Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

CONDITIONAL NON-LATTICE INTEGRATION, PRICING, AND SUPERHEDGING

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Abstract. Closely motivated by financial considerations, we develop an integration theory which is not classical i.e. it is not necessarily associated to a measure. The base space, denoted by $\mathcal{S}$ and called a trajectory space, substitutes the set Ω in probability theory and provides a fundamental structure via conditional subsets $\mathcal{S}_{(S,j)}$ that allows the definition of conditional integrals. The set $\mathcal{S}$ naturally embodies a weak no-arbitrage hypothesis which in turn is key to establishing the basic properties needed to develop the theory of integration. The constructed conditional integrals can be interpreted as the required investment, at each conditioning node, to hedge an integrable function, the latter characterized a.e. and in the limit as we increase the number of portfolios used. The a.e. notion is defined by means of financial considerations and it does not rely on measure theory. The integral is not classical due to the fact that the elementary vector space of portfolio payoffs is not a vector lattice. In contrast to a classical stochastic setting, where price processes are associated to conditional expectations (with respect to risk neutral measures), we uncover a theory where prices are naturally given by conditional non-lattice integrals. In particular, no measurability assumptions are needed. Besides the central role of $\mathcal{S}$, key ingredients in our approach are conditional superhedging operators that act as conditional outer integrals, the associated superhedging norms provide countable subadditivity and allow to define null events.

1. Introduction

The paper develops a non-classical theory of integration in a financial trading context. The abstract non-lattice integration theory, that we rely on, was developed first in [17] and then refined in [16] (henceforth referred as the Leinert-König theory). We extend the Leinert-König framework to a conditional version by taking advantage of a natural conditioning structure present in our trading setting.

The abstract Leinert-König theory is here employed to set up a trajectorial framework for trading in a financial market in infinite discrete time. To gain perspective, our approach is located between the model-free arbitrage and superhedging theory in finite discrete time studied, e.g., by [1, 7, 8, 9, 10] and the pathwise superhedging approach to mathematical finance in continuous time initiated by

2020 *Mathematics Subject Classification.* Primary 28C05; Secondary 60G42, 91G80.

Key words and phrases. Non-lattice Daniell integration, superhedging, trajectorial market models.

The research of S. E. Ferrando is supported in part by an NSERC grant.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
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Vovk [22] and further developed, e.g., in [2, 3, 4, 18, 23]. The infinite discrete time framework also makes our approach closely related to the game-theoretic approach to probability of Shafer, Vovk, and co-authors which is detailed in [19].

The Leinert-König theory mimics, and also generalizes, the classical (measure-based) Lebesgue-Daniell theory of integration in the sense that the integral is first defined for a class of elementary functions and then extended to a larger class of integrands by continuity arguments. However, it dispenses with the property that the (abstract) class of elementary functions forms a vector lattice by only requiring the vector space property. For this reason, their constructions can lead to non-classical integrals, i.e., to integrals which are not associated to measures. The Leinert-König theory relies on two operators, which we call norm operator and outer integral in the following discussion (their non-conditional versions are denoted $\overline{I}$ and $\overline{\sigma}$ respectively). The norm operator can be considered as the analogue of the L^1-seminorm of the classical theory. It is countable-subadditive and can be used to generalize the crucial continuity-from-below property of the elementary integral, on which the Lebesgue-Daniell integration relies. This generalization of continuity-from-below is Leinert's continuity property in the non-lattice framework, which was later strengthened by König, cp. Definition 2.7 below. Null functions and null sets are defined in terms of the norm operator. On the other hand, the outer integral $\overline{\sigma}$ is subadditive but not necessarily countable-subadditive, and it is an extension of the elementary integral. The corresponding space of integrable functions is the largest subspace of functions on which it acts linearly. The Leinert-König integral satisfies, for example, the Beppo-Levi theorem for series of non-negative functions (under Leinert's condition) and the monotone convergence theorem for increasing sequences of non-negative functions (under König's condition). The validity of Lebesgue's dominated convergence theorem requires additional assumptions connected to the lattice property. A crucial difference with the classical theory is that the non-lattice integral need not coincide, in the general case, with the norm operator on the cone of non-negative integrable functions. Thus, both operators (norm operator and outer integral) play different roles.

To apply the Leinert-König theory in our financial context, we follow [13] and start with a set $\mathcal{S}$ of trajectories which models the possible future price evolution of a risky stock (discounted in terms of a tradable numeraire). The payoffs of simple portfolios (i.e., of finite linear combinations of one-period buy-and-hold strategies) form the vector space of elementary functions on $\mathcal{S}$ and the corresponding hedging cost defines the elementary integral. The need to dispense with the lattice property in this trading context is due to the fact that the space of portfolio payoffs is not, in the general case, a vector lattice. Note that each simple portfolio has a finite investment horizon. In order to trade in infinite time, the norm operator and the outer integral apply the idealization of trading with a countable superposition of simple portfolios (whose associated sequence of trading horizons may be unbounded). The norm operator requires that the payoff of each of the simple portfolios is non-negative for every trajectory. Even though the operator $\overline{I}$ is derived from the framework of the Leinert-König theory, it can be considered as

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the natural analogue of Vovk's outer measure [22] in the infinite discrete-time case. The definition of the operator $\overline{\sigma}$ relaxes the positivity requirement on the payoff of one the simple portfolios in the superposition and represents the minimal superhedging cost under the idealization of trading with countably many portfolios. Note that the use of countably many portfolios is crucial to detect several subsets of trajectories, on which rather obvious arbitrage opportunities exist, as null sets (as illustrated in the examples of Subsection 2.4).

In this paper, we study a conditional version of the superhedging outer integral $\overline{\sigma}$, when the investor enters the market at time $j \in \mathbb{N}$ and the past stock prices are represented by the initial segment $(S_0, \dots S_j)$ of some trajectory. We will introduce a notion of conditional integrability, which depends on the null sets of the unconditional norm operator and means that a function f is conditionally integrable, if and only if seller and buyer agree on a unique price for an option with payoff function f (when contracted at time j) in almost every scenario of the past stock prices. We will illustrate by some examples, that, in general, this notion of integrability can be satisfied even if there is no perfect hedge (up to null sets and under the idealization of trading with countably many portfolios). However, under a suitable conditional formulation of König's stronger continuity condition, we can prove the classical characterization (see, e.g., [15]) that an option can be perfectly hedged, if and only if it has a uniquely determined price (which then is the hedging cost). Note that the conditional superhedging outer integral naturally gives rise to the notion of a trajectorial supermartingale which is studied in [6].

We stress that our trajectorial setting works under very mild assumptions. It provides a meaningful superhedging outer integral if Leinert's continuity from below condition holds. The latter property can be viewed as a very weak no-arbitrage assumption. If it fails, one can then construct for every initial wealth $V < 0$ an idealized trading strategy (consisting of a superposition of countably many simple portfolios as described above) which yields a terminal wealth of at least one unit of currency on every trajectory, see the discussion right before Definition 2.8. Leinert's continuity condition does not require topological properties on the trajectory set (which are usually required in continuous time pathwise finance) or the existence of a martingale measure for the stock price model (which plays a central role in the finite discrete-time model-free finance literature). We refer to Section 2.5 for a detailed comparison to the literature, but mention now that the way we deal with null sets is an important difference to most of the discrete-time pathwise finance literature: While the quasi-sure approach of [7], and similarly the pathwise approach of [9, 10], fixes a family of probability measures and considers a set to be negligible if it has zero probability under each of these measures, the trajectorial approach avoids any reference to a-priori given probability measures for determining null sets. To be more specific, Example 2 in Subsection 2.4 illustrates how the superhedging outer integral acts on a trajectory space without a martingale measure via a combination of detecting null sets and of passing to the topological closure on the remaining subset of trajectories in order to come up with an arbitrage-free model. This flexibility makes the trajectory set based superhedging

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outer integral attractive. In particular, one can apply it to trajectory sets which are constructed using historical data by means of combinatorial recombinations and worst case constraints (as in [14]). These data-based constructions can be at odds with no-arbitrage modeling constraints. Removing a-priori such arbitrage opportunities, e.g., via the arbitrage aggregator of [8, 10], may cause fundamental problems as the whole model may collapse into the empty set (see the discussion in Section 2.5 below). In such situations, the trajectorial superhedging outer integral can still deliver relevant price bounds. Thus, in combination with the algorithm proposed in [11], the trajectorial approach could lead to an algorithmic procedure for computing option price intervals from historical data.

The paper is organized as follows: Section 2 introduces the trajectorial setting (Subsections 2.1–2.2), discusses the main result on the characterization of conditionally integrable functions as payoff functions of replicable options (Subsection 2.3), and illustrates our framework through several examples (Subsection 2.4). Moreover, we discuss the relation and the differences to the literature on model-free finance in discrete and continuous time (Subsection 2.5). Section 3 provides sufficient criteria for appropriate conditional versions of the continuity conditions of Leinert and König. While Leinert's condition is a minimum requirement for the outer integral to extend the elementary integral, König's stronger conditions will be required for some of our results. Section 4 is devoted to a detailed study of the space of conditionally integrable functions. Next to the main result on the characterization of conditionally integrable functions, it contains versions of the Beppo-Levi theorem and the monotone convergence theorem for the conditional superhedging integral as well as a norm completion characterization of the space of integrable functions. Appendices A–E provide auxiliary material related to the proofs of the main results, while some additional examples are presented in Appendix F.

2. Discussion of the setting and of the main result

2.1. **Trajectorial setting and null sets.** We first introduce the basic components of a trajectorial market model in infinite discrete time (with time index $i \in \mathbb{N}_0$) and explain the relation to the theory of non-lattice integration developed in [17, 16].

The market consists of a risky stock whose possible price fluctuations are modeled (in discounted units) by a trajectory set. Additionally, the investor can trade into a money market account with zero interest rate.

Definition 2.1 (Trajectory Set). Given a real number s_0, a *trajectory set*, denoted by $\mathcal{S} = \mathcal{S}(s_0)$, is a subset of

$$\mathcal{S}_\infty(s_0) = \{S = (S_i)_{i \in \mathbb{N}_0} : S_i \in \mathbb{R},\ S_0 = s_0\}.$$

We make fundamental use of the following *conditional spaces*; for $S \in \mathcal{S}$ and $j \in \mathbb{N}_0$ let:

$$\mathcal{S}_{(S,j)} \equiv \{\tilde{S} \in \mathcal{S} : \tilde{S}_i = S_i,\ 0 \le i \le j\}.$$

The conditional space $\mathcal{S}_{(S,j)}$ contains the possible future price evolution of the stock, provided the investor enters the market at time j and the past stock prices

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are given by $(S_0, \ldots, S_j)$. For simplicity, sometimes we will refer to the space $\mathcal{S}_{(S,j)}$ through a pair (S, j) with $S \in \mathcal{S}$ and $j \geq 0$ which will be called a *node*.

Remark 2.2. In practice, the coordinates S_i are multidimensional in order to allow for multiple sources of uncertainty ([14]), for simplicity we restrict to $S_i \in \mathbb{R}$. One can also extend the theory to allow for several traded assets S_i^k (as in [12]).

We next describe the set of portfolio positions available to the investor, when entering the market at time j in a given stock price scenario (S, j).

Definition 2.3. For any fixed $S \in \mathcal{S}$ and $j \geq 0$, $\mathcal{H}_{(S,j)}$ will be a set of sequences of functions $H = (H_i)_{i \geq j}$, where $H_i : \mathcal{S}_{(S,j)} \to \mathbb{R}$ are non-anticipative in the following sense: for all $\tilde{S}, \hat{S} \in \mathcal{S}_{(S,j)}$ such that $\tilde{S}_k = \hat{S}_k$ for $j \leq k \leq i$, then $H_i(\tilde{S}) = H_i(\hat{S})$ (i.e. $H_i(\tilde{S}) = H_i(\tilde{S}_0, \ldots, \tilde{S}_i)$).

