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FINITE GROUPS IN WHICH SOME MAXIMAL SUBGROUPS ARE MNP-GROUPS

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Abstract. A finite group G is called an MNP-group if all maximal subgroups of the Sylow subgroups of G are normal in G. The aim of this paper is to give a necessary and sufficient condition for a group to be an MNP-group, characterize the structure of finite groups whose maximal subgroups (respectively, maximal subgroups of even order) are all MNP-groups, and determine finite non-abelian simple groups whose second maximal subgroups (respectively, maximal subgroups of even order) are all MNP-groups.

1. Introduction

All groups considered in this paper are finite and notions and notations are standard.

One of the important topics in group theory is to characterize the structure of groups by applying some properties of local subgroups. Since Srinivasan [13] gave two sufficient conditions for supersolvability of groups by considering the maximal subgroups of their Sylow subgroups, a large number of scholars have researched on this topic and the results have been generalised frequently. Among them, Walls [15] called the group whose maximal subgroups of the Sylow subgroups are normal MNP-group and gave its characterization. The first aim of this paper is to give a necessary and sufficient condition for a group to be an MNP-group (see section 3).

Recently, Meng et al. [10] studied the structure of groups all of whose maximal subgroups of even order are MS-groups (A group G is called an MS-group if all minimal subgroups of G permute with every Sylow subgroup of G). Lu et al. [7] studied the structure of groups by replacing MS-groups with SB-groups (A group G is called an SB-group if every subgroup of G is either permutable with every Sylow subgroup of G or abnormal in G).

The second aim of this paper is to investigate the structure of groups in which some subgroups are MNP-groups. Since the property of MNP-groups is not inherited in subgroups, we call a group G a *sub-MNP-group* if every maximal subgroup of G is an MNP-group but G is not. It is obviously different from the non-MNP-group

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whose all proper subgroups are MNP-groups. In section 4, we classify completely the sub-MNP-groups. In section 5, we characterize groups all of whose maximal subgroups of even order are MNP-groups. In section 6, we determine non-abelian simple groups all of whose second maximal subgroups of even order (respectively, second maximal subgroups) are MNP-groups.

2. Preliminary results

We give some necessary lemmas as shown below.

Lemma 2.1. [11, 5.2.15] *Let G be a group. If $\Phi(G) \leq H \trianglelefteq G$ and $H/\Phi(G)$ is nilpotent, then H is nilpotent.*

Lemma 2.2. [15, Theorem 6] *A group G is an MNP-group if and only if $G = H\langle x\rangle$, where H is a normal nilpotent Hall subgroup of G, and every generator of every Sylow subgroup of $\langle x\rangle$ induces a power automorphism of order dividing a prime in $H/\Phi(H)$.*

Recall that $\Omega_1(G)$ and $\Omega_2(G)$ denote the subgroups generated by all elements x of the p-group G such that $x^p = 1$ and $x^{p^2} = 1$, respectively.

Lemma 2.3. [6, IV, Satz 5.12] *Suppose that a p'-group H acts on a p-group G. Let*

$$\Omega(G) = \begin{cases} \Omega_1(G), & p > 2, \\ \Omega_2(G), & p = 2. \end{cases}$$

If H acts trivially on $\Omega(G)$, then H acts trivially on G as well.

Lemma 2.4. *Let G be an MNP-group and $N \trianglelefteq G$. Then G/N is also an MNP-group.*

Proof. It is obviously true by checking. □

Lemma 2.5. *Let G be a group, $P \in \mathrm{Syl}_p(G)$ and $P \trianglelefteq G$ for a prime p. Then $\Phi(P) = \Phi(G)_p$.*

Proof. Let $B = \Phi(G)_p = P \cap \Phi(G)$. It is clear that $B \trianglelefteq G$ and $\Phi(P) \leq B \leq P$.

Now we prove $B \leq \Phi(P)$. By the Schur-Zassenhaus theorem, there exists a Hall p'-subgroup K of G and $G = PK$. Let M be a maximal subgroup of P. If $B \nleq M$, then $BM = P$ and $G = PK = BMK = MK$. However, it is obvious that $MK < G$ by order considerations, a contradiction that shows that $B \leq \Phi(P)$. □

Lemma 2.6. [13, Theorem 1] *If a group G is an MNP-group, then G is supersolvable.*

Lemma 2.7. [15, Lemma 1] *A Hall subgroup of an MNP-group must be an MNP-group.*

Lemma 2.8. *Let $G = P \rtimes Q$ be a sub-MNP-group with $P = \langle x\rangle$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Then P is of order p.*

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Proof. If $x^p \neq 1$ and Q is non-cyclic, then $\langle x^p\rangle Q$ is an MNP-group by hypothesis, and so it is nilpotent by Lemma 2.2. However, Lemma 2.3 implies that G is nilpotent clearly, a contradiction. Hence P is of order p.

