda = 3$.* In this case, according to Theorem 3.2, the 2-cocycle (3.3) can be expressed as follows:

$$c(X_F, X_G) = c_{1,1,0}^{2,2,10} \gamma(X_F, X_G),$$

where

$$\gamma(X_F, X_G) = \bar{\eta}(F'')\bar{\eta}(G'').$$

By (3.5), we deduce that, up to a scalar factor, $c_{1,1,0}^{2,2,10} = (-1)^{|F|}$. Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to subsection 4.1, we can see that, any coboundary $\delta(B) \in B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ can be expressed as follows:

$$\delta(B) = \tau \delta(\mathfrak{J}_4^\lambda), \quad \tau \in \mathbb{C}, \quad (4.12)$$

where

$$\mathfrak{J}_4^\lambda(X_G) = -2(2\lambda + 1) \left(2(-1)^{|G|} \bar{\eta}(G^{(3)}) \bar{\eta} - 2G^{(3)} \bar{\eta}^2 - \lambda G^{(4)} \right) - 12(-1)^{|G|} \bar{\eta}(G'') \bar{\eta}^3.$$

The expression (4.12) reads

$$\delta(B)(X_F, X_G) = \tau \left(\mathfrak{L}_{X_F}^{\lambda, \lambda+3} \mathfrak{J}_4^\lambda(X_G) - (-1)^{|F||G|} \mathfrak{L}_{X_G}^{\lambda, \lambda+3} \mathfrak{J}_4^\lambda(X_F) - \mathfrak{J}_4^\lambda([X_F, X_G]) \right).$$

Using the graded Leibniz formula:

$$\bar{\eta}^j \circ F = \sum_{i=0}^j \binom{j}{i}_s (-1)^{|F|(j-i)} \bar{\eta}^i(F) \bar{\eta}^{j-i}, \quad \text{where} \quad \binom{j}{i}_s = \begin{cases} \binom{\lfloor \frac{j}{2} \rfloor}{\lfloor \frac{i}{2} \rfloor} & \text{if } i \text{ is even or } j \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

we can check the following equation

$$\delta(B)(X_F, X_G) = \tau (-1)^{|F|} 6\lambda(2\lambda + 5) \gamma(X_F, X_G).$$

So, for $\lambda \neq 0, -\frac{5}{2}$, we can see that the coefficient $c_{1,1,0}^{2,2,10}$ can be eliminated by adding a coboundary: indeed, the constant τ can be chosen such that $\Upsilon = \delta(B)$. Hence, the cohomology is zero-dimensional. While for $\lambda = 0$ or $-\frac{5}{2}$, the coefficient $c_{1,1,0}^{2,2,10}$ cannot be eliminated by adding a coboundary. Hence, the cohomology space is one-dimensional.

4.2.2. *The case where $\mu - \lambda = \frac{7}{2}$.* In this case, according to Theorem 3.2, the 2-cocycle (3.3) can be expressed as follows:

$$c(X_F, X_G) = c_{1,1,1}^{2,2,11} \gamma(X_F, X_G),$$

where

$$\gamma(X_F, X_G) = \bar{\eta}(F'')\bar{\eta}(G'')\bar{\eta} + (-1)^{|G|} \frac{\lambda}{3} \left((-1)^{|F|} F^{(3)} \bar{\eta}(G'') - \bar{\eta}(F'') G^{(3)} \right).$$

By (3.8), we deduce that, up to a scalar factor, $c_{1,1,1}^{2,2,11} = (-1)^{|G|}$. Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to subsection 4.1, we can see that any coboundary $\delta(B) \in B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ can be expressed as follows:

$$\delta(B) = \tau \delta(\mathfrak{J}_{\frac{9}{2}}^\lambda), \quad \tau \in \mathbb{C}, \quad (4.13)$$

where

$$\mathfrak{J}_{\frac{9}{2}}^\lambda(X_G) = (-1)^{|G|} \left(4G^{(3)} \bar{\eta}^3 + (2\lambda + 1) G^{(4)} \bar{\eta} \right) - 6\bar{\eta}(G'') \bar{\eta}^4 - (2\lambda + 1) \left(4\bar{\eta}(G^{(3)} \bar{\eta}^2 + \lambda \bar{\eta}(G^{(4)})) \right).$$