$H \in \mathcal{H}_{(S,j)}$ may be referred to as a *conditional portfolio*. We think of $H_i(\tilde{S})$ as the number of shares of the risky stock held by the investor between time i and time $i+1$. Our setting admits certain portfolio restrictions, which are summarized in the following conditions:

- (H.1) The sets $\mathcal{H}_{(S,j)}$ are assumed to be vector spaces $\alpha \mathcal{H}_{(S,j)} + \mathcal{H}_{(S,j)} \subseteq \mathcal{H}_{(S,j)}$ for all $\alpha \in \mathbf{R}$.
- (H.2) The portfolios $H^c = (H_i^c)_{i \geq j}$ where H_i^c are constant valued -1, 0 or 1, on $\mathcal{S}_{(S,j)}$, are assumed to belong to $\mathcal{H}_{(S,j)}$. So the null portfolio ($\mathbf{0} = (H_i^{\mathbf{0}} \equiv 0)_{i \geq j}$) belongs to $\mathcal{H}_{(S,j)}$.
- (H.3) If $(H_i)_{i \geq j} \in \mathcal{H}_{(S,j)}$ and $k \geq j$, then $((H_i)|_{\mathcal{S}_{(S,k)}})_{i \geq k} \in \mathcal{H}_{(S,k)}$.
- (H.4) Let $(H_i)_{i \geq j} \in \mathcal{H}_{(S,j)}$, if $G_i \equiv H_i$ for $j \leq i \leq k$ and $G_i = 0$ for $i > k$, then
  $G \in \mathcal{H}_{(S,j)}$ as well.

Conditions (H.1)–(H.4) are assumed for the rest of this paper without further notice.

Remark 2.4. As a special case of a set of portfolios that fulfill (H.1)–(H.4) one can take all portfolios $(H_i)_{i \geq j}$ of the form $H_i(\hat{S}) = h_i(S_0, \ldots, S_j, \hat{S}_{j+1}, \ldots, \hat{S}_i)$ for every $\hat{S} \in \mathcal{S}_{(S,j)}$, where $h_i : \mathbb{R}^{i+1} \to \mathbb{R}$ are $\mathcal{A}_i/\mathcal{B}$-measurable with respect to any fixed σ-field $\mathcal{A}_i$ of $\mathbb{R}^{i+1}$ and the Borel σ-field $\mathcal{B}$ of $\mathbb{R}$.

For a node (S, j), $H \in \mathcal{H}_{(S,j)}$, $V \in \mathbb{R}$ and $n \geq j$ we define the *portfolio payoff* $\Pi_{j,n}^{V,H} : \mathcal{S}_{(S,j)} \to \mathbb{R}$ as

$$\Pi_{j,n}^{V,H}(\tilde{S}) \equiv V + \sum_{i=j}^{n-1} H_i(\tilde{S})\ \Delta_i \tilde{S},\ \tilde{S} \in \mathcal{S}_{(S,j)},\ \text{where } \Delta_i \tilde{S} = \tilde{S}_{i+1} - \tilde{S}_i,\ i \geq j.$$

$\Pi_{j,n}^{V,H}$ is the payoff of the portfolio H with initial investment V and investment horizon n when the past of the stock price up to time j is described by the node (S, j). Note that the definition of $\Pi^{V,H}$ reflects the usual self-financing condition, see, e.g. [15]. Functions $f : \mathcal{S}_{(S,j)} \to \mathbb{R}$, which can be represented as a portfolio

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

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payoff, will be called *elementary functions*, and we write

$$\mathcal{E}_{(S,j)} = \{f = \Pi^{V,H}_{j,n} : H \in \mathcal{H}_{(S,j)},\ V \in \mathbb{R} \text{ and } n \in \mathbb{N}\}$$

for the set of those elementary functions. If $f = \Pi^{V,H}_{j,n} \in \mathcal{E}_{(S,j)}$, we say that the simple portfolio (V, n, H) is a *perfect hedge* for f. The hedging price is the initial endowment V to set up this hedge, leading to the operator

$$I_{(S,j)} : \mathcal{E}_{(S,j)} \to \mathbb{R},\ f \mapsto V, \quad \text{if } f = \Pi^{V,H}_{j,n} \in \mathcal{E}_{(S,j)}. \tag{2.1}$$

Note that the operator $I_{(S,j)}$ is a well-defined, linear, and isotone operator for every node (S, j), if and only if every node is 0-neutral in the sense of the following definition (see Appendix A for the details).

Definition 2.5. Given a trajectory space $\mathcal{S}$ and a node (S, j):

- (S, j) is called a 0-*neutral node* if
$$\sup_{\tilde{S}\in\mathcal{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) \geq 0 \quad \text{and} \quad \inf_{\tilde{S}\in\mathcal{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) \leq 0.$$
- (S, j) is called an *up-down node* if
$$\sup_{\tilde{S}\in\mathcal{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) > 0 \quad \text{and} \quad \inf_{\tilde{S}\in\mathcal{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) < 0. \tag{2.2}$$
- (S, j) is called a *flat node* if
$$\sup_{\tilde{S}\in\mathcal{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) = 0 = \inf_{\tilde{S}\in\mathcal{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j). \tag{2.3}$$
- (S, j) is called an arbitrage-free node if (2.2) or (2.3) hold, otherwise it is called an *arbitrage node*. An arbitrage node (S, j) is said to be of type I if there exist $\hat{S} \in \mathcal{S}_{(S,j)}$ such that $\hat{S}_{j+1} = S_j$, otherwise it is said to be of type II.

We will, thus, assume from now on that every node (S, j) is zero neutral (but, see also Remark 2.11). Then, each operator $I_{(S,j)}$ is an elementary integral in the sense of the non-lattice integration developed in [16, 17]. Since the operator depends on the past through the node (S, j), we will refer to $I_{(S,j)}$ as the *elementary hedging integral conditionally on* (S, j).

The hedging price interpretation in (2.1) requires that $f(\tilde{S}) = \Pi^{V,H}_{j,n}(\tilde{S})$ for every $\tilde{S} \in \mathcal{S}_{(S,j)}$. This is in contrast to the classical setting where the hedge only needs to hold outside a null set with respect to a reference probability measure $\mathbb{P}$. How null events are introduced in model-free settings represents a crucial theoretic construction with substantial implications. We explain, in Section 2.5, the differences between our approach and the literature (e.g. [10] in finite discrete time or [22] in continuous time). Our framework relies on the general constructions of the Leinert-König theory to define null sets. Denote by $\mathcal{E}^{+}_{(S,j)}$ the cone of non-negative

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functions in $\mathcal{E}_{(S,j)}$ and consider

$$\begin{aligned}\overline{I}_{(S,j)}f &\equiv \inf\left\{\sum_{m=1}^{\infty} I_{(S,j)}f_m :\ f_m\in\mathcal{E}^+_{(S,j)},\ f(\tilde S)\le\sum_{m=1}^{\infty} f_m(\tilde S)\ \forall\tilde S\in\mathcal{S}_{(S,j)}\right\}\\ &=\inf\left\{\sum_{m=1}^{\infty} V^m :\ \Pi^{V^m,H^m}_{j,n_m}\in\mathcal{E}^+_{(S,j)},\ f(\tilde S)\le\sum_{m=1}^{\infty}\Pi^{V^m,H^m}_{j,n_m}(\tilde S)\ \ \forall\tilde S\in\mathcal{S}_{(S,j)}\right\},\end{aligned} \tag{2.4}$$

which is defined on all functions $f:\mathcal{S}_{(S,j)}\to[0,\infty]$. $\overline{I}_{(S,j)}f$ can be interpreted as the minimal superhedging price when trading takes place with the idealization of a superposition of countably many simple portfolios with non-negative portfolio payoff. The non-negativity assumption ensures that $\overline{I}_{(S,j)}$ is countable-subadditive.

Let Q denote the set of all functions from $\mathcal{S}$ to $[-\infty,\infty]$ and $P\subset Q$ denotes the set of non-negative functions (taking values in $[0,\infty]$). The following conventions are in effect: $0\cdot\infty=0$, $\infty+(-\infty)=\infty$, $u-v\equiv u+(-v)\ \forall\ u,v\in[-\infty,\infty]$, and $\inf\emptyset=\infty$ (following [16]).

For $f\in Q$, we define the *conditional norm operator*

$$\|f\|_j(S):=\overline{I}_{(S,j)}(|f_{|(S,j)}|),\quad S\in\mathcal{S},$$

where $f_{|(S,j)}$ denotes the restriction of f on the node (S,j).

Definition 2.6. Fix a node (S,j). A function $f\in Q$ is said to be a *null function conditionally on* (S,j), if $\|f\|_j(S)=0$, and a subset $E\subset\mathcal{S}$ is a *null set conditionally on* (S,j), if $\|\mathbf{1}_E\|_j(S)=0$, i.e., if $\mathbf{1}_E$ is a null function conditionally on (S,j). We say that a property holds *a.e. on* $\mathcal{S}_{(S,j)}$, if the subset of $\mathcal{S}$ on which the property does not hold is a null set conditionally on (S,j).

Note that $\mathcal{S}_{(S,0)}=\mathcal{S}$ for every $S\in\mathcal{S}$. Hence, $\|\cdot\|_0(S)$ does not depend on S, and we write $\|\cdot\|$ in place of $\|\cdot\|_0(S)$. (Unconditional) *null functions* f and *null sets* E are defined with respect to $\|\cdot\|$, i.e., via the identities $\|f\|=0$ and $\|\mathbf{1}_E\|=0$, respectively. We will sometimes call these null sets and null functions global ones. We say that a property holds *a.e.*, if it is valid outside a global null set. Note that, although the above notion of null sets is adapted from the abstract theory of [17, 16], this construction is completely analogous to the construction of null sets in continuous-time model-free finance via Vovk's outer measure [22]. We stress that by the countable-subadditivity of the $\overline{I}$ operator, countable unions of null sets are again null sets (analogously for their conditional versions).

2.2. **The conditional superhedging outer integrals.** The next step is to extend the conditional hedging integrals $I_{(S,j)}$ to account for the following issues:

1) Ensure that hedging only has to take place outside a conditional null set.
2) Apply some idealization which ensures that trading can take place up to infinite time (noting that the elementary functions are based on trading with finite investment horizon.)

This will lead to our notion of a conditional superhedging outer integral.

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Recall that classical integral constructions can be based on continuous extensions of elementary integrals with respect to some (semi-)norm. However, the space of elementary functions $\mathcal{E}_{(S,j)}$ does not satisfy the lattice property (except for some simple special cases such as a binomial tree structure of the trajectory set). Therefore, care must be taken when formulating the continuity requirement of the elementary integral in a proper way. Leinert [17] and König [16] show how to build a non-lattice integration theory based on the following continuity properties.

Definition 2.7. Leinert Condition: $I_{(S,j)}(f) \leq \bar{I}_{(S,j)}(f^+), \forall f \in \mathcal{E}_{(S,j)}$.

König Condition ($K_{(S,j)}$): $I_{(S,j)}(f)+\bar{I}_{(S,j)}(f^-) = \bar{I}_{(S,j)}(f^+), \forall f \in \mathcal{E}_{(S,j)}$.