If $x^p \neq 1$ and $Q = \langle y\rangle$ is cyclic, then $\langle x^p\rangle Q$ is an MNP-group by hypothesis, and so $\langle x^p\rangle\langle y^q\rangle$ is nilpotent. By Lemma 2.3, $P\langle y^q\rangle$ is nilpotent. Thus, G is an MNP-group, a contradiction. Hence P is of order p. □

Lemma 2.9. [5, Lemma 2.9] *If a q-group G of order q^n has a unique non-cyclic maximal subgroup, then G is isomorphic to one of the following groups:*

(i) $C_{q^{n-1}} \times C_q = \langle y, z \mid y^{q^{n-1}} = z^q = 1, [y, z] = 1\rangle$, *where $n \geq 3$;*

(ii) $M_{q^n} = \langle y, z \mid y^{q^{n-1}} = z^q = 1, [y, z] = y^{q^{n-2}}\rangle$, *where $n \geq 3$, and $n \geq 4$ if $q = 2$.*

Lemma 2.10. [14, Corollary 1] *Every minimal simple group is isomorphic to one of the following groups:*

(i) $\mathrm{PSL}(3, 3)$;

(ii) *the Suzuki group* $\mathrm{Sz}(2^q)$, *where q is an odd prime;*

(iii) $\mathrm{PSL}(2, p)$, *where p is a prime with $p > 3$ and $p^2 \not\equiv 1 (\mathrm{mod}\ 5)$;*

(iv) $\mathrm{PSL}(2, 2^q)$, *where q is a prime;*

(v) $\mathrm{PSL}(2, 3^q)$, *where q is an odd prime.*

Lemma 2.11. [8, Theorem 10.1.4] *If a group G has a fixed-point-free automorphism of order 2, then G is abelian.*

3. A necessary and sufficient condition of MNP-groups

In this section, we give a necessary and sufficient condition of MNP-groups.

Theorem 3.1. *A group G is an MNP-group if and only if $G/\Phi(G)$ is an MNP-group.*

Proof. This necessity is clear by Lemma 2.4.

Now we prove the sufficiency.

If $G/\Phi(G)$ is an MNP-group, then assume that $G/\Phi(G) = H_1/\Phi(G)\cdot K/\Phi(G)$ by Lemma 2.2, where $H_1/\Phi(G)$ is a nilpotent normal Hall π-subgroup of $G/\Phi(G)$, and $K/\Phi(G)$ is a cyclic Hall π'-subgroup of $G/\Phi(G)$. By Lemma 2.1, H_1 is nilpotent, and so there exists a nilpotent normal Hall π-subgroup H of H_1 such that H char $H_1 \trianglelefteq G$. Furthermore, $H \trianglelefteq G$ and it is also a normal Hall π-subgroup of G.

Let $K/\Phi(G) = \langle y\Phi(G)\rangle = \langle y\rangle\Phi(G)/\Phi(G) \cong \langle y\rangle/\langle y\rangle\cap\Phi(G)$ be a Hall π'-subgroup of $G/\Phi(G)$. Now $G = H_1K = HK = \langle H, y, \Phi(G)\rangle = H\langle y\rangle$. If the order of y contains both π-number and π'-number, then there exists a Hall π'-subgroup $\langle x\rangle$ of $\langle y\rangle$ such that it is also a π'-Hall subgroup of G. Thus, $G = H\langle y\rangle = H\langle x\rangle$.

Let p be a prime such that $p \mid |H|$ and let P be a Sylow p-subgroup of H. Then $P \trianglelefteq G$ and P is the unique Sylow p-subgroup of G. By Lemma 2.5, we have that $P \cap \Phi(G) = \Phi(P)$. If M is a maximal subgroup of P, then $P \cap \Phi(G) \leq M$ and $M\Phi(G)/\Phi(G)$ is a maximal subgroup of $P\Phi(G)/\Phi(G)$. Therefore $M\Phi(G) \trianglelefteq G$ and M is a Sylow p-subgroup of $M\Phi(G)$. By the Frattini argument, $G = N_G(M)M\Phi(G) = N_G(M)M = N_G(M)$ and $M \trianglelefteq G$.

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Let p be a prime dividing $|\langle x\rangle|$ and let P be a Sylow p-subgroup of G. Then P is cyclic and the unique maximal subgroup of P is $\Phi(P) = M$. Now we have $P \cap \Phi(G) \leq M$, and as before, $M \trianglelefteq G$. Consequently, G is an MNP-group. □

4. A complete classification of sub-MNP-groups

In this section, we classify groups with the property that all of whose maximal subgroups are MNP-groups.

Theorem 4.1. *Let G be a sub-MNP-group. Then*

(I) *$G = P \rtimes Q$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with $p \neq q$, $|G| = p^a q^b$ and at least one of a and b is more than 1;*

(II) *at least one of P and Q is cyclic.*

Proof. By hypothesis and Lemma 2.6, G is either supersolvable or minimal non-supersolvable. So G is solvable and it has a normal Sylow p-subgroup P by a result in [4]. Let $\{P_1, P_2, \ldots, P_s\}$ be a Sylow basis of G. Without loss of generality, assume that the maximal subgroup P_{11} of P_1 is not normal in G by hypothesis. If $s \geq 3$, then P_1P_j $(j = 2, \ldots, s)$ are MNP-groups by Lemma 2.7. Thus, P_{11} is normal in P_1P_j, and so P_{11} is normal in G. This contradiction leads to $|\pi(G)| = 2$ and (I) holds.