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The expression (4.13) reads

$$\delta(B)(X_F, X_G) = \tau \left((-1)^{|F|} \mathfrak{L}_{X_F}^{\lambda, \lambda + \frac{7}{2}} \mathfrak{J}_{\frac{9}{2}}^{\lambda}(X_G) - (-1)^{(|F|+1)|G|} \mathfrak{L}_{X_G}^{\lambda, \lambda + \frac{7}{2}} \mathfrak{J}_{\frac{9}{2}}^{\lambda}(X_F) - \mathfrak{J}_{\frac{9}{2}}^{\lambda}([X_F, X_G]) \right).$$

Using the graded Leibniz formula, we can check the following expressions for nonzero $\beta_{\varepsilon}^{k_1, k_2, 11}$:

$$\beta_{1,1,1}^{2,2,11} = \tau(-1)^{|G|} 3(2\lambda + 3), \quad \beta_{1,0,0}^{2,3,11} = -\tau\lambda(2\lambda + 3), \quad \beta_{0,1,0}^{3,2,11} = \tau(-1)^{|F|} \lambda(2\lambda + 3).$$

Thus, we deduce the following equation

$$\delta(B)(X_F, X_G) = \tau(-1)^{|G|} 3(2\lambda + 3) \gamma(X_F, X_G).$$

So, for $\lambda \neq -\frac{3}{2}$, we can see that the coefficient $c_{1,1,1}^{2,2,11}$ can be eliminated by adding a coboundary. Hence, the cohomology is zero-dimensional. Now, for $\lambda = -\frac{3}{2}$, clearly the coefficient $c_{1,1,1}^{2,2,11}$ cannot be eliminated by adding a coboundary. Hence the cohomology is one-dimensional.

4.2.3. *The case where $\mu - \lambda = 4$.* In this case, according to Theorem 3.2, the 2-cocycle (3.3) can be expressed as follows:

$$c(X_F, X_G) = c_{1,0,1}^{2,3,12} \gamma(X_F, X_G),$$

where

$$\begin{aligned} \gamma(X_F, X_G) &= \bar{\eta}(F'') G^{(3)} \bar{\eta} - (-1)^{|F|} F^{(3)} \bar{\eta}(G'') \bar{\eta} + 6(-1)^{|G|} \bar{\eta}(F'') \bar{\eta}(G'') \partial_x \\ &\quad - (-1)^{|G|} 2\lambda \left(\bar{\eta}(F^{(3)}) \bar{\eta}(G'') + \bar{\eta}(F'') \bar{\eta}(G^{(3)}) \right). \end{aligned}$$

By (3.5), we deduce that, up to a scalar factor, $c_{1,0,1}^{2,3,12} = (-1)^{|F|+|G|}$. Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to subsection 4.1, we can see that, any coboundary $\delta(B) \in B^2(\mathcal{K}(1); \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$ can be expressed as follows:

$$\delta(B) = \tau \delta(\mathfrak{J}_5^{\lambda}), \quad \tau \in \mathbb{C},$$

where

$$\begin{aligned} \mathfrak{J}_5^{\lambda}(X_G) &= 20(\lambda + 1) \left(-2(-1)^{|G|} \bar{\eta}(G^{(3)}) \bar{\eta}^3 + G^{(3)} \bar{\eta}^4 \right) - 5(\lambda + 1)(2\lambda + 1) \left((-1)^{|G|} \bar{\eta}(G^{(4)}) \bar{\eta} - 2G^{(4)} \bar{\eta}^2 \right) \\ &\quad - 30(-1)^{|G|} \bar{\eta}(G'') \bar{\eta}^5 + 2\lambda(\lambda + 1)(2\lambda + 1) G^{(5)}. \end{aligned}$$