Clearly, *König's condition* is stronger than *Leinert's condition*, which in turn implies (see Proposition 3.4) the classical continuity requirement:

$$|I_{(S,j)}(f)| \leq \bar{I}_{(S,j)}(|f|), \quad f \in \mathcal{E}_{(S,j)}.$$

Following the same idealization of a superposition of countably many simple portfolio as in the definition of $\bar{I}_{(S,j)}$, but relaxing the positivity assumption on the portfolio payoff of one of them, the *superhedging outer integral conditional on* (S,j) is defined to be

$$\begin{aligned}\overline{\sigma}_{(S,j)}f &\equiv \inf\left\{\sum_{m=0}^{\infty} I_{(S,j)}f_m : \ f_0 \in \mathcal{E}_{(S,j)},\ f_m \in \mathcal{E}^+_{(S,j)}\ (m\geq 1),\ f(\tilde{S}) \leq \sum_{m=0}^{\infty} f_m(\tilde{S})\ \forall \tilde{S} \in \mathcal{S}_{(S,j)}\right\}\\ &= \inf\Bigg\{\sum_{m=0}^{\infty} V^m : \ \Pi^{V^0,H^0}_{j,n_0} \in \mathcal{E}_{(S,j)},\ \Pi^{V^m,H^m}_{j,n_m} \in \mathcal{E}^+_{(S,j)}\ (m\geq 1),\\ &\qquad f(\tilde{S}) \leq \sum_{m=1}^{\infty} \Pi^{V^m,H^m}_{j,n_m}(\tilde{S})\ \ \forall \tilde{S} \in \mathcal{S}_{(S,j)}\Bigg\}.\end{aligned}$$

Clearly, $\overline{\sigma}_{(S,j)}f \leq I_j(f)$ for any $f \in \mathcal{E}_{(S,j)}$, and we think of $\overline{\sigma}_{(S,j)}f$ as the superhedging price for a financial derivative with payoff function f when trading takes place with the idealization of a countable superposition of simple portfolios and the investor enters the market at time j in the scenario (S,j). The importance of *Leinert condition* lies in the following dichotomy: If *Leinert condition* holds, then $\overline{\sigma}_{(S,j)}f = I_{(S,j)}(f)$ for any $f \in \mathcal{E}_{(S,j)}$ (and, hence, the idealization of trading with a superposition of countably many portfolios does not change the price of payoffs $\Pi^{V,H}_{j,n}$ generated by 'simple' portfolios). If, however, *Leinert condition* fails, then $\overline{\sigma}_{(S,j)}f = -\infty$ for any $f \in \mathcal{E}_{(S,j)}$ (and, hence, the idealized trading strategies generate arbitrage). See p. 262 in [17] for the first claim and p. 449 in [16] for the second one in the abstract setting of non-lattice integration.

Suppose that a trader enters the market at time j. In view of the previous discussion, it is tempting to suppose that *Leinert condition* holds at every node (S,j). This would ensure that the trader arrives in an economically meaningful environment in every scenario. However, assuming *Leinert condition* at every node is too strong, as it rules out the possibility of arbitrage nodes of type II. This will

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

become clear from the following short discussion of equivalent characterizations of *Leinert condition.*

Definition 2.8 (($L_{(S,j)}$))**.** For a given node (S,j), $j \geq 0$, the following property will be called (conditional) *continuity from below.*

$$(L_{(S,j)}) \qquad f \leq \sum_{m\geq 1} f_m \text{ on } \mathcal{S}_{(S,j)},\ f \in \mathcal{E}_{(S,j)}, f_m \in \mathcal{E}^+_{(S,j)} \implies I_j f(S) \leq \sum_{m\geq 1} I_j f_m(S).$$

Proposition 3.4 shows, among other things, that the properties $\overline{\sigma}_j 0(S) = 0$, $(L_{(S,j)})$ and *Leinert condition* (from Definition 2.7) are equivalent. Henceforth, we will loosely refer to any of these equivalent properties as Leinert's condition.

The following lemma shows that Leinert's condition cannot hold at an arbitrage node of type II.

Lemma 2.9. *Assume that* (S,j) *is an arbitrage node of type II. Then,*

$$\overline{\sigma}_j f(S) = -\infty \quad \textit{for any } f \in Q.$$

In particular, Leinert's condition fails at (S,j)*.*

Proof. We may consider the case when $\tilde{S}_{j+1} > S_j$ for all $\tilde{S} \in \mathcal{S}_{(S,j)}$. Take then, for all $m \geq 1$: $H_j^m(\tilde{S}) = 1$ and $H_i^m(\tilde{S}) = 0$ for all $i > j$, $V^m = 0$. Also, $H_i^0 = 0$ for all $i \geq j$. Then, for any $V^0 \in \mathbb{R}$: $f(\tilde{S}) \leq V^0 + \infty = V^0 + \sum_{m\geq 1} H_j^m(\tilde{S})\ \Delta_j \tilde{S}$ holds for any $\tilde{S} \in \mathcal{S}_{(S,j)}$ and $f \in Q$. The first claim then follows. In particular $\overline{\sigma}_j 0(S) = -\infty$, i.e. Leinert's condition fails. □

Therefore, under the presence of type II nodes one cannot uphold the property that Leinert's condition is satisfied at at all nodes (S,j). The way to allow for type II nodes is to weaken the latter property to hold only a.e. as in the following definition.

Definition 2.10 (Properties (L_j)-a.e. and (L)-a.e.)**.** For a fixed $j \geq 0$, the following two statements will be referred as property (L_j)-a.e.:

i) $(L_{(S,0)})$ holds,

ii) $\mathcal{N}^{(L_j)} \equiv \{S \in \mathcal{S} : \ (L_{(S,j)}) \text{ fails}\}$ is a null set.

If, in addition to item *i*), display (2.5) below holds, we will say that (L)-a.e. holds:

$$\{S \in \mathcal{S} : \exists k \geq 0 \ s.t.\ (L_{(S,k)}) \text{ fails}\} \text{ is a null set.} \tag{2.5}$$

Item *i*) guarantees that $\overline{I}(\mathbf{1}_{\mathcal{S}}) = 1$ and so if (L_j)-a.e. holds, then $\mathcal{S} \setminus \mathcal{N}^{(L_j)} \neq \emptyset$ and in fact $\overline{I}(\mathbf{1}_{\mathcal{S}\setminus\mathcal{N}^{(L_j)}}) = 1$. If (L)-a.e. holds, then $\mathcal{N}^{(L_j)}$ can be replaced by $\bigcup_{j\in\mathbb{N}} \mathcal{N}^{(L_j)}$ in the previous identity. Under appropriate conditions, Theorem 3.10 and Corollary 3.11 establish property (L)-a.e.

Remark 2.11. As a side remark, we note that we could dispense with our standing assumption that $\mathcal{S}$ is 0-neutral which in turn makes I_j well defined at all nodes (S,j). Namely, we can drop the hypothesis of 0-neutrality and replace it for (L)-a.e. the latter implies that I_j is only well defined (and linear and isotone) at almost

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every node (S, j). This property is enough to develop the results in the paper, for simplicity we have refrained from adopting such more general framework.

Definition 2.10 provides a first instance of a situation, where the nodewise defined conditional operators are combined with global null sets. In order to deal with such situations it is convenient to work with the 'global' notation set forth in the following remark.

Remark 2.12. Fix $j \geq 0$. $H = (H_i)_{i \geq j}$ with $H_i : \mathcal{S} \to \mathbb{R}$ is called a global portfolio, if, for every $S \in \mathcal{S}$, the restriction $((H_i)_{|(S,j)})_{i \geq j}$ belongs to $\mathcal{H}_{(S,j)}$ The set of global portfolios is denoted $\mathcal{H}_j$. We say that a function $g \in Q$ is j-nonanticipative, if $g(S) = g(\tilde{S})$, whenever the nodes (S, j) and $(\tilde{S}, j)$ coincide. $f : \mathcal{S} \to \mathbb{R}$ is called a global elementary function starting at j, if there are j-nonanticipative functions V, n, and a global portfolio $H \in \mathcal{H}_j$ such that

$$f(S) = V(S) + \sum_{i=j}^{n(S)-1} H_i(S)\Delta_i(S)$$

for every $S \in \mathcal{S}$. Here, n takes values in $\{j, j+1, \ldots\}$ and V takes values in $\mathbb{R}$. In this case, we write $\Pi^{V,H}_{j,n} \equiv f$ and think of this expression as portfolio payoff of a global (simple) portfolio with investment horizon n and initial endowment V. These portfolio payoffs can equivalently be characterized as the elements of the set

$$\mathcal{E}_j = \{f : \mathcal{S} \to \mathbb{R} : \ f_{|(S,j)} \in \mathcal{E}_{(S,j)} \text{ for every } S \in \mathcal{S}\}.$$

For $f = \Pi^{V,H}_{j,n} \in \mathcal{E}_j$, we define $I_j f(S) \equiv I_{(S,j)}(f_{|(S,j)}) = V(S)$. Writing $\mathcal{E}^+_j$ for the cone of non-negative functions in $\mathcal{E}_j$, we, then, observe that for $f \in P$ and $S \in \mathcal{S}$,

$$\begin{aligned}\overline{I}_j f(S) \equiv& \overline{I}_{(S,j)}(f_{|(S,j)}) \\ =& \inf\left\{\sum_{m=1}^{\infty} V^m(S) : \ \Pi^{V^m,H^m}_{j,n_m} \in \mathcal{E}^+_j, \ f(\tilde{S}) \leq \sum_{m=1}^{\infty} \Pi^{V^m,H^m}_{j,n_m}(\tilde{S}) \ \ \forall \tilde{S} \in \mathcal{S}_{(S,j)}\right\}.\end{aligned}$$

Hence, we may think of $\bar{I}_j$ as an operator mapping P to the space of j-nonanticipative functions. In the same way, we may interpret $\overline{\sigma}_j f(S) \equiv \overline{\sigma}_{(S,j)}(f_{|(S,j)})$, which is defined for $f \in Q$. Note that, in the case $j = 0$, the operators I_0, $\overline{I}_0$, and $\overline{\sigma}_0$ do not depend on S and we will often abbreviate them by omitting the subscript 0. We remark that $\overline{\sigma}_j$ acts on all functions $f \in Q$ and not just on measurable ones. This is in line with the theory of outer integrals with respect to probability measures, see, e.g., [20].

2.3. **Replicable claims and conditionally integrable functions.** In view of the discussion in the previous subsection, we may think of a generalized portfolio as an element of the vector space spanned by global portfolios and countable superpositions of global portfolios with non-negative portfolio payoffs. More precisely, let us define

$$\mathcal{M}_j \equiv \left\{\sum_{m=1}^{\infty} f_m, \ f_m \in \mathcal{E}^+_j \ (j \geq 1), \ \left[\sum_{m=1}^{\infty} I_j f_m < \infty \ a.e.\right]\right\}.$$

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Elements in $\mathcal{M}_j$ represent the payoff of a superposition of global portfolios with nonnegative portfolio payoffs, when the investor enters the market at time j. The condition $\sum_{m=1}^{\infty} I_j f_m < \infty$ *a.e.* means that the total initial endowment required for this trading strategy is finite outside a global null set. The *payoff of a generalized portfolio* starting at time j is a function $g \in Q$ of the form $g = h + u - v$ for $h \in \mathcal{E}_j$ and $u, v \in \mathcal{M}_j$. We say that $f \in Q$ can be *replicated by a generalized portfolio starting at time* j, if there are $h \in \mathcal{E}_j$, $u, v \in \mathcal{M}_j$ and $N_1, N_2 \subset \mathcal{S}$ such that

$$f(S) = h(S) + u(S) - v(S), \quad S \notin (N_1 \cup N_2)$$

where N_1 is a global null set and N_2 is a null set conditional on $(\tilde{S}, j)$ for every $\tilde{S} \notin N_1$. The roles of the global null set and the conditional null set are as follows: Elements $S^1 \in N^1$ may be thought to correspond to exceptional scenarios (S^1, j), when entering the market at time j, in which the investor does not need to set up a replicating portfolio (e.g., because the candidate for a replicating portfolio requires infinite initial endowment). If $\tilde{S} \notin N_1$, then the investor starts the replication in the conditional model $\mathcal{S}_{(\tilde{S},j)}$, but it must hold only outside the null set $N_2 \cap \mathcal{S}_{(\tilde{S},j)}$ of this conditional model. In the present section, we write $\mathcal{R}_j$ for the space of all the functions that can be replicated by a generalized portfolio starting at time j.

We next introduce the set of *conditionally integrable functions* with respect to the superhedging outer integral $\overline{\sigma}_j$ as

$$\mathcal{L}_{\overline{\sigma}_j} = \{f \in Q : \ \overline{\sigma}_j f + \overline{\sigma}_j(-f) = 0 \ a.e.\}.$$

Defining the *conditional inner integral* via $\underline{\sigma}_j f = -\overline{\sigma}_j(-f)$, the integrability condition above can be re-written as $\overline{\sigma}_j f - \underline{\sigma}_j f = 0$ a.e.

It turns out that the restriction of $\overline{\sigma}_j$ to $\mathcal{L}_{\overline{\sigma}_j}$ is linear in the sense that

$$\overline{\sigma}_j(\alpha f_1 + f_2) = \alpha \overline{\sigma}_j(\alpha f_1) + \overline{\sigma}_j(f_2) \ a.e.$$

for every $\alpha \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{L}_{\overline{\sigma}_j}$ (see Proposition 4.1).