Assume that neither P nor Q is cyclic, and let Q_1, Q_2 be two maximal subgroups of Q and H be any maximal subgroup of P. Then PQ_1 and PQ_2 are MNP-groups by hypothesis, and so H is normal in not only PQ_1 but also PQ_2. Thus, H is normal in G. The arbitrariness of H induces that AQ and BQ are MNP-groups by hypothesis for two maximal subgroups A and B of P. By Lemma 2.2, Q is normal in $G = \langle A, B, Q\rangle$, a contradiction. Hence either P or Q is cyclic and (II) holds. □

Theorem 4.2. *Let p and q be distinct prime divisors of the order of a group G, $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Then G is a sub-MNP-group if and only if G is isomorphic to one of the following types:*

(I) *$G = \langle x, y \mid x^p = y^{q^n} = 1, y^{-1}xy = x^i\rangle$, where $q \mid p-1$, $i^q \not\equiv 1 (\mathrm{mod}\ p)$, $i^{q^2} \equiv 1 (\mathrm{mod}\ p)$, $n \geq 2$ and $1 < i < p$;*

(II) *$G = \langle x, y \mid x^{pq} = y^q = 1, y^{-1}xy = x^i\rangle$, where $q \mid p-1$, $i \equiv 1 (\mathrm{mod}\ q)$, $i^q \equiv 1 (\mathrm{mod}\ p)$ with $1 < i < pq$;*

(III) *$G = \langle x, y \mid x^{4p} = 1, y^2 = x^{2p}, y^{-1}xy = x^{-1}\rangle$ with $p \neq 2$;*

(IV) *$G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1, y^{-1}xy = x^i, [x, z] = 1, [y, z] = 1\rangle$, where $q \mid p-1$, $i^q \equiv 1 (\mathrm{mod}\ p)$, $1 < i < p$ and $n \geq 3$;*

(V) *$G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1, y^{-1}xy = x^i, [x, z] = 1, [y, z] = y^{q^{n-2}}\rangle$, where $q \mid p-1$, $i^q \equiv 1 (\mathrm{mod}\ p)$, $1 < i < p$, $n \geq 3$, and $n \geq 4$ if $q = 2$;*

(VI) *$G = P \rtimes Q$, where $P/\Phi(P) = R/\Phi(P) \times K/\Phi(P) = \langle \bar{a}, \bar{b}\rangle$ is an elementary abelian p-group of order p^2, $Q = \langle y\rangle$ is cyclic of order q^r with $q \mid p-1$, and y induces two power automorphisms of order dividing q in $R/\Phi(R)$ and $K/\Phi(K)$ respectively. Define $\bar{a}^y = \bar{a}^i$, $\bar{b}^y = \bar{b}^j$, $[P, y^q] = 1$ and $r \geq 1$, where $i \not\equiv j (\mathrm{mod}\ p)$ and $0 < i, j < p$;*

(VII) *$G = P \rtimes Q$, where $Q = \langle y\rangle$ is cyclic of order $q^r > 1$, with $q \nmid p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle y^q\rangle$ in Q;*

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(VIII) *$G = P \rtimes Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, $Q = \langle y\rangle$ is cyclic of order $q^r > 1$, y induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and y centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$;*

(IX) *$G = P \rtimes Q$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle y\rangle$ is cyclic of order q^r, $q \parallel p-1$ and $r > 1$. Define $a_j^y = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^y = a_0^i$, where i is a primitive q-th root of unity modulo p;*

(X) *$G = P \rtimes Q$, where $P = \langle a_0, a_1\rangle$ is an extra-special group of order p^3 with exponent p, $Q = \langle y\rangle$ is a cyclic group of order 2^r with $2 \parallel p-1$ and $r > 1$. Define $a_0^y = a_1$ and $a_1^y = a_0^{-1}x$, where $x \in \langle [a_0, a_1]\rangle$.*

Proof. Assume that G is a sub-MNP-group, let $G = P \rtimes Q$, and at least one of P and Q is cyclic by Theorem 4.1, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$.

We discuss from the following four types.

(1) Assume that P and Q are cyclic. By Lemma 2.8, if $P = \langle x\rangle$, then $x^p = 1$. By hypothesis, $Q = \langle y\rangle$ and $\langle y^q\rangle$ is not normal in G and $\langle y^{q^2}\rangle$ is normal in $\langle x\rangle\langle y^q\rangle$.

Conjugation in P by $\langle y\rangle$ yields a nontrivial homomorphism from Q to $\mathrm{Aut}(P)$. Hence $y^{-1}xy = x^i$ where $1 < i < p$. Since $\mathrm{Aut}(P)$ has order $p-1$ and y yields a homomorphism of order multiple of q, we have that $q \mid p-1$. Moreover, $\langle y^q\rangle$ is not normal and $\langle y^{q^2}\rangle$ is normal in $\langle x\rangle\langle y^q\rangle$. Hence $x^{i^q} \neq x$ and $x^{i^{q^2}} = x$. It follows that $i^q \not\equiv 1 (\mathrm{mod}\ p)$ and $i^{q^2} \equiv 1 (\mathrm{mod}\ p)$. Therefore, G is of type (I).