Using the same arguments as before, we obtain the expressions for nonzero $\beta_{\varepsilon}^{k_1, k_2, 12}$

$$\begin{aligned} \frac{1}{6}(-1)^{|G|} \beta_{1,1,0}^{2,2,12} &= (-1)^{|F|+1} \beta_{0,1,1}^{2,3,12} = \beta_{1,0,1}^{2,3,12} = -5\tau(-1)^{|F|+|G|} (2\lambda^2 + 7\lambda + 2), \\ \beta_{1,1,0}^{3,2,12} &= \beta_{1,1,0}^{2,3,12} = 10\tau(-1)^{|F|} \lambda(2\lambda^2 + 7\lambda + 2). \end{aligned}$$

Thus, we deduce the following equation

$$\delta(B)(X_F, X_G) = -\tau(-1)^{|F|+|G|} 10 \left(\lambda + \frac{7 + \sqrt{33}}{4} \right) \left(\lambda + \frac{7 - \sqrt{33}}{4} \right) \gamma(X_F, X_G)$$

So, for $\lambda \neq \frac{-7 \pm \sqrt{33}}{4}$, we can see that the coefficient $c_{1,0,1}^{2,3,12}$ can be eliminated by adding a coboundary. Hence, the cohomology is zero-dimensional. While for $\lambda = \frac{-7 + \sqrt{33}}{4}$ or $\frac{-7 - \sqrt{33}}{4}$, the coefficient $c_{1,0,1}^{2,3,12}$ cannot be eliminated by adding a coboundary. Hence, the cohomology space is one-dimensional.

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4.2.4. *The case where $\mu - \lambda \geq \frac{9}{2}$.* Here, a straightforward computation shows that the 2-cocycle condition is equivalent to formulas (3.4)–(3.6) and (3.8)–(3.1) corresponding to $\mathfrak{osp}(1|2)$ -invariant operators together with the following systems:

For $n = 2(\mu - \lambda + 2)$ even:

$$\begin{aligned} & (-1)^{|F|+|G|} \left(\binom{\alpha+a}{\alpha} c_{1,1,0}^{\beta,\gamma,n} + \binom{\beta+a}{\beta} c_{1,1,0}^{\alpha,\gamma,n} + \binom{\gamma+a}{\gamma} c_{1,1,0}^{\alpha,\beta,n} \right) + \\ & + \binom{\alpha+\beta}{\alpha} c_{0,1,1}^{\alpha+\beta,\gamma,n} + \binom{\alpha+\gamma}{\alpha} c_{0,1,1}^{\alpha+\gamma,\beta,n} + \binom{\beta+\gamma}{\beta} c_{0,1,1}^{\beta+\gamma,\alpha,n} = 0, \end{aligned} \quad (4.14)$$

where $a + \alpha + \beta + \gamma = \frac{n}{2} - 1$, $\alpha \geq \beta \geq \gamma \geq 2$ and $a \in \{0, 1\}$, which is obtained from the coefficient of $\bar{\eta}(F^{(\alpha)})\bar{\eta}(G^{(\beta)})\bar{\eta}(H^{(\gamma)})\bar{\eta}\partial_x^a$,

$$\begin{aligned} & 2\left(\frac{a}{\alpha} + \lambda\right)\binom{\alpha+\alpha-1}{\alpha-1}c_{1,1,0}^{\beta,\gamma,n} + (-1)^{|F|+|G|} \left(\left(\frac{a}{\beta} + 2\lambda\right)\binom{\alpha+\beta-1}{\beta-1}c_{0,1,1}^{\alpha,\gamma,n} + \left(\frac{a}{\gamma} + 2\lambda\right)\binom{\alpha+\gamma-1}{\gamma-1}c_{0,1,1}^{\alpha,\beta,n} \right) + \\ & + (2 - \frac{a}{\beta})\binom{\alpha+\beta-1}{\alpha}c_{1,1,0}^{\alpha+\beta-1,\gamma,n} + (2 - \frac{a}{\gamma})\binom{\alpha+\gamma-1}{\alpha}c_{1,1,0}^{\alpha+\gamma-1,\beta,n} - (-1)^{|F|}\binom{\beta+\gamma}{\beta}c_{0,0,0}^{\beta+\gamma,\alpha,n} = 0, \end{aligned} \quad (4.15)$$