Notice that the inner integral $\underline{\sigma}_j f \equiv -\overline{\sigma}_j(-f)$ corresponds to the maximal initial endowment for subhedging f with a generalized portfolio. Hence, integrability means that buyer and seller agree on the same price for trading f. In parallel to the classical model-based theory (e.g., [15]), one expects that this is the case, if and only if f can be perfectly replicated. Example 1 below illustrates, however, that $\mathcal{L}_{\overline{\sigma}_j}$ can be larger than $\mathcal{R}_j$ in our model-free setting, even if (L)-a.e. holds. However, the classical relation can be restored, if we assume that König's condition is satisfied in the following sense:

Definition 2.13 ((K_j)-a.e.)**.** Given a fixed $j \geq 0$, we say that condition (K_j)*-a.e.* is in force, if $(K_{(S,0)})$ holds and if there is a global null set $\mathcal{K}$ such that $(K_{(S,j)})$ holds for every $S \notin \mathcal{K}$. In other words,

$$I_j(f)(S) + \bar{I}_j(f^-)(S) = \bar{I}_j(f^+)(S) \ \forall f \in \mathcal{E}_j \text{ and all } S \in \mathcal{S} \setminus \mathcal{K}.$$

Similarly, we say that (K)-a.e. holds if $(K_{(S,0)})$ holds and there is a global null set $\mathcal{K}$ such that $(K_{(S,k)})$ holds for all $k \geq 0$ and for every $S \notin \mathcal{K}$.

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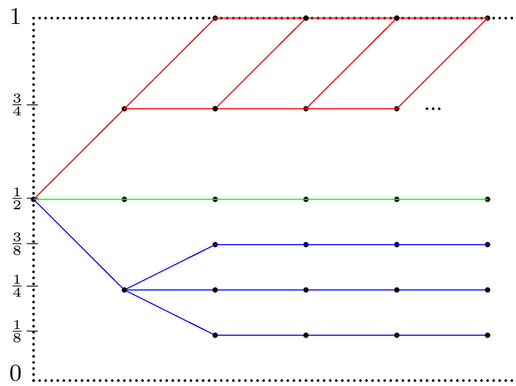

FIGURE 1. Trajectory set for Example 1.

The following theorem is a special case of Theorem 4.15 below and can be considered as a conditional version of König's characterization of integrable functions in [16].

Theorem 2.14. *Suppose (K)-a.e. holds. Then, $\mathcal{L}_{\overline{\sigma}_j} = \mathcal{R}_j$ for every $j \geq 0$.*

Remark 2.15. Note that, in contrast to the theory of model-free finance in continuous time (e.g., the discussion in Remark 2.6 of [3]), we do not impose any topological assumptions on the trajectory set $\mathcal{S}$, but instead study the role of continuity conditions on the (conditional) hedging price operators $I_{(S,j)}$. While Leinert's continuity condition is the minimal requirement for setting up a reasonable framework for trading with the class of generalized portfolios introduced above, Theorem 2.14 can be seen as an indication that König's stronger continuity condition is a cornerstone to recover analogues of classical results in the trajectorial framework in infinite discrete time.

2.4. **Examples.** In this subsection, we discuss Theorem 2.14 and the trajectoral setting through some examples. The first one is a counterexample, which shows that the characterization of replicable functions in Theorem 2.14 may fail if König's condition is replaced by the weaker continuity condition of Leinert.

Example 1. We assume that there are no portfolio restrictions, i.e., $\mathcal{H}_{(S,j)}$ is the space of all non-anticipative sequences. The trajectory set $\mathcal{S} = \mathcal{S}^0 \cup \mathcal{S}^+ \cup \mathcal{S}^-$,

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illustrated in Figure 1, is partitioned into the three subsets:

$$\mathcal{S}^{+} = \{S^{+,n} : n \in \mathbb{N}\}, \quad S_i^{+,n} = \begin{cases} 1/2, & i = 0, \\ 3/4, & i = 1, \dots, n, \\ 1, & i \geq n+1. \end{cases}$$

$$\mathcal{S}^{0} = \{S^{0}\}, \quad S_i^{0} = 1/2,\, i \geq 0.$$

$$\mathcal{S}^{-} = \{S^{-,n} : n = -1, 0, 1\}, \quad S_i^{+,n} = \begin{cases} 1/2, & i = 0, \\ 1/4, & i = 1, \\ 1/4 + n/8, & i \geq 2. \end{cases}$$

We will show the following items:

(1) $\mathcal{S}^{+}$ is the largest global null set, i.e., $\|\mathbf{1}_{\mathcal{S}^{+}}\| = 0$, but $\|\mathbf{1}_{\{S\}}\| > 0$ for every $S \notin \mathcal{S}^{+}$.
(2) $\overline{\sigma} f = f(S^0)$ for every $f \in Q$, and, hence $\mathcal{L}_{\overline{\sigma}} = Q$ (i.e., all functions are integrable).
(3) There are functions $f \in Q$, which do not belong to $\mathcal{R}_0$, i.e. f cannot be replicated by a generalized portfolio.
(4) Leinert's condition $(L_{(S,j)})$ holds for every $S \notin \mathcal{S}^{+}$ and $j \geq 0$, and, thus, in view of item (1), (L)-a.e. holds.

When verifying these items, we will make use of the following notation. For $f \in \mathcal{E}_0$, f has the form $f(S) = V + \sum_{i=0}^{n-1} H_i(S)\Delta_i S$ for some portfolio H, initial endowment V and investment horizon n. For $k \geq 0$, we then write $f_k(S) = V + \sum_{i=0}^{(n\wedge k)-1} H_i(S)\Delta_i S$. Note that, for every $f \in \mathcal{E}_0^{+}$ and $k \geq 0$, $f_k \in \mathcal{E}_0^{+}$ by Lemma 1 in [13], because all nodes are zero neutral.

(1) Define $f^{+,n} = \mathbf{1}_{\{S^{+,n}\}}$ for $n \geq 1$. Since, $f^{+,n}(S) = 0 + 4 \cdot \mathbf{1}_{(S^{+,n},n)}(S)\Delta_n S$ for every $S \in \mathcal{S}$, we conclude, that $f^{+,n} \in \mathcal{E}_0^{+}$ and $\|\mathbf{1}_{\mathcal{S}^{+}}\| \leq \sum_{m\geq 1} \|\mathbf{1}_{\{S^{+,n}\}}\| = 0$ by the σ-subaddditivity of the norm operator. To show that the singletons $\{S\}$ for $S \notin \mathcal{S}^{+}$ are not null sets, we provide the argument for $S = S^{-,1}$, noting that the other cases can be treated analogously. Suppose $(f_m)_{m\geq 1}$ is a sequence in $\mathcal{E}_0^{+}$ such that

$$\sum_{m\geq 1} f_m \geq \mathbf{1}_{\{S^{-,1}\}} \text{ on } \mathcal{S} \text{ and } V \equiv \sum_{m\geq 1} I(f_m) < \infty.$$

Then, in particular,

$$(I): \sum_{m\geq 1} f_{m,2}(S^{-,1}) \geq 1, \quad (II): \sum_{m\geq 1} f_{m,2}(S^{-,-1}) \geq 0, \quad (III): \sum_{m\geq 1} f_{m,1}(S^{+,1}) \geq 0.$$

By the Aggregation Lemma (Lemma C.5), the superposition of the portfolio positions can be aggregated into a single portfolio position at up-down nodes. Hence, there are real constants a, b (the aggregated portfolio position at the initial node $(S^{-,1}, 0)$ and in the down-branch at time 1 $(S^{-,1}, 1)$ such that

$$(I): V - a/4 + b/8 \geq 1, \quad (II): V - a/4 - b/8 \geq 0, \quad (II): V + a/4 \geq 0.$$

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Considering the inequality $(I)+(II)+2(III)$ leads to $V \geq 1/4$. Hence, $\|\mathbf{1}_{\{S^{-,1}\}}\| \geq 1/4$.

(2) Fix some arbitrary $f \in Q$ and denote $f^* := 4|f(S^0)| + 4\max_{n=-1,0,1}|f(S^{-,n})|$. Define $f_0 \in \mathcal{E}_0$ via

$$f_0(S) \equiv f(S^0) - f^*(S_1 - S_0) = \begin{cases} f(S^0), & S = S^0, \\ f(S^0) - f^*/4 \geq -f^*/2, & S \in \mathcal{S}^+, \\ f(S^0) + f^*/4 \geq f(S), & S \in \mathcal{S}^-. \end{cases}$$

Let $f_m = (f^*/2) \cdot f^{+,m}$ for $m \geq 1$, with $f^{+,m} \in \mathcal{E}_0^+$ defined in item (1). Then, $\sum_{m\geq 0} f_m \geq f$ on $\mathcal{S}$ and $\sum_{m\geq 0} I(f_m) = f(S^0)$. Hence, $\overline{\sigma} f \leq f(S^0)$. As the trajectory S^0 is constant, it is clear that superhedging f (with a generalized portfolio) requires at least the initial endowment of $f(S^0)$, leading to $\overline{\sigma} f \geq f(S^0)$. In particular, $\overline{\sigma} f + \overline{\sigma}(-f) = f(S^0) - f(S^0) = 0$, i.e. $f \in \mathcal{L}_{\overline{\sigma}}$.

(3) By item (1), $f \in \mathcal{R}_0$, if and only if there are $h \in \mathcal{E}_0$ and $u, v \in \mathcal{M}_0$ such that $f(S) = (h + u - v)(S)$ for every $S \notin \mathcal{S}^+$. Applying the same aggregation argument as in item (1) to the portfolio positions of u and v at the up-down-nodes $(S^{-,1}, 0)$ and $(S^{-,1}, 1)$ and taking into account that the trajectories become constant, we observe: $f \in \mathcal{R}_0$ if and only if there are real constants V, a, b such that

$$f(S) = V + a(S_1 - S_0) + b(S_2 - S_0), \quad S \notin \mathcal{S}^+.$$

Hence, restricting the functions in $\mathcal{R}_0$ to the complement of $\mathcal{S}^+$ yields a three-dimensional vector space, while restricting the functions in Q to the complement of $\mathcal{S}^+$ leads to a four-dimensional vector space. In particular, $\mathcal{R}_0 \neq Q = \mathcal{L}_{\overline{\sigma}}$, i.e., the assertion of Theorem 2.14 does not hold in this example.

(4) Consider the space $\mathcal{S}_{(S,j)}$ for some $j \geq 0$ and $S \notin \mathcal{S}^+$. Then, the trajectory $S^j = (S_0, \ldots, S_j, S_j, S_j, \ldots)$, which is constant after time j, belongs to $\mathcal{S}_{(S,j)}$. The same argument as for the lower bound in item (2) implies $\overline{\sigma}_{(S,j)} 0 \geq 0$. Hence, $(L_{(S,j)})$ holds by Proposition 3.4.

In view of Theorem 2.14, we observe that (K)-a.e. is not satisfied. We can make this failure of König's condition more explicit by considering the function $f = \mathbf{1}_{\mathcal{S}^-} - \mathbf{1}_{\mathcal{S}^+}$, which belongs to $\mathcal{E}_0$ and satisfies $If = 0$, because $f(S) = 0 - 4(S_1 - S_0)$ for every $S \in \mathcal{S}$. Then, $f^- = \mathbf{1}_{\mathcal{S}^+}$ is a null function by item (1) and $\overline{I} f^+ \geq \overline{I}\mathbf{1}_{\{S^{-,1}\}} > 0$ by the same item. Hence,

$$I(f) + \overline{I} f^- = 0 < \overline{I} f^+,$$

which shows that $(K_{(S,0)})$ fails.