(2) Assume that P is cyclic and Q is non-cyclic. By Lemma 2.8, P is of order p. If Q has two non-cyclic maximal subgroups Q_1 and Q_2, then PQ_1 and PQ_2 are both nilpotent by Lemma 2.2, and so $Q = Q_1Q_2$ is normal in G, a contradiction. Therefore, every maximal subgroup of Q is cyclic or Q has a unique non-cyclic maximal subgroup.

Case 1 Every maximal subgroup of Q is cyclic.

It is clear that Q is either elementary abelian of order q^2 or the quaternion group of order 8.

Let Q be an elementary abelian group of order q^2. Since $\mathrm{Aut}(P)$ is a cyclic group of order $p-1$, we have that $Q/C_Q(x)$ is a cyclic group of order $q \mid p-1$. There exists $a \notin C_Q(x)$ and $C_Q(x) = \langle b\rangle$. Hence $Q = \langle a\rangle \times \langle b\rangle$ and G is of type (II).

Let Q be the quaternion group of order 8. Since $Q/C_Q(x)$ is a cyclic group, we have that $Q/C_Q(x)$ has order $2 \mid p-1$. Hence $C_Q(x) = \langle a\rangle$ is a cyclic subgroup of Q or order 4 and $p \neq 2$. There exists $b \in Q$ of order 4 such that $x^b = x^{-1}$, and also $a^b = a^{-1}$. Hence the elements xa and b generate G and G is of type (III).

Case 2 Let $Q = \langle a\rangle \rtimes \langle b\rangle$ such that $|\langle a\rangle| = q^{n-1}$ and $|\langle b\rangle| = q$, and $P = \langle x\rangle$. Since $\mathrm{Aut}(P)$ is a cyclic group of order $p-1$, we have that $Q/C_Q(x)$ is a cyclic group of order $q^r \mid p-1$.

Since $\langle a\rangle P$ is an MNP-group, we have that $a^q \in C_Q(x)$ and $Q/C_Q(x)$ is a cyclic group of order q. Moreover, $\langle a^q\rangle\langle b\rangle P$ is an MNP-group. Thus $\langle a^q\rangle\langle b\rangle$ is normal in $\langle a^q\rangle\langle b\rangle P$. Hence $b \in C_Q(x)$. Consequently, G is of type (IV) or (V).

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(3) Assume that P is non-cyclic, $Q = \langle y \rangle$ and that G is supersolvable.

Let $1 \lhd \cdots \lhd \Phi(P) \lhd N \lhd \cdots \lhd R \lhd P \lhd \cdots \lhd G$ be a principal series of G. Since $N/\Phi(P) \nleq \Phi(G/\Phi(P)$, there exists a maximal subgroup M of G such that $MN = G$. By Baer's theorem [2, Theorem 1.1.7], the group $G/\mathrm{Core}_G(N)$ has a unique minimal normal subgroup $N\mathrm{Core}_G(M)/\mathrm{Core}_G(M)$ and $C_G(N\mathrm{Core}_G(M)/\mathrm{Core}_G(M)) = N\mathrm{Core}_G(M)$. Since $P/\Phi(P)$ is an elementary abelian group, it is clear that $C_G(N\mathrm{Core}_G(M)/\mathrm{Core}_G(M)) = C_G(N/\Phi(P)) \geq P$. Then $K = P \cap \mathrm{Core}_G(M)$ is a normal subgroup of G and $|P : K| = |P\mathrm{Core}_G(M) : \mathrm{Core}_G(M)| = |N\mathrm{Core}_G(M) : \mathrm{Core}_G(M)| = |N : \Phi(P)| = p$. By hypothesis, $R\langle y \rangle$ and $K\langle y \rangle$ are both MNP-groups, and so $\langle y^q \rangle$ is normal in $G = \langle R, K, y \rangle$.

Now we prove $s = 2$. Without loss of generality, assume $s = 3$ and let $P/\Phi(P) = \langle \bar{a_1} \rangle \times \langle \bar{a_2} \rangle \times \langle \bar{a_3} \rangle$, where $a_1, a_2 \in R$, $a_2, a_3 \in K$. Since $R\langle y \rangle$ is an MNP-group, we have $(r\Phi(R))^y = r^l\Phi(R)$ by Lemma 2.2 for every $r \in R \backslash \Phi(R)$, where l is a positive integer. Thus, $(r\Phi(P))^y = r^l\Phi(P)$ for every $r \in R\backslash\Phi(P)$. Similarly, $(k\Phi(P))^y = k^m\Phi(P)$ for every $k \in K\backslash\Phi(P)$, where m is a positive integer. Furthermore, $a_2^l\Phi(P) = (a_2\Phi(P))^y = a_2^m\Phi(P)$, and so $l \equiv m (\mathrm{mod}\ p)$. Hence $(a_n\Phi(P))^y = a_n^l\Phi(P)$ for $n = 1, 2, 3$. By Lemma 2.2, G is an MNP-group, this contradiction leads to $s = 2$.

Now we let $P/\Phi(P) = R/\Phi(P) \times K/\Phi(P) = \langle \bar{a} \rangle \times \langle \bar{b} \rangle$, where $a \in R, b \in K$, $\bar{a}^y = \bar{a}^i$ and $\bar{b}^y = \bar{b}^j$. Clearly, $i \not\equiv j (\mathrm{mod}\ p)$, P has only two maximal subgroups R and K which are normal in G. So G is of type (VI).