where $a + \alpha + \beta + \gamma = \frac{n}{2}$, $\alpha \geq \beta \geq \gamma \geq 2$, $\alpha \geq 3$ and $a \in \{0, 1\}$, which is obtained from the coefficient of $F^{(\alpha)}\bar{\eta}(G^{(\beta)})\bar{\eta}(H^{(\gamma)})\partial_x^a$.

For $n = 2(\mu - \lambda + 2)$ odd:

$$\begin{aligned} & \left(\frac{a}{\alpha} + 2\lambda\right)\binom{\alpha+\alpha-1}{\alpha-1}c_{1,1,1}^{\beta,\gamma,n} + \left(\frac{a}{\beta} + 2\lambda\right)\binom{\alpha+\beta-1}{\beta-1}c_{1,1,1}^{\alpha,\gamma,n} + \left(\frac{a}{\gamma} + 2\lambda\right)\binom{\alpha+\gamma-1}{\gamma-1}c_{1,1,1}^{\alpha,\beta,n} \\ & - (-1)^{|F|+|G|} \left(\binom{\alpha+\beta}{\alpha} c_{0,1,0}^{\alpha+\beta,\gamma,n} + \binom{\alpha+\gamma}{\alpha} c_{0,1,0}^{\alpha+\gamma,\beta,n} + \binom{\beta+\gamma}{\beta} c_{0,1,0}^{\beta+\gamma,\alpha,n} \right) = 0, \end{aligned} \quad (4.16)$$

where $a + \alpha + \beta + \gamma = \frac{n-1}{2}$, $\alpha \geq \beta \geq \gamma \geq 2$ and $a \in \{0, 1\}$, which is obtained from the coefficient of $\bar{\eta}(F^{(\alpha)})\bar{\eta}(G^{(\beta)})\bar{\eta}(H^{(\gamma)})\partial_x^a$,

$$\begin{aligned} & (-1)^{|F|+|G|} \left(\left(\frac{a}{2\alpha} + \lambda\right)\binom{\alpha+\alpha-1}{\alpha-1}c_{0,0,1}^{\beta,\gamma,n} - \left(\frac{a}{\beta} + \lambda\right)\binom{\alpha+\beta-1}{\beta-1}c_{1,0,0}^{\alpha,\gamma,n} + \left(\frac{a}{\gamma} + \lambda\right)\binom{\alpha+\gamma-1}{\gamma-1}c_{1,0,0}^{\alpha,\beta,n} \right. \\ & \left. + \left(\frac{a}{2\alpha} - 1\right)\binom{\alpha+\beta-1}{\alpha-1}c_{1,0,0}^{\alpha+\beta-1,\gamma,n} - \left(\frac{a}{2\alpha} - 1\right)\binom{\alpha+\gamma-1}{\alpha-1}c_{1,0,0}^{\alpha+\gamma-1,\beta,n} + (-1)^{|F|}\left(\frac{\gamma}{\beta} - 1\right)\binom{\beta+\gamma-1}{\beta-1}c_{0,1,0}^{\beta+\gamma-1,\alpha,n} \right) = 0 \end{aligned} \quad (4.17)$$

where $a + \alpha + \beta + \gamma = \frac{n+1}{2}$, $\alpha \geq 2$, $\beta \geq \gamma \geq 3$ and $a \in \{0, 1\}$, which is obtained from the coefficient of $\bar{\eta}(F^{(\alpha)})G^{(\beta)}H^{(\gamma)}\partial_x^a$.

Of course, these systems have at least one solution in which the solutions $c_{\varepsilon}^{k_1, k_2, n}$ are just the coefficients $\beta_{\varepsilon}^{k_1, k_2, n}$ of the coboundary (4.11).