Remark 2.16. It is easy to check that the point mass $\mathbb{Q}$ on S^0 is the unique martingale measure in the previous example. In particular, the superhedging outer integral $\overline{\sigma}$ coincides with the expectation under the unique martingale measure $\mathbb{Q}$, leading to a superhedging duality in the context of this simple example. The subtle point is that the minimal superhedge $(f_m)_{m\geq 0}$ for f constructed in item (2) of the previous example may fail to be a perfect hedge, because it typically will over-replicate on the set $\mathcal{S}^-$, which is not a null set with respect to the norm

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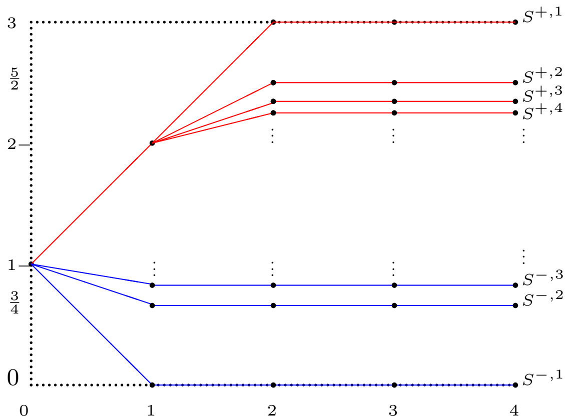

FIGURE 2. No-martingale measure example.

operator $\|\cdot\|$ on the trajectorial model. Hence, perfect hedging is not possible outside a global null set despite of the uniqueness of the martingale measure. This failure results from the inconsistency between the null sets of the trajectorial model defined via the Vovk-type approach and the null sets of the martingale measure. This problem can (at least partially) be mended by imposing König's condition which enforces the consistency between the null sets of the norm operator and the null sets the superhedging outer integral, see Corollary 4.14 below (which is based on Bemerkung 1.8. in [16]).

The second example illustrates the role of $\overline{\sigma}$ in an extreme situation of a trajectory set without martingale measure.

Example 2. Consider the following trajectory set taken from Example 2 in [6], which is sketched in Figure 2. Here,

$$\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-, \quad \mathcal{S}^\pm = \{S^{\pm,n} | n \in \mathbb{N}\},$$

where,

$$S_i^{+,n} = \begin{cases} 1, & i = 0, \\ 2, & i = 1 \\ 2 + \frac{1}{n}, & i \geq 2, \end{cases}, \qquad S_i^{-,n} = \begin{cases} 1, & i = 0, \\ 1 - \frac{1}{n^2}, & i \geq 1 \end{cases}.$$

We, again, assume that there are no portfolio restrictions. It is shown in [6] that (L)-a.e. holds in this example, while, obviously, there is no probability measure on the power set of $\mathcal{S}$ that makes the projection process $T_k(S) \equiv S_k$ into a martingale.

We first show that, for every $f \in Q$,

$$\overline{\sigma} f = \limsup_{n\to\infty} f(S^{-,n}). \tag{2.6}$$

We write $f^* = \limsup_{n\to\infty} f(S^{-,n})$. The same aggregation argument as in the previous example combined with the fact that trajectories in $\mathcal{S}^-$ become constant

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after time 1 implies the following: If $\overline{\sigma} f < V \in \mathbb{R}$, then there is a constant $H_0 \in \mathbb{R}$ such that

$$V + H_0(S^{-,n} - 1) \geq f(S^{-,n}), \quad n \geq 1.$$

Passing with n to infinity along a subsequence which approaches the lim sup, we obtain: $V \geq f^*$. Hence, $\overline{\sigma} f \geq f^*$.

For the converse inequality, fix $\epsilon > 0$ and choose $N_0 = N_0(\epsilon)$ such that $f(S^{-,n}) \leq f^* + \epsilon$ for every $n \geq N_0$. Let $C \equiv \max\{|f(S^{-,n})| : \ n \leq N_0\}$ and choose $H_0 > 0$ sufficiently large such that $f^* + \epsilon - H_0(S^{-,N_0} - 1) \geq C$. Then, for every $n \leq N_0$,

$$f^* + \epsilon - H_0(S^{-,n} - 1) \geq f^* + \epsilon - H_0(S^{-,N_0} - 1) \geq C \geq f(S^{-,n}).$$

Moreover, for $n > N_0$,

$$f^* + \epsilon - H_0(S^{-,n} - 1) \geq f^* + \epsilon \geq f(S^{-,n}).$$

Hence, $f_0 \in \mathcal{E}_0$ defined via $f_0(S) = f^* + \epsilon - H_0(S_1 - S_0)$ for $S \in \mathcal{S}$ satisfies $f_0 \geq f$ on $\mathcal{S}^-$. For $m \geq 1$, consider $f^{+,m}(S) = 0 + 1(S_2 - S_1)$, $S \in \mathcal{S}$. Then, $f^{+,m} \in \mathcal{E}^+$, noting that this function vanishes on $\mathcal{S}^-$ and is strictly positive on $\mathcal{S}^+$. In particular $\sum_{m=1}^\infty f^{+,m} = \infty \cdot \mathbf{1}_{\mathcal{S}^+}$. Therefore, $f_0 + \sum_{m=1}^\infty f^{+,m} \geq f$ on $\mathcal{S}$, which implies $\overline{\sigma} f \leq f^* + \epsilon$. Passing with ϵ to zero finishes the proof of (2.6).

We collect some consequences of (2.6):

(1) The space of integrable functions is

$$\mathcal{L}_{\overline{\sigma}} = \{f \in Q : \lim_{n\to\infty} f(S^{-,n}) \text{ exists in } \mathbb{R}\}.$$

In particular, $\mathcal{R}_0 \subsetneq \mathcal{L}_{\overline{\sigma}} \subsetneq Q$. The first inclusion can be seen to be strict as in the previous example, by noting that the restriction of the functions in $\mathcal{R}_0$ to $\mathcal{S}^-$ forms a two-dimensional vector space and that $\|\mathbf{1}_{\{S^{-,n}\}}\| > 0$ for every $n \geq 1$.

(2) If $f \in Q$ is of the form $f(S) = F(S_2)$ for some function $F : \mathbb{R} \to \mathbb{R}$ which is left-continuous at 1, then, as a consequence of item (1), $f \in \mathcal{L}_{\overline{\sigma}}$ and $\overline{\sigma} f = F(1)$.

(3) $\overline{\sigma}$ fails to be countable-subadditive on the set nonnegative integrable functions, as, e.g.,

$$\overline{\sigma}\Big(\sum_{m=1}^\infty \mathbf{1}_{\{S^{-,n}\}}\Big) = \overline{\sigma}(\mathbf{1}_{\mathcal{S}^-}) = 1 > 0 = \sum_{n=1}^\infty \overline{\sigma}(\mathbf{1}_{\{S^{-,n}\}}).$$

Here the identity $\overline{\sigma}(\mathbf{1}_{\mathcal{S}^-}) = 1$ can, e.g., be derived by applying item (2) to the function $F(x) = \mathbf{1}_{(-\infty,1]}(x)$. In particular, there is no subset $\mathcal{P}_0$ of the set of all probability measures on the power set of $\mathcal{S}$ such that $\overline{\sigma}(\cdot) = \sup_{\mathbb{P}\in\mathcal{P}_0} E_{\mathbb{P}}[\cdot]$.

Let us explain, what intuitively happens in this example. The $\bar{I}$-operator detects the up-branch $\mathcal{S}^+$ as a null set, but provides positive mass (actually mass 1) to the down-branch $\mathcal{S}^-$. The topological closure of the down-branch is $\mathrm{cl}(\mathcal{S}^-) = \mathcal{S}^- \cup \{S^0\}$, where $S^0 = (1, 1, \ldots)$ is the constant trajectory. The integrability condition in item (1) just means that f can be continuously extended to the enlarged

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

trajectory set $\mathcal{S}' \equiv \mathcal{S}^+ \cup \mathrm{cl}(\mathcal{S}^-)$ via $f(S^0) \equiv \lim_{n\to\infty} f(S^{-,n})$. Note that the enlarged trajectory set $\mathcal{S}'$ has a unique martingale measure namely the point mass $\mathbb{Q}$ on the new trajectory S^0. Hence, for an integrable function f, $\overline{\sigma} f$ coincides with expectation of the continuous extension of f under the unique martingale measure $\mathbb{Q}$ on $\mathcal{S}'$.

Summarizing, in this example, the non-lattice framework first removes the null sets and then makes the remaining model 'reasonable' by passing to the topological closure.

Remark 2.17. The previous example is in striking contrast to the classical theory, in that the trajectory set $\mathcal{S}$ does not support any martingale measures. We now sketch another extreme case, where $\overline{\sigma}$, in fact, coincides with a classical integral: Suppose that the trajectory set consists of an (infinite-time) binomial tree with up-down nodes only. Then, the elementary functions $\mathcal{E}_0$ form a vector-lattice (under the assumption of no portfolio restrictions) and the superhedging outer integral $\overline{\sigma}$ corresponds to a classical integral with respect to a measure μ as detailed on p. 261 in [17]. In this case, it is not difficult to check, that the measure μ coincides with the unique martingale measure $\mathbb{Q}$ for the model and that the conditional superhedging operator $\overline{\sigma}_j$ is a version of the conditional expectation $E_{\mathbb{Q}}[\cdot|\mathcal{F}_j]$ where $(\mathcal{F}_j)_{j\geq 0}$ is the filtration generated by the coordinate process $T_j(S) = S_j$. Example 7 in Appendix F below explains how to construct integrable functions in the limit (as time goes to infinity) in a trinomial tree, where the $\overline{\sigma}$-operator does not agree with a classical integral.

The next example illustrates the interplay between the global null set and the conditional null set for replication in the presence of portfolio constraints.

Example 3. We now assume that portfolios are deterministic in the following sense: $H = (H_i)_{i\geq j}$ belongs to $\mathcal{H}_{(S,j)}$, if and only if there is a sequence $(h_i)_{i\geq j}$ of real numbers such that $H_i(\hat{S}) = h_i$ for every $\hat{S} \in \mathcal{S}_{(S,j)}$. Hence, the investor fixes all future portfolio positions when entering the market in the scenario (S, j) and does not react to future price movements of the stock. Clearly this constraint satisfies (H.1)–(H.4). We define the trajectory set $\mathcal{S} = \mathcal{S}^0 \cup \mathcal{S}^+ \cup \mathcal{S}^-$ (which in fact is a three-period model) via

$$\mathcal{S}^0 = \{S^{0,n} :\ n = -1, 0, 1\}, \quad S_j^{0,n} = \begin{cases} 10, & j = 0, \\ 5, & j = 1, \\ 5 + n, & j \geq 2, \end{cases}$$

$$\mathcal{S}^+ = \{S^{+,n} :\ n = 0, 1, 2\}, \quad S_j^{+,n} = \begin{cases} 10, & j = 0, 1, \\ 11, & j = 2, \\ 11 + n, & j \geq 3, \end{cases}$$

$$\mathcal{S}^- = \{S^{-,n} :\ n = 0, -1, -2\}, \quad S_j^{-,n} = \begin{cases} 10, & j = 0, 1, \\ 9, & j = 2, \\ 9 + n, & j \geq 3. \end{cases}$$

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
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The initial node $(S, 0)$ is an arbitrage node of type 1, and the arbitrage can be exploited by shortening the stock, e.g., $f = \mathbf{1}_{\mathcal{S}^0} = 0 - 1/5 \cdot (S_1 - S_0) \in \mathcal{E}_0^+$. This shows that $N_1 = \mathcal{S}^0$ is a global null set. In an analogous way, one can exploit the type I arbitrage at the node $(S^{+,0}, 2)$ in the conditional space $\mathcal{S}_{(S^{+,0},2)}$ (by buying the stock) and the type I arbitrage at the node $(S^{-,0}, 2)$ in the conditional space $\mathcal{S}_{(S^{-,0},2)}$ (by shortening the stock). Hence, $N_2 = \{S^{+,n}, S^{-,-n} : n = 1, 2\}$ is a null set conditional on $(S, 2)$ for every $S \notin N_1$. Note, however, that these arbitrages at the nodes $(S^{+,0}, 2)$ and $(S^{-,0}, 2)$ cannot be realized by a trader who enters the market at time 0. This is because, due to the portfolio constraint, such a trader has to apply the same portfolio position at time 2 at both nodes $(S^{+,0}, 2)$ and $(S^{-,0}, 2)$. Hence, N_2 is not a global null set. The definition of the set of replicable functions $\mathcal{R}_2$ at time 2 (based on global null sets and conditional ones) now only requires that perfect replication is possible on the set $\mathcal{S} \setminus (N_1 \cup N_2) = \{S^{+,0}, S^{-,0}\}$. Hence, $Q = \mathcal{R}_2$. Indeed, if $f \in Q$, then $g \in \mathcal{E}_2$ defined by $g = f(S^{+,0})\mathbf{1}_{(S^{+,0},2)} + f(S^{-,0})\mathbf{1}_{(S^{-,0},2)}$ clearly coincides with f on $\mathcal{S} \setminus (N_1 \cup N_2)$. It is also not difficult to check directly, that $Q = \mathcal{L}_{\overline{\sigma}_2}$ in this example.