(4) Assume that G is minimal non-supersolvable and $Q = \langle y \rangle$ is cyclic.

It is easy to prove that G has only two kinds of maximal subgroups $P\langle y^q \rangle^g$ and $\Phi(P)\langle y \rangle^g$ for $g \in G$ by applying the property of minimal non-supersolvable groups [4].

Case 1 If G is also minimal non-nilpotent, then G is of either type (VII) or type (VIII) by a result in [3, Theorem 3].

Case 2 If G is not minimal non-nilpotent with P abelian, by applying [1, Theorem 9, 10], assume that $G = PQ$, where $P = \langle a_0, a_1, \ldots, a_{q-1} \rangle$ is elementary abelian of order p^q, $Q = \langle y \rangle$ is cyclic of order q^r, q^f is the highest power of q dividing $p - 1$ and $r > f \geq 1$. Define $a_j^y = a_{j+1}$ for $0 \leq j < q - 1$ and $a_{q-1}^y = a_0^i$, where i is a primitive q^f-th root of unity modulo p.

By hypothesis and Lemma 2.2, y^q induces a power automorphism of order q in P. Hence, $a_0^{i^q} = a_0^{y^{q^2}} = a_0$. Thus $i^q \equiv 1 (\mathrm{mod}\ p)$ and $f = 1$. So G is of type (IX).

Case 3 If G is not minimal non-nilpotent with P non-abelian, by applying [1, Theorem 9, 10], assume that $G = PQ$ such that $P = \langle a_0, a_1 \rangle$ is an extra-special group of order p^3 with exponent p, $Q = \langle y \rangle$ is cyclic of order 2^r with 2^f the largest power of 2 dividing $p - 1$ and $r > f \geq 1$, and $a_0^y = a_1$ and $a_1^y = a_0^i x$, where $x \in \langle [a_0, a_1] \rangle$ and i is a primitive 2^f-th root of unity modulo p.

Since $a_0^{y^2} = a_1^y = a_0^i x \neq a_0$, then $P\langle y^2 \rangle$ is non-nilpotent. By hypothesis, $[x, y^2] = 1$ and $a_0^{y^4} = a_0^{i^2} x^{i+1} = a_0$. Thus, $a_0^{i^2-1} = (x^{i+1})^{-1}$. Hence $i \equiv -1 (\mathrm{mod}\ p)$. By computations, $a_1^{y^4} = a_1$, $[a_0, a_1]^{y^2} = [a_0^i, a_1^i] = [a_0, a_1]^{i^2} = [a_0, a_1]$. So G is of type (X).

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Conversely, it is clear that a group satisfying one of types (I)–(X) is a sub-MNP-group. □

5. Groups all of whose maximal subgroups of even order are MNP-groups

In this section, we determine groups with the property that all of whose maximal subgroups of even order are MNP-groups.

Theorem 5.1. *Let G be a group of even order. Suppose that all maximal subgroups of G of even order are MNP-groups. Then G is solvable.*

Proof. Let G be a counterexample of minimal order. Then the following statements about G are true.

(1) G is a minimal simple group.

If $1 \lhd N \lhd G$, then G/N satisfies the hypotheses. Hence G/N is a solvable group by the minimality of G. If N does not have even order, then N is solvable and G is solvable. If N has even order, there is a maximal subgroup M of G whose order is even and such that $N \trianglelefteq M$. By Lemma 2.6, M is supersolvable, and then N is solvable and so G is solvable, a contradiction.

(2) Final contradiction.

Now we claim that G is not isomorphic to one of the simple groups listed in Lemma 2.10. Note that every proper subgroup of G of even order must be supersolvable by Lemma 2.6, but each of $\mathrm{PSL}(2,p)$, $\mathrm{PSL}(2,3^q)$ and $\mathrm{PSL}(3,3)$ contains a subgroup isomorphic to a non-supersolvable alternating group A_4 of degree 4, a contradiction. If G is isomorphic to $\mathrm{PSL}(2,2^q)$ or $\mathrm{Sz}(2^q)$, then G is a Zassenhaus group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group. This implies that G has a non-supersolvable subgroup of even order, a contradiction. □

Theorem 5.2. *Let G be a non-MNP-group of even order. If all maximal subgroups of G of even order are MNP-groups, then $|\pi(G)| \leq 3$.*

Proof. By Theorem 5.1, G is solvable. Let $\pi(G) = \{p_1, p_2, \dots, p_s\}$ with $p_1 = 2$ and $\{P_1, P_2, \dots, P_s\}$ be a Sylow basis of G. If G is a sub-MNP-group, then $|\pi(G)| = 2$ by Theorem 4.1.

Now we assume that G is not a sub-MNP-group. By hypothesis, G possesses a maximal subgroup M of odd order which is not an MNP-group. Without loss of generality, let $M = P_2 \cdots P_s$. Then there exists a positive integer j and a maximal subgroup P_{j1} of P_j such that P_{j1} is not normal in M. Without loss of generality, we can let $j = s$. If $s \geq 4$, then $P_1 P_i P_s$ $(i = 2, \dots, s-1)$ is a proper Hall subgroup of G. By hypothesis and Lemma 2.7, P_{j1} is normal in $P_1 P_i P_s$, and so P_{j1} is normal in $G = P_1 P_2 \cdots P_s$, a contradiction. Hence $|\pi(G)| \leq 3$. □

We first determine the non-MNP-groups having two prime divisors.