4.2.5. *The case where $\mu - \lambda = \frac{9}{2}$.* In this case, according to Theorem 3.2, the space of solutions is spanned by $c_{1,1,1}^{2,2,13}$ and $c_{1,0,0}^{2,4,13}$. Moreover, by (4.16), we readily obtain:

$$\lambda c_{1,1,1}^{2,2,13} - 3(-1)^{|F|+|G|} c_{0,1,0}^{4,2,13} = 0.$$

According to formula (3.4), we deduce that the coefficients of every 2-cocycle is expressed in terms of $c_{1,1,1}^{2,2,13}$. But this general formula may contain coboundaries. We explain how the coboundaries can be removed. Consider any coboundary given as in (4.11). We discuss the following cases:

1) $\lambda \neq -2$. Then, by Lemma 4.3 part (i), the coefficient $c_{1,1,1}^{2,2,13}$ can be eliminated by adding the coboundary (4.11) because $\beta_{1,1,1}^{2,2,13}$ is nonzero. Hence, the cohomology is zero-dimensional.

2) $\lambda = -2$. Then the constant $\beta_{1,1,1}^{2,2,13}$ vanishes. Hence the coefficient $c_{1,1,1}^{2,2,13}$ cannot be eliminated by adding the coboundary (4.11). Therefore the cohomology is one-dimensional.

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4.2.6. *The case where $\mu - \lambda = 5$.* In this case, according to Theorem 3.2, together with formula (4.14), we check that the coefficients of every 2-cocycle are expressed in terms of $c_{0,0,0}^{4,3,14}$ and $c_{1,0,1}^{2,4,14}$. On the other hand, by Lemma 4.3 part (i), one of the coefficients $c_{0,0,0}^{4,3,14}$, $c_{1,0,1}^{2,4,14}$ can be eliminated by adding the coboundary (4.11) because $\beta_{0,0,0}^{4,3,14}$ or $\beta_{1,0,1}^{2,4,14}$ is nonzero. Hence, the cohomology is one-dimensional.

4.2.7. *The case where $\mu - \lambda = \frac{11}{2}$.* In this case, by Theorem 3.2 together with formula (4.16), we check that the coefficients of every 2-cocycle are expressed in terms of $c_{1,1,1}^{2,2,15}$ and $c_{0,0,1}^{4,3,15}$. We discuss the following cases:

1) $\lambda \neq -\frac{5}{2}$. Then, by Lemma 4.3 part (i), one of the coefficients $c_{1,1,1}^{2,2,15}$, $c_{0,0,1}^{4,3,15}$ can be eliminated by adding the coboundary (4.11) because $\beta_{1,1,1}^{2,2,15}$ or $\beta_{0,0,1}^{4,3,15}$ is nonzero. Hence, the cohomology is one-dimensional.

2) $\lambda = -\frac{5}{2}$. Then, by Lemma 4.3 part (i), the coefficients $\beta_{1,1,1}^{2,2,15}$ and $\beta_{0,0,1}^{4,3,15}$ vanish simultaneously. Hence the coefficients $c_{1,1,1}^{2,2,15}$ and $c_{0,0,1}^{4,3,15}$ cannot be eliminated by adding the coboundary (4.11). Therefore the cohomology is two-dimensional.

4.2.8. *The case where $\mu - \lambda = p - 2$ with $p \in \mathbb{N} + 8$.* According to Theorem 3.2 together with formulas (4.14)–(4.15), we check that the coefficients of every 2-cocycle are expressed in terms of $c_{1,0,1}^{2,4,2p}$. On the other hand, by Lemma 4.3 part (ii), we can see that

$$\beta_{1,0,1}^{2,4,2p} = 0 \text{ if } \lambda = -\frac{2p-5 \pm \sqrt{8p-15}}{4}.$$

So, in the same way as before, we deduce that the cohomology space is one-dimensional for $\lambda = -\frac{2p-5 \pm \sqrt{8p-15}}{4}$ and zero-dimensional otherwise.