2.5. **Relation to the literature.** In this section, we will discuss the relation of our setting (the *non-lattice approach* based on [17, 16]) to existing model-independent settings in the literature, distinguishing between the *finite discrete-time setting* ([1, 7, 8, 9, 10]), *the continuous-time approach* ([22, 2, 3, 4]), and the *game-theoretic infinite discrete-time approach* ([21, 19]).

2.5.1. *Null Sets as Unlikely Financial Events.* There is a rich literature on model independent versions of the first fundamental theorem of asset pricing and superhedging duality results in finite discrete-time, e.g., [1, 7, 8, 9, 10]. Some of these references [8, 9, 10] rely on aggregation of local arbitrage opportunities at the nodes, in order to prescribe events that are negligible. This approach leads to a notion of null sets which, a priori, does not depend on superhedging arguments with nonanticipative portfolios. Then, e.g., in superhedging duality results, the super-replication only needs to hold outside a null set. In contrast, the non-lattice approach (which we adopt) and similarly the continuous-time framework (initiated in [22]) require the superhedge to work pointwise on every trajectory and, a posteriori, define a set N to be a null set if the indicator function of N can be superhedged with zero (or arbitrarily small positive) initial capital. It is crucial that we rely upon countable superpositions of positive simple portfolio payoffs. Otherwise, zero-neutral arbitrage nodes of type II, as in Example 2, or arbitrages that appear in the long run, as in Example 1, cannot be detected by trading with simple portfolios only. Compare with the construction of the sequences $\{f^{+,m}\}_{m\geq 1}$ in these examples.

Let us have a closer look at the backward recursion, which is applied in Eq. (14) of [10] to identify null sets. The following discussion adapts their construction to our setting (but we stress that [10] covers a general multi-asset framework). As [10] works in finite discrete time, we assume that the model has finite maturity, in the sense that all trajectories stay constant after some fixed time $N \in \mathbb{N}$ (independent of the trajectory). Starting from the last time period, between time $N - 1$ and

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

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N, they first remove all trajectories S which pass through an arbitrage node at time $N-1$, unless $S_N = S_{N-1}$. The same procedure is then applied to the reduced trajectory set at time $N-2$ and repeated by backward induction until one reaches time 0. It turns out that the trajectories S in the remaining submodel (after stripping away inductively all local arbitrage opportunities) are exactly those for which the singleton $\{S\}$ has strictly positive probability under at least one martingale measure $\mathbb{Q}$. Hence, either the remaining submodel is empty or it only has up-down-nodes and flat nodes. The theory developed in [10] supposes that this remaining submodel is non-empty (via assuming existence of a martingale measure for the original model) and then restricts all superhedging arguments to this remaining submodel. We note in passing that it is not straightforward to generalize this backward recursion procedure to a genuine infinite time horizon.

Revisiting Example 2 (which is in fact a two-period model, since all trajectories become constant after time 2), we can illustrate the difference between the approach in [10] and ours. Following the backward induction algorithm of [10], the (red) upper branch $\mathcal{S}^+$ is removed from the model at time 1, because these trajectories pass through an arbitrage node of type II at time 1. The reduced model then consists of the blue lower branch $\mathcal{S}^-$ only. However, in this reduced model the initial node (at time 0) is an arbitrage node of type II, and, is, thus, removed in the next step of the backward induction. The final remaining model is empty, corresponding to the fact that the original model has no martingale measure. In contrast, the non-lattice approach applies pure superhedging arguments to detect null sets and, thus, finds that the lower branch $\mathcal{S}^-$ is not a null set. It then, moves to the topological closure of $\mathcal{S}^-$ to come up with an 'arbitrage-free' model, which is applied for pricing purposes (as detailed in Example 2).

2.5.2. *Comparison to Vovk's Approach.* The continuous-time pathwise framework initiated by Vovk [22] is based on a superhedging operator which is nowadays called Vovk's outer measure and which determines the null sets of the model, see, e.g., [4] for a discussion of Vovk's outer measure and variations thereof. This approach is very close to ours in that the superhedging operator is the primary object, superhedges must hold on every path, and null sets are derived by pathwise superhedging arguments. Model-independent superhedging dualities in continuous time (e.g., [4, 2, 3]) either require that the trajectory set consists of all continuous paths [4] or the trajectory set (sometimes called prediction set in the continuous time framework) is assumed to satisfy some topological properties (as in [3]) including some compactness requirements. A typical assumption is that any stopped path belongs to the prediction set, see, e.g. [2, 3] or [23]. Translated into our setting, the analogous property is the following one: For any $S \in \mathcal{S}$ and $j \geq 0$, the trajectory $(S_0, \ldots, S_{j-1}, S_j, S_j, \ldots)$ belongs to $\mathcal{S}$. Example 4 (found at the end of Section 3) shows that this property implies that König's condition $(K_{(S,j)})$ holds at every node (S, j). Moreover, it rules out arbitrage nodes of type II and, more generally, the possibility to recoup losses at later times as illustrated in Example 1. Compared to the continuous-time framework, our infinite discrete-time setting is

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

conceptually simpler and it is promising in order to gain new insights by studying relevant questions through the continuity conditions of non-lattice integration.

We also comment on the relation to the game-theoretic approach in infinite discrete-time as outlined in Chapter 7 of Shafer and Vovk [19]. Their setting requires to choose a local outer expectation operator at each node, e.g., in the application to mathematical finance, the minimal superhedging cost for the one-period submodel starting at an up-down node. Then, it proceeds to construct a global operator $\overline{\mathbb{E}}$, which is closely related to our $\overline{\sigma}$, from the local outer expectations. They impose, however, conditions which ensure that the local outer expectation and the global one coincide on functions which depend on the next time step only, see Lemma 7.6 in [19]. Hence, situations as in Example 1 cannot be accommodated in their framework: Superhedging the indicator function of the event $\mathcal{S}^+ = \{S \in \mathcal{S} : \ S_1 > S_0\}$ in the one-period trinomial model starting at time 0 requires an initial capital of $1/8 > 0$. However, the one-period submodel cannot see that all trajectories in $\mathcal{S}^+$ will eventually increase. This arbitrage in the long run results in the fact that its indicator function can be superhedged with zero initial capital in the dynamic infinite-time model – leading to an inconsistency between the local one-period superhedging price and the global one. While our non-lattice approach allows for trajectories in which losses at an earlier time can be recuperated at later times (either by realizing an arbitrage at an type II arbitrage node as in Example 2 or by exploiting an arbitrage in the long run as in Example 1), such a situation is not possible under the assumptions in the game-theoretic approach in [19]. We also refer to [6] for a more detailed comparison between the game-theoretic approach and the non-lattice approach.

3. On Leinert's condition and König's condition

In this section, we provide criteria on how to check the continuity conditions of Leinert and König. Before doing so, we collect some elementary properties of the superhedging operators $\overline{I}$ and $\overline{\sigma}$.

3.1. Some properties of the superhedging operators.

The proofs of the following elementary propositions, which are adapted from [16, 17] to our setting will be provided in Appendix B. All appearing equalities and inequalities are valid for all points in the spaces where the functions are defined unless qualified by an explicit a.e.

Proposition 3.1 (see proof in Appendix B)**.** *Let (S, j) be a fixed node:*

(1) *Consider $f, g \in P$. If $f(\tilde{S}) \leq g(\tilde{S})$ $\forall \tilde{S} \in \mathcal{S}_{(S,j)}$, then $\overline{I}_j f(S) \leq \overline{I}_j g(S)$.*
(2) *If $f \in P$ and $\alpha \in \mathbb{R}^+$ then $\overline{I}_j(\alpha f)(S) = \alpha \overline{I}_j f(S)$.*
(3) *If $g, g_k \in P$, for $k \geq 1$ satisfying $g(\tilde{S}) \leq \sum_{k\geq 1} g_k(\tilde{S})$ $\forall \tilde{S} \in \mathcal{S}_{(S,j)}$, then*
$$\overline{I}_j g(S) \leq \sum_{k\geq 1} \overline{I}_j g_k(S).$$
(4) *$\overline{I}_j f(S) \leq I_j f(S)$, for $f \in \mathcal{E}_j^+$.*

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

The next proposition is concerned with the norm operator and with null sets.

Proposition 3.2 (see proof in Appendix B). *Consider $f, g : \mathcal{S} \to [-\infty, \infty]$ and a fixed node (S, j), then:*

(1) $\|g\|_j(S) = 0$ *iff* $g = 0$ *a.e. on* $\mathcal{S}_{(S,j)}$.
(2) *If* $\|g\|_j(S) < \infty$ *then* $|g| < \infty$ *a.e. on* $\mathcal{S}_{(S,j)}$.
(3) *If* $|f| \le |g|$ *a.e. on* $\mathcal{S}_{(S,j)}$ *then* $\|f\|_j(S) \le \|g\|_j(S)$. *Therefore, if* $|f| = |g|$ *a.e. on* $\mathcal{S}_{(S,j)}$ *then* $\|f\|_j(S) = \|g\|_j(S)$.
(4) *The countable union of conditional null sets is a conditionally null set.*
(5) *For* $f \in P$ *and* $0 \le j \le k$: $0 \le \overline{I}_j(\overline{I}_k f) \le \overline{I}_j f$. *Therefore, if* $g \in Q$ *is conditionally null at* $\mathcal{S}_{(S,j)}$ *then* $\overline{I}_k(|g|) = 0$ *a.e. on* $\mathcal{S}_{(S,j)}$.

We now turn to the superhedging outer integral $\overline{\sigma}$.

Proposition 3.3 (see proof in Appendix B). *Unless indicated otherwise, consider $f, g \in Q$ and let (S, j) denote a generic node.*

(1) $\overline{\sigma}_j(f + g)(S) \le \overline{\sigma}_j f(S) + \overline{\sigma}_j g(S)$.
(2) $\overline{\sigma}_j f(S) \le \overline{I}_j f(S)$ *if* $f \in P$.
(3) $\overline{\sigma}_j f(S) \le I_j f(S)$ *if* $f \in \mathcal{E}_j$.
(4) *If* $f \le g$ *a.e. on* $\mathcal{S}_{(S,j)}$, *then* $\overline{\sigma}_j f(S) \le \overline{\sigma}_j g(S)$. *Therefore If* $f = g$ *a.e. on* $\mathcal{S}_{(S,j)}$, *then* $\overline{\sigma}_j f(S) = \overline{\sigma}_j g(S)$.
(5) $\overline{\sigma}_j(gf)(S) = g(S)\ \overline{\sigma}_j(f)(S)$ *if* $g(S) = g(S_0, \ldots, S_j) > 0$.
(6) $\overline{\sigma}_j f(S) \le \overline{\sigma}_j(|f - g|)(S) + \overline{\sigma}_j g(S)$.
(7) *Either: a)* $\overline{\sigma}_j 0 = 0$ *or b)* $A \equiv \{\overline{\sigma}_j 0 < 0\} \ne \emptyset$ *and* $\overline{\sigma}_j f(S) = \pm\infty$ *for all* $S \in A$ *and for all* $f \in Q$.
(8) *If* $\overline{\sigma}_j f(S) < \infty$ *then* $f < \infty$ *conditionally a.e. at* (S, j).
(9) *If* $\overline{\sigma}_j 0(S) = 0$, $\overline{\sigma}_j f(S) < \infty$ *and* $\underline{\sigma}_j f(S) > -\infty$, *then:*
$\overline{\sigma}_j f(S) > -\infty$ *and* $\underline{\sigma}_j f(S) < \infty$ *and* $|f| < \infty$ *a.e. on* $\mathcal{S}_{(S,j)}$.