Theorem 5.3. *Let G be a non-MNP-group and $\pi(G) = \{2, p\}$. Then all maximal subgroups of G of even order are MNP-groups if and only if G is a sub-MNP-group.*

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Proof. Clearly, the maximal subgroup of G of odd order (if exists) is a Sylow subgroup, and so it is an MNP-group. Therefore, G is a sub-MNP-group. The rest is clear. □

We next determine the non-MNP-groups having three prime divisors.

Theorem 5.4. *Let G be a non-MNP-group of even order and $|\pi(G)| = 3$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $R \in \mathrm{Syl}_2(G)$ with $p > q > r = 2$. Suppose that all maximal subgroups of G of even order are MNP-groups. Then R is of order 2 and one of the following statements holds:*

(I) *$G = M \times R$, where M is a sub-MNP-group;*

(II) *$G = M \rtimes R = (P \rtimes Q) \rtimes R$ with $1 < C_M(R) < M$, where M is a sub-MNP-group with Q cyclic, R induces two power automorphisms of order dividing 2 in $P/\Phi(P)$ and Q respectively;*

(III) *$G = M \rtimes R = (Q \rtimes P) \rtimes R$ with $1 < C_M(R) < M$, where M is a minimal non-nilpotent group with Q non-cyclic, R induces two power automorphisms of order dividing 2 in P and $Q/\Phi(Q)$ respectively;*

(IV) *$G = M \rtimes R = (P \rtimes Q) \rtimes R$ with $C_M(R) = Q$, where M is a sub-MNP-group with P of order p and Q non-cyclic.*

Proof. Since $|\pi(G)| = 3$, G is not a sub-MNP-group by Theorem 4.1. Then there exists a maximal subgroup M of G of odd order such that M is not an MNP-group by hypothesis. By Theorem 5.1, G is solvable. Then we can let M be a Hall $2'$-subgroup of G and $G = MR$.

We first prove that R is of order 2 and M is the normal 2-complement from two cases as follows.

Case 1 $O_2(G) \neq 1$.

If $O_2(G) < R$, then $M < MO_2(G) < MR = G$, which contradicts that M is a maximal subgroup of G. So $O_2(G) = R$ is the normal Sylow 2-subgroup of G. Note that M is non-nilpotent, if $|R| > 2$, then M_1R is a maximal subgroup of G of even order for any maximal subgroup M_1 of M. By hypothesis, M_1R is an MNP-group, and so $M_1R = M_1 \times R$ by Lemma 2.6. Furthermore, $MR = M \times R$. For any non-trivial maximal subgroup R_1 of R, it makes $M < MR_1 < MR = G$, a contradiction. So R is of order 2.

Case 2 $O_2(G) = 1$.

Since $O_2(G) = 1$ and the solvability of G, we have $O_{2'}(G) \neq 1$.

Let $N/O_{2'}(G)$ be a minimal normal subgroup of $G/O_{2'}(G)$. Since $N/O_{2'}(G)$ has even order and $O_{2'}(G) \leq M$, we have that $MN = G$ and $M \cap N = O_{2'}(G)$. Hence $N/O_{2'}(G)$ is a Sylow subgroup of $G/O_{2'}(G)$, $RO_{2'}(G) = N$ and $G = N_G(R)N = N_G(R)O_{2'}(G)$. Let H be a Hall $2'$-subgroup of $N_G(R)$. By hypothesis and Lemma 2.6, H is normal in $N_G(R)$, and R_1H is a subgroup of G for every maximal subgroup R_1 of R. So $G = R(HO_{2'}(G))$ and let $M = HO_{2'}(G)$. Moreover, we can see that $R_1HO_{2'}(G) = R_1M > M$ if $R_1 > 1$, which implies that $G = R_1M$. It follows that $R = R_1$. This contradiction induces that $|R| = 2$ and so M is the normal 2-complement.

We next complete the rest of the proof as follows.

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Suppose $C_M(R) = 1$. Then an automorphism of R acting on M is both of order 2 and fixed point-free. Lemma 2.11 implies that M is abelian, and so it is an MNP-group. This contradiction leads to $C_M(R) > 1$.

If $C_M(R) = M$, then $MR = M \times R$, and M_1R is an MNP-group by hypothesis for any maximal subgroup M_1 of M. By Lemma 2.7, M_1 is an MNP-group, and so M is a sub-MNP-group. Therefore, G is of type (I).

If $1 < C_M(R) < M$, then PR and QR are MNP-groups by hypothesis and Lemma 2.7, and so R induces two power automorphisms of order dividing 2 in $P/\Phi(P)$ and $Q/\Phi(Q)$ respectively by Lemma 2.2.