4.2.9. *The case where $\mu - \lambda = p - \frac{3}{2}$ with $p \in \mathbb{N} + 8$.* According to Theorem 3.2 together with formulas (4.16)–(4.17), we check that the coefficients of every 2-cocycle are expressed in terms of $c_{1,1,1}^{2,2,2p+1}$. On the other hand, by Lemma 4.3 part (ii), we can see that $\beta_{1,1,1}^{2,2,2p+1} = 0$ if $\lambda = 1 - \frac{p}{2}$. So, in the same way as before, we deduce that the cohomology space is one-dimensional for $\lambda = 1 - \frac{p}{2}$ and zero-dimensional otherwise. This completes the proof of Theorem 3.3.

5. EXTENSIONS OF $\mathcal{K}(1)$ BY $\mathfrak{D}_{\lambda,\mu}$

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$ -module. The second space $H^2(\mathfrak{g}, V)$ classifies the nontrivial extensions of the Lie algebra $\mathfrak{g}$ by the module V :

$$0 \longrightarrow V \longrightarrow \mathfrak{g}_V \longrightarrow \mathfrak{g} \longrightarrow 0,$$

the Lie structure on $\mathfrak{g}_V = \mathfrak{g} \oplus V$ being given by

$$[(g_1, a), (g_2, b)] = ([g_1, g_2], g_1 \cdot b - g_2 \cdot a + c(g_1, g_2)),$$

where c is a 2-cocycle with values in V .

Here we consider a natural class of "non-central" extensions of $\mathcal{K}(1)$, namely extensions by the module $\mathfrak{D}_{\lambda,\mu}$ of linear differential operators acting on weighted densities. We will be interested in the projectively invariant extensions which are given by projectively invariant 2-cocycles c . The cocycle c in this case represents a non-trivial cohomology class of the second cohomology space $H_{\text{diff}}^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$. We mention that the same problem was considered in [18, 19] in the classical setting.

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Corollary 5.1. 1) For $(\lambda, \mu) \in J$ or $\mu - \lambda = 5$ or $\mu - \lambda = \frac{11}{2}$ and $\lambda \neq -\frac{5}{2}$ or $\mu - \lambda = p - \frac{3}{2}$ and $\lambda = 1 - \frac{p}{2}$ with $p \in \mathbb{N} + 8$ or

$$\mu - \lambda = p - 2 \text{ and } \lambda = -\frac{2p - 5 \pm \sqrt{8p - 15}}{4} \text{ with } p \in \mathbb{N} + 8,$$

there exists a unique non-trivial extension of $\mathcal{K}(1)$ by $\mathfrak{D}_{\lambda, \mu}$, where

$$J = \left\{ (0, 3), \left(-\frac{5}{2}, \frac{1}{2}\right), \left(-\frac{3}{2}, 2\right), \left(-2, \frac{5}{2}\right), \left(\frac{-7+\sqrt{33}}{4}, \frac{9+\sqrt{33}}{4}\right), \left(\frac{-7-\sqrt{33}}{4}, \frac{9-\sqrt{33}}{4}\right) \right\}.$$

2) For $\mu - \lambda = \frac{11}{2}$ and $\lambda = -\frac{5}{2}$, there exists two non-isomorphic non-trivial extensions of $\mathcal{K}(1)$ by $\mathfrak{D}_{\lambda, \mu}$.

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LABORATOIRE DE RECHERCHE: PHYSIQUE MATHÉMATIQUE, MODÉLISATION QUANTIQUE ET CONCEPTION MÉCANIQUE, UNIVERSITÉ DE CARTHAGE, TUNISIA

Email address: `raoufghallabi@gmail.com`

INSTITUT PRÉPARATOIRE AUX ÉTUDES D’INGÉNIEUR DE NABEUL, LR: PHYSIQUE MATHÉMATIQUE, MODÉLISATION QUANTIQUE ET CONCEPTION MÉCANIQUE, UNIVERSITÉ DE CARTHAGE, TUNISIA

Email address: `benfracj_nizar@yahoo.fr`

DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES DE SFAX, BP 802, 3038 SFAX, TUNISIA

Email address: `saalem.faidi@yahoo.fr`

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