The next proposition provides several equivalent formulations of Leinert's condition.

Proposition 3.4 (see proof in Appendix B). *For a given node (S, j), $j \ge 0$, consider the following items:*

(1) $\overline{\sigma}_j 0(S) = 0$.
(2) *Property* $(L_{(S,j)})$.
(3) *Leinert condition (from Definition 2.7).*
(4) $\underline{\sigma}_j f(S) = I_j f(S) = \overline{\sigma}_j f(S)$ *for* $f \in \mathcal{E}_{(S,j)}$.
(5) $|I_j f(S)| \le \|f(S)\|_j$ *for* $f \in \mathcal{E}_{(S,j)}$.
(6) $I_j f(S) = \overline{I}_j f(S)$ *for* $f \in \mathcal{E}^+_{(S,j)}$.

Then, items (1)–(4) above are equivalent. Moreover, (4) ⇒ (5) ⇒ (6).

If Leinert's condition is in force, the outer integral has the following additional properties.

Corollary 3.5 (see proof in Appendix B). *Given a node (S, j), $j \ge 0$, assume $\overline{\sigma}_j 0(S) = 0$. Then, for $f \in Q$,*

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

(1) $0 \leq \overline{\sigma}_j f(S) + \overline{\sigma}_j(-f)(S)$ *and* $\underline{\sigma}_j f(S) \leq \overline{\sigma}_j f(S)$.
(2) $|\overline{\sigma}_j f(S)| \leq \overline{\sigma}_j |f|(S)$.
(3) *If* $f = 0$ *a.e. on* $\mathcal{S}_{(S,j)}$ *then* $\underline{\sigma}_j f(S) = \underline{\sigma}_j |f|(S) = 0 = \overline{\sigma}_j |f|(S) = \overline{\sigma}_j f(S)$, *which in turn implies* $\underline{\sigma}_j f^+(S) = \underline{\sigma}_j f^-(S) = 0 = \overline{\sigma}_j f^-(S) = \overline{\sigma}_j f^+(S)$.
(4) *If* $0 \leq g$ *a.e. on* $\mathcal{S}_{(S,j)}$, *then* $0 \leq \underline{\sigma}_j g(S)$.

3.2. **Leinert's condition.** In view of item (7) of Proposition 3.3, we observe that condition (L)-a.e., introduced in Definition 2.10 is crucial to have an appropriate conditional outer integral defined by $\overline{\sigma}_j$ at all time points $j \in \mathbb{N}_0$. In order to obtain conditions for its validity, we discuss some properties concerning the behavior of trajectories as time approaches infinity.

Given a sequence $\{S^n\}_{n \geq 0} \subset \mathcal{S}_{(S,j)}$ satisfying

$$S_i^n = S_i^{n+1} \quad 0 \leq i \leq n, \ \forall \ n, \tag{3.1}$$

define

$$\overline{S} = \{\overline{S}_i\}_{i \geq 0} \text{ by } \overline{S}_i \equiv S_i^i. \text{ We will use the notation } \overline{S} = \lim_{n \to \infty} S^n.$$

Notice that $\overline{S}_i = S_i, \ 0 \leq i \leq j$ because $S^i \in \mathcal{S}_{(S,j)}$. Moreover

$$\overline{S}_i = S_i^n, \ 0 \leq i \leq n, \ \forall \ n \geq 0.$$

Let $\overline{\mathcal{S}_{(S,j)}}$ be the set of such $\overline{S}$, clearly $\mathcal{S}_{(S,j)} \subset \overline{\mathcal{S}_{(S,j)}}$, because for $\tilde{S} \in \mathcal{S}_{(S,j)}$ and we can take $\tilde{S}^n = \tilde{S}$ for all $n \geq 0$. We say that $\mathcal{S}_{(S,j)}$ is (trajectorially) *complete* if $\mathcal{S}_{(S,j)} = \overline{\mathcal{S}_{(S,j)}}$.

Remark 3.6. (1) The process of going from $\mathcal{S}_{(S,j)}$ to $\overline{\mathcal{S}_{(S,j)}}$ does not alter the properties of nodes (being 0-neutral, no-arbitrage, etc), see [13] for some details. Moreover, portfolios $\Pi_{j,n}^{V,H}$ can be extended, in an obvious way, from acting on $\mathcal{S}_{(S,j)}$ to act on $\overline{\mathcal{S}_{(S,j)}}$; we will freely make use of this fact below without further comments. Note, however, that completing the model by passing to $\overline{\mathcal{S}_{(S,j)}}$ is not as harmless, as it might appear at first glance. In the context of Example 1, $\overline{\mathcal{S}} = \mathcal{S} \cup \{(1/2, 3/4, 3/4, ...)\}$. Then, the up-branch $\mathcal{S}^+ \cup \{(1/2, 3/4, 3/4, ...)\}$ in the completed model is not a null set anymore. So the completion process may change null sets into non-null sets.
(2) The game-theoretic infinite discrete time approach of [19] assumes that the analogue of trajectorial completeness is satisfied in their context by the very definition of the sample space (which replaces our trajectory space), see p. 147 in [19].

We next introduce a weaker condition than trajectorial completeness, concerning the limit behavior of trajectories.

Definition 3.7 (Reversed Fatou Property (RFP))**.** We will say that $\mathcal{S}_{(S,j)}$ satisfies the Reversed Fatou Property (RFP, for short) if: for any $f_0 \in \mathcal{E}_{(S,j)}, f_m = \Pi_{j,n_m}^{V^m,H^m} \in \mathcal{E}^+_{(S,j)}$, $m \geq 1$, satisfying $\sum_{m \geq 1} V^m(S) < \infty$, and $\overline{S} \in \overline{\mathcal{S}_{(S,j)}}$ there exists

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
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at least one $\{S^n\}_{n\geq 0} \subseteq \mathcal{S}_{(S,j)}$ such that $\overline{S} = \lim_{n\to\infty} S^n$ and

$$\sum_{m\geq 0} \limsup_{n\to\infty} f_m(S^n) \geq \limsup_{n\to\infty} \sum_{m\geq 0} f_m(S^n). \tag{3.2}$$

Remark 3.8. Given a sequence $\{S^n\}_{n\geq 0}$ satisfying (3.1): $\sum_{m\geq 0} \limsup_{n\to\infty} f_m(S^n) = \sum_{m\geq 0} \limsup_{n\to\infty} f_m(S_0^n, \ldots, S_{n_m}^n) = \sum_{m\geq 0} f_m(S_0^{n_m}, \ldots, S_{n_m}^{n_m}) = \sum_{m\geq 0} f_m(\overline{S})$. Thus, Fatou's lemma in conjunction with (3.2) implies $\lim_{n\to\infty} \sum_{m\geq 0} f_m(S^n) = \sum_{m\geq 0} f_m(\overline{S})$.

Clearly (3.2) holds if $\overline{S} \in \mathcal{S}_{(S,j)}$ by taking $S^n \equiv \overline{S}$ for all n, hence (3.2) is a condition on elements of $\overline{\mathcal{S}_{(S,j)}} \setminus \mathcal{S}_{(S,j)}$. In particular, if $\mathcal{S}_{(S,j)}$ is (trajectorially) complete it then satisfies the RFP. An example in Appendix F shows that RFP is weaker than (trajectorial) completeness.

Note that RFP is a local (i.e. nodewise) property which depends on the trajectory set and on the sets of admissible portfolios. We may also consider the following global version (i.e. defined only at the initial node $(S, 0)$) which is independent of the portfolio sets.

Definition 3.9. We will say that $\mathcal{S}$ satisfies the Global Reversed Fatou Property (GRFP, for short) if for every sequence $(V^m)_{m\geq 1}$ of non-negative reals, $(n_m)_{m\geq 1}$ of non-negative integers, and $(H^m)_{m\geq 1}$ of non-anticipating sequences $H_i^m : \mathcal{S} \to \mathbb{R}$ such that $\Pi_{0,n_m}^{V^m,H^m} \geq 0$ the following holds: If $\sum_{m\geq 1} V^m < \infty$ and $\overline{S} \in \overline{\mathcal{S}}$ then there exists at least one $\{S^n\}_{n\geq 0} \subseteq \mathcal{S}$ such that $\overline{S} = \lim_{n\to\infty} S^n$ and

$$\sum_{m\geq 1} \limsup_{n\to\infty} f_m(S^n) \geq \limsup_{n\to\infty} \sum_{m\geq 1} f_m(S^n).$$

The following theorem and its corollary weaken the completeness assumption of Theorem 7.1 in [6] and extend it to the case of portfolio restrictions.

Theorem 3.10. *Suppose GRFP holds and the following condition on arbitrage nodes of type II is satisfied: If (S, j) is an arbitrage node of type II, then $j \geq 1$, $(S, j-1)$ is an up-down node and for every $\epsilon > 0$ there are $S^{\epsilon,1}, S^{\epsilon,2} \in \mathcal{S}_{(S,j-1)}$ such that*

$$S_j^{\epsilon,1} - S_{j-1} \geq -\epsilon, \quad S_j^{\epsilon,2} - S_{j-1} \leq \epsilon$$

and such that $(S^{\epsilon,1}, j)$, $(S^{\epsilon,2}, j)$ are not type II arbitrage nodes.

Then, $(L_{(S,j)})$ holds at a node (S, j), if and only the node (S, j) is not an arbitrage node of type II.

Proof. If (S, j) is a type II arbitrage node, then $L_{(S,j)}$ fails by Theorem 2.9. Conversely, suppose that (S, j) is not an arbitrage node of type II. We aim at showing that $L_{(S,j)}$ holds. By Lemma C.1, we may assume that there are no portfolio restrictions. In this situation GRFP implies that RFP holds at every node by Lemma C.3, and, in particular, at the fixed node (S, j). Lemma C.2 now allows to move from the original trajectory set to its completion $\overline{\mathcal{S}}$, which, of course, is trajectorially complete. The condition on the arbitrage nodes of type II, which is imposed

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

in Theorem 3.10 for $\mathcal{S}$, is inherited by $\overline{\mathcal{S}}$ thanks to Remark 3.6. Summarizing the foregoing, the problem has been reduced to a complete trajectory set without portfolio constraints, which is covered by Theorem 7.1 in [6]. □

Corollary 3.11 (see proof in Appendix C)**.** (1) *Suppose there are no portfolio restrictions. Then, (L)-a.e. holds under the assumptions of Theorem 3.10.*

(2) *Under the standing assumptions (H.1)–(H.4), assume that $\mathcal{S}$ satisfies GRFP and has no-arbitrage nodes of type II. Then, $(L_{(S,j)})$ holds at every node (S,j).*

The role of the reversed Fatou property for the validity of these results is illustrated in Examples 5–6 in Appendix F.

3.3. **König's condition.** As illustrated by Example 1, the (L)-a.e. formulation of Leinert's condition is not sufficient to guarantee the characterization of replicable claims as integrable functions. Therefore, we now discuss criteria to check that the stronger continuity condition of König holds.

The next result provides a relation between both continuity conditions. It relies on the following additional property of aggregation of portfolios:

(H.5) For any sequence $\{H^m\}_{m\geq 1}$ in $\mathcal{H}_{(S,j)}$ the portfolio H given by
$$H_i(\tilde{S}) := \begin{cases} \sum_{m=1}^{\infty} H_i^m(\tilde{S}), & \text{if the series is convergent in } \mathbb{R} \\ 0, & \text{otherwise} \end{cases}$$
belongs to $\mathcal{H}_{(S,j)}$.

Theorem 3.12. *Suppose Leinert's condition $(L_{(S,j)})$ holds at every node (S,j). Then, König's condition $(K_{(S,j)})$ is also valid at every node (S,j) under each of the following additional assumptions:*

(1) *There are no portfolio restrictions;*
(2) *The portfolios satisfy (H.1)–(H.5) and the trajectory set does not have any arbitrage nodes.*

Sketch of the Proof. A proof in the case (1) is provided in [5], see their Theorem 5.1. It consists of two key steps. In the first step it is shown that all involved portfolios may be assumed to have the same finite maturity. The corresponding manipulations make use of (H.3) and (H.4) only. In the second step, a superposition of countably many portfolios is accumulated. This step requires (H.5) and Lemma C.5. In the absence of arbitrage nodes of type I, the technical ramifications of the proof of Theorem 5.1 in [5] involving (null) sets $\mathcal{N}_n$ and a random time τ become trivial, and so the identical proof works for case (2) as well. □

Combining this theorem with Theorem 3.10 and Corollary 3.11, we obtain the following criteria.