Let $1 \lhd \cdots \lhd K \lhd PQ \trianglelefteq G$ be a principle series of G. Since G is solvable, then one of P and Q is contained in K, and there exists a maximal subgroup H of G of even order such that $KR \leq H$. By hypothesis and Lemma 2.6, H is supersolvable. If $Q \leq K$, then Q is either cyclic or normal in K by hypothesis and Lemma 2.2. Furthermore, if Q is normal in K, then Q is normal in G as Q char $K \trianglelefteq G$. Similarly, if $P \leq K$, then P is normal in G by the supersolvability of K. We discuss from three cases as shown below.

(1) Q is cyclic.

Clearly, $P \trianglelefteq G$. We prove that PQ is a sub-MNP-group.

By hypothesis, $P\Phi(Q)R$ is an MNP-group, and so $P\Phi(Q)$ is an MNP-group by Lemma 2.7. Let $P_1Q \lessdot PQ$ with $P_1 \leq P$. Clearly, $\Phi(P) \leq P_1$ and PR is an MNP-group, so $R \leq N_G(P_1)$. Thus, P_1QR is an MNP-group by hypothesis, and so P_1Q is an MNP-group by Lemma 2.7. Therefore, PQ is a sub-MNP-group and G is of type (II).

(2) Q is non-cyclic and $Q \trianglelefteq G$.

If P is non-cyclic, then P_1QR is an MNP-group by hypothesis for any maximal subgroup P_1 of P. By Lemma 2.7 and Lemma 2.2, P_1Q is nilpotent, and so PQ is nilpotent, a contradiction. Thus, P is cyclic and $\Phi(P)Q$ is nilpotent as the fact $\Phi(P)QR$ is an MNP-group. Let $PQ_1 \lessdot PQ$ with $Q_1 \leq Q$. Clearly, $\Phi(Q) \leq Q_1$ and PQ_1R is a maximal subgroup of G of even order. By hypothesis, PQ_1R is an MNP-group, so it is supersolvable by Lemma 2.6. Hence $P \trianglelefteq PQ_1$. It follows that PQ_1 is nilpotent from the fact Q_1 is subnormal in G, and so PQ is minimal non-nilpotent. Thus, G is of type (III).

(3) P is normal in G.

If Q is cyclic, then G is of type (II) as the arguments in (1).

If Q is non-cyclic, then PQ_1R, PQ_2R are MNP-groups by hypothesis for two different maximal subgroups Q_1, Q_2 of Q. So P_1 is normal in $G = \langle PQ_1R, PQ_2R\rangle$ for any maximal subgroup P_1 of P.

We first prove that P is cyclic. If not, then P_1QR, P_2QR are MNP-groups by hypothesis for two different maximal subgroups P_1, P_2 of P. It follows that Q is normal in $G = \langle P_1QR, P_2QR\rangle$ by Lemma 2.2, and so PQ is nilpotent, a contradiction.

We next prove that P is of order p. If not, then $\Omega_1(P) \leq \Phi(P)$ and $\Phi(P)QR$ is an MNP-group by hypothesis, so $\Phi(P)Q$ is nilpotent by Lemma 2.2. Therefore, PQ is nilpotent by Lemma 2.3, a contradiction.

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Obviously, PQ_1R is an MNP-group by hypothesis, and so PQ_1 is an MNP-group by Lemma 2.7 for any maximal subgroup Q_1 of Q. Hence PQ is a sub-MNP-group.

By examining Theorem 4.2, M is isomorphic to one of types (II), (IV), (V) in Theorem 4.2, and so there exists a maximal subgroup PQ_1R of G such that PQ_1R is an MNP-group by hypothesis, and PQ_1 is non-nilpotent for some maximal subgroup Q_1 of Q. By Lemma 2.2 again, Q_1R is nilpotent. Furthermore, QR is nilpotent by the fact that QR is an MNP-group. Hence G is of type (IV). □

6. Simple groups all of whose second maximal subgroups of even order are MNP-groups

In this section, we determine non-abelian simple groups all of whose second maximal subgroups of even order (respectively, maximal subgroups) are MNP-groups.

Theorem 6.1. *Let G be a group all of whose second maximal subgroups of even order are MNP-groups. Then G is a non-abelian simple group if and only if G is isomorphic to one of the following types:*

(I) A_5;

(II) $\mathrm{PSL}(2,p)$, *where p is a prime with $p \geq 13$, $5 \nmid p^2-1$, $16 \nmid p^2-1$, and only one of $(p+1)/4$ and $(p-1)/4$ is a prime;*

(III) $\mathrm{PSL}(2,2^q)$, *where q is an odd prime and 2^q-1 is a prime;*

(IV) $\mathrm{PSL}(2,3^q)$, *where q is an odd prime and $(3^q+1)/4$ is a prime.*

Proof. Suppose that G is a non-abelian simple group. Let M be a maximal subgroup of G. If M is a group of odd order, then M is solvable. If M is a group of even order, then M is either an MNP-group or a non-MNP-group whose all maximal subgroups of even order are MNP-groups by hypothesis. By Lemma 2.6 and Theorem 5.1, M is solvable. Hence all proper subgroups of G are solvable. So G is a minimal simple group. By Lemma 2.10, we know that G is isomorphic to one of the following simple groups:

(i) $\mathrm{PSL}(3,3)$;

(ii) the Suzuki group $\mathrm{Sz}(2^q)$, where q is an odd prime;

(iii) $\mathrm{PSL}(2,p)$, where p is a prime with $p>3$ and $p^2 \not\equiv 1 (\mathrm{mod}\ 5)$;

(iv) $\mathrm{PSL}(2,2^q)$, where q is a prime;

(v) $\mathrm{PSL}(2,3^q)$, where q is an odd prime.

Case 1 $G \ncong \mathrm{PSL}(3,3)$;

Suppose $G \cong \mathrm{PSL}(3,3)$. Then G contains a maximal subgroup which is isomorphic to S_4, and so G contains a second maximal subgroup isomorphic to A_4. But A_4 is not an MNP-group, a contradiction. Hence $G \ncong \mathrm{PSL}(3,3)$.

Case 2 $G \ncong \mathrm{Sz}(2^q)$, where q is an odd prime.

Suppose $G \cong \mathrm{Sz}(2^q)$. Then G contains a Frobenius maximal subgroup M with a cyclic complement H of order 2^q-1 and kernel K of order 2^{2q}. Since K is non-abelian, then $\Phi(K)H$ is contained in a second maximal subgroup N of G of even order which is an MNP-group. By Lemma 2.6, N is supersolvable, and so $\Phi(K)H$ is supersolvable. Hence $\Phi(K)H$ is nilpotent, a contradiction. So G is not isomorphic to $\mathrm{Sz}(2^q)$.

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Case 3 $G \cong A_5$ or $G \cong \mathrm{PSL}(2,p)$, where p is a prime with $p \geq 13$, $5 \nmid p^2-1$, $16 \nmid p^2-1$, and only one of $(p+1)/4$ and $(p-1)/4$ is a prime.

Suppose $G \cong \mathrm{PSL}(2,p)$. If $p = 5$, then $G \cong A_5$, so G is of type (I). If $p^2 \equiv 1 (\mathrm{mod}\ 16)$, then G has a maximal subgroup which is isomorphic to S_4 by [9, Corollary 2.2], and so A_4 is a second maximal subgroup of G which is not an MNP-group, a contradiction. If $p \geq 13$, then G has maximal subgroups which are isomorphic to dihedral groups D_{p-1} and D_{p+1}, a Frobenius group of order $p(p-1)/2$, A_4 by [9, Corollary 2.2]. Furthermore, 4 must divide the order of either of D_{p-1} or D_{p+1}, say A. Clearly, the Sylow 2-subgroup of A is non-cyclic and A is not an MNP-group by Lemma 2.2, so A is either a sub-MNP-group or a group whose all maximal subgroups of even order are MNP-groups by hypothesis. Note $4 \| A|$, then A must be a sub-MNP-group by Theorems 5.3 and 5.4. By examining Theorem 4.2, G is of type (II).

Case 4 $G \cong A_5$ or $G \cong \mathrm{PSL}(2,2^q)$, where q and 2^q-1 are odd primes.

Suppose $G \cong \mathrm{PSL}(2,2^q)$. Then by [9, Corollary 2.2], G has maximal subgroups: the dihedral groups of order $2(2^q \pm 1)$; the Frobenius group H of order $2^q(2^q-1)$; A_4 if $q = 2$. Similar arguments as in Case 3, H must be a sub-MNP-group if $q > 2$. By examining Theorem 4.2, we have that 2^q-1 must be a prime. So G is of type (III). Clearly, $G \cong A_5$ if $q = 2$, so G is of type (I).

Case 5 $G \cong \mathrm{PSL}(2,3^q)$, where $(3^q+1)/4$ is a prime.

Suppose $G \cong \mathrm{PSL}(2,3^q)$. Then by [9, Corollary 2.2], G has maximal subgroups: the dihedral groups of order $3^q \pm 1$; the normalizer H of the Sylow 3-subgroup of G of order $3^q(3^q-1)/2$; A_4. Clearly, $4 \mid 3^q+1$, $3^q(3^q-1)/2$ is odd number and the dihedral group of order 3^q-1 is an MNP-group, so we only consider the dihedral group D of order 3^q+1. Similar arguments as in Case 3, D must be a sub-MNP-group. By examining Theorem 4.2, $(3^q+1)/4$ is a prime, so G is of type (IV).

Conversely, it is clear that a group of one of types (I)–(IV) is satisfied with the condition of this theorem. □

Corollary 6.2. *Let G be a group all of whose second maximal subgroups are MNP-groups. Then G is a non-abelian simple group if and only if G is isomorphic to one of the following types:*

(I) A_5;

(II) $\mathrm{PSL}(2,p)$, *where p is a prime with $p \geq 13$, $5 \nmid p^2-1$, $16 \nmid p^2-1$, only one of $(p+1)/4$ and $(p-1)/4$ is a prime, and $(p-1)/2$ is square-free if $(p+1)/4$ is a prime;*

(III) $\mathrm{PSL}(2,2^q)$, *where q is an odd prime, and 2^q-1 is a prime;*

(IV) $\mathrm{PSL}(2,3^q)$, *where q is an odd prime, $(3^q+1)/4$ and $(3^q-1)/2$ are primes.*

Proof. Similar arguments as the proof in Theorem 6.1. □

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