Corollary 3.13 (see proof in Appendix C)**.** (1) *Suppose (H.1)–(H.5), the global reversed Fatou property GRFP, and that the trajectory set does not have any arbitrage nodes. Then, $(K_{(S,j)})$ holds at every node (S,j).*

(2) *Suppose there are no portfolio restrictions, GRFP is satisfied and the trajectory*

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
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set does not have any arbitrage nodes of type II. Then, $(K_{(S,j)})$ *holds at every node* (S, j).

Example 4. Suppose the trajectory set is closed against stopping in the sense that for every trajectory $S \in \mathcal{S}$ and $k \geq 0$, the stopped trajectory $S^{[k]} \equiv (S_{j\wedge k})_{j\geq 0} = (S_1, \ldots, S_{k-1}, S_k, S_k, \ldots)$ also belongs to $\mathcal{S}$. Note that the analogue of this property is often imposed in the continuous-time pathwise approach [2, 3]. Under this assumption, for every node (S, j) and every $f \in \mathcal{E}_{(S,j)}$, $f(S^{[j]}) = I_j f(S)$. Hence, if there are $f_0 \in \mathcal{E}_{(S,j)}$ and $f_m \in \mathcal{E}^+_{(S,j)}$, $m \geq 1$ such that $\sum_{m\geq 0} f_m(\tilde{S}) \geq 0$ for every $\tilde{S} \in \mathcal{S}_{(S,j)}$, then, by choosing $\tilde{S} = S^{[j]}$, we obtain $\sum_{j\geq 0} I_j f(S) = \sum_{j\geq 0} f(S^{[j]}) \geq 0$. Hence, $\overline{\sigma}_j(0)(S) \geq 0$, which, by Proposition 3.4, implies that $(L_{(S,j)})$ holds.

Thus, by Theorem 3.12, $(K_{(S,j)})$ is also valid at every node (S, j), if the model satisfies the portfolio restrictions (H.1)–(H.5) and has no-arbitrage nodes of type I or if the model has no portfolio restrictions at all. For the sake of concreteness, let us assume that all nodes are trinomial with one up-branch, one down-branch, and one branch, which continues constantly. If $\mathcal{S}$ is the set of all those trajectories S passing through these nodes such that S eventually becomes constant (but the time of the last up-move or down-move depends on S), then the model is clearly trajectorially incomplete, but satisfies $(K_{(S,j)})$ at every node (S, j).

4. Conditionally Integrable Functions and Their Characterization

In this section we study the space of conditionally integrable functions and, in particular, prove a generalization of Theorem 2.14. Recall that the space $\mathcal{L}_{\overline{\sigma}_j}$ of conditionally integrable functions describes the payoffs of those financial derivatives, for which the seller and buyer agree on a unique (finite) price by superhedging arguments when the derivative is traded at time j. It is formally defined to be

$$\mathcal{L}_{\overline{\sigma}_j} \equiv \{f : \mathcal{S} \to [-\infty, \infty] : \ \overline{\sigma}_j f - \underline{\sigma}_j f = 0 \ a.e.\}.$$

Most results in this section will fix $j \geq 0$ and assume property (L_j)-a.e. (introduced in Definition 2.10). Property (K_j)-a.e. (cp. Definition 2.13) will only be required in some of the results

Due to the fact that by Corollary 3.5 item (1), $0 \leq \overline{\sigma}_j f(S) + \overline{\sigma}_j(-f)(S) = \overline{\sigma}_j f(S) - \underline{\sigma}_j f(S)$ holds a.e., we have: $f \in \mathcal{L}_{\overline{\sigma}_j}$ if and only if $\overline{\sigma}_j f - \underline{\sigma}_j f \leq 0$ *a.e.* which, in turn, implies $\overline{\sigma}_j f \leq \underline{\sigma}_j f$ a.e. Moreover, the statements are equivalent if and only if $\underline{\sigma}_j f$ and $\overline{\sigma}_j f$ are finite a.e. Therefore, $f \in \mathcal{L}_{\overline{\sigma}_j}$ implies $-\infty < \overline{\sigma}_j f(S) < \infty$ and $-\infty < \underline{\sigma}_j f(S) < \infty$, each set of inequalities holding a.e. Proposition 3.3, item (9) implies that for each such S we have $-\infty < f < \infty$ a.e. on $\mathcal{S}_{(S,j)}$. For the special case of $j = 0$, $\overline{\sigma}_0 f - \underline{\sigma}_0 f \leq 0$ *a.e.* if and only if $\overline{\sigma}_0 f(S_0) - \underline{\sigma}_0 f(S_0) \leq 0$.

$f \in \mathcal{L}_{\overline{\sigma}_j}$ will be called a (conditionally) *integrable function*, for such a function we set the *conditional integral* notation:

$$\int_j f \equiv \overline{\sigma}_j f,$$

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

this defines conditional integrals everywhere on $\mathcal{S}$ and avoids the introduction of classes of equivalence of functions defined a.e. That being said, we emphasize that for $f \in \mathcal{L}_{\overline{\sigma}_j}$, $-\infty < \int_j f(S) = \underline{\sigma}_j f(S) = \overline{\sigma}_j f(S) < \infty$ a.e. We also write:

$$\int_j f \doteq \underline{\sigma}_j f, \text{ where we will use } \doteq \text{ to denote equality a.e.}$$

We will mix the equivalent uses of a.e. and $\doteq$ as we see most convenient for display purposes. The case $j = 0$ is denoted by $\int f \equiv \int_0 f$ which is a constant defined everywhere on $\mathcal{S}$.

The next two results (Proposition 4.1 and Theorem 4.3) are known for the unconditional non-lattice integral, see [16]. Their generalization to the conditional integrals requires to take care of the global null sets that feature in the definition of conditional integrability.

Proposition 4.1 (see proof in Appendix D. Compare with König's Behauptung 2.1 in [16]). *Given $j \geq 0$, assume that property (L_j)-a.e. holds. Then,*

1. $\mathcal{E}_j \subset \mathcal{L}_{\overline{\sigma}_j}$.
2. *If $f \in \mathcal{L}_{\overline{\sigma}_j}$ and $[f = g$ a.e. at $(S, j)]$ holds a.e. then:* $g \in \mathcal{L}_{\overline{\sigma}_j}$ *and* $\int_j f \doteq \int_j g$.
3. *Consider $f \in \mathcal{L}_{\overline{\sigma}_j}$ and $c \in \mathbb{R}$. Then $cf \in \mathcal{L}_{\overline{\sigma}_j}$ and: $\int_j cf = c \int_j f$ if $c > 0$ and $\int_j cf = (-c) \int_j (-f) \doteq c \int_j f$ if $c \leq 0$.*
4. *If $f, g \in \mathcal{L}_{\overline{\sigma}_j}$, then $f + g \in \mathcal{L}_{\overline{\sigma}_j}$ and $\int_j (f + g) \doteq \int_j f + \int_j g$.*

Remark 4.2. If $[g = 0$ *a.e.* at $(S, j)]$ holds a.e., item (2) above gives $g \in \mathcal{L}_{\overline{\sigma}_j}$ and $\int_j g \doteq 0$. The same argument also implies that $|g|, g^+, g^- \in \mathcal{L}_{\overline{\sigma}_j}$ and that all the conditional integrals are zero a.e.

The next theorem provides an alternative description of integrable functions which is closer to Lebesgue's classical approach, as a closed space under the norm $\| \, \|_j$. The main hypothesis contains the qualifier *a.e.*, even if it were strengthened to every S, the conclusion will still hold only a.e.

Theorem 4.3 (Compare with König's Satz 2.9 in [16]). *Given $j \geq 0$, assume that property $(L_j) -$ a.e. holds and let $f \in Q$. If there exist $\{f_n\}_{n \geq 1} \subset \mathcal{L}_{\overline{\sigma}_j}$ such that $\lim_{n \to \infty} \|f - f_n\|_j = 0$ a.e., then $f \in \mathcal{L}_{\overline{\sigma}_j}$ and $\lim_{n \to \infty} \int_j f_n \doteq \int_j f$.*

Proof. The following holds a.e. $0 \leq \lim_{n \to \infty} \overline{\sigma}_j(|f - f_n|) \leq \lim_{n \to \infty} \|f - f_n\|_j = 0$; by Proposition 3.3 item (6) it follows that, $\overline{\sigma}_j f \leq \overline{\sigma}_j f_n + \overline{\sigma}_j(|f - f_n|)$ and $\overline{\sigma}_j(-f) \leq \overline{\sigma}_j(-f_n) + \overline{\sigma}_j(|-f + f_n|)$. Thus, since $f_n \in \mathcal{L}_{\overline{\sigma}_j}$, $\overline{\sigma}_j(-f_n) = -\underline{\sigma}_j f_n$ and Corollary 3.5 item (1) (the latter available because (L_j)-a.e. holds),

$$[0 \leq \overline{\sigma}_j f - \underline{\sigma}_j f = \overline{\sigma}_j f + \overline{\sigma}_j(-f) \leq 2\overline{\sigma}_j(|f - f_n|)] \ a.e.,$$

from which we obtain $f \in \mathcal{L}_{\overline{\sigma}_j}$. From the assumption that (L_j)-a.e. holds, we can rely on Proposition 4.1 item (4) and Corollary 3.5 item (2) to compute,

$$\left|\int_j f - \int_j f_n\right| \doteq |\overline{\sigma}_j(f - f_n)| \leq \overline{\sigma}_j(|f - f_n|) \xrightarrow[n \to \infty]{} 0 \ a.e.,$$

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

which implies that $\lim_{n\to\infty} \int_j f_n \doteq \int_j f$. □

4.1. **Characterization of Conditional Integrability.** This section starts by introducing the basic functions that are integrable and culminates in Theorem 4.15 that characterizes all possible integrable functions.

Definition 4.4. For $j \geq 0$ given, define the following set of functions:

$$\mathcal{M}_j \equiv \{v = \sum_{m=1}^{\infty} f_m,\ f_m \in \mathcal{E}_j^+ \ \forall\ m \geq 1,\ [\sum_{m=1}^{\infty} I_j f_m < \infty,\ a.e.]\}.$$

$\mathcal{M} \equiv \mathcal{M}_0$ (denoted by $M(I|\mathcal{E})$ in [16]). Moreover for fixed (S,j) define

$$\mathcal{M}_j(S) \equiv \{v = \sum_{m=1}^{\infty} f_m,\ f_m \in \mathcal{E}_{(S,j)}^+ \ \ \forall\ m \geq 1,\ \sum_{m=1}^{\infty} I_j f_m(S) < \infty\}.$$

Therefore $\mathcal{M}_0 = \mathcal{M}_0(S)$ is valid for all S. Observe that $\mathcal{E}_j^+ \subset \cap_{S\in\mathcal{S}} \mathcal{M}_j(S) \subset \mathcal{M}_j$. In particular $v \in \mathcal{M}_j$ iff there exist $F \subset \mathcal{S}$ with null complement such that $v \in \cap_{S\in F}\mathcal{M}_j(S)$.

Proposition 4.5 (see proof in Appendix D)**.** *Let $v = \sum_{m=1}^{\infty} f_m,\ f_m \in \mathcal{E}_{(S,j)}^+$ for a given node (S,j) such that $(L_{(S,j)})$ holds, then:*

$$\underline{\sigma}_j v(S) = \overline{\sigma}_j v(S) = \overline{I}_j v(S) = \sum_{m\geq 1} I_j f_m(S). \tag{4.1}$$

Therefore, if we assume that property (L_j)-a.e. holds: $\mathcal{M}_j \subseteq \mathcal{L}_{\overline{\sigma}_j}^+$ and if $v \in \mathcal{M}_j$ then, $\int_j v \doteq \sum_m \int_j f_m$.

Moreover, if $v \in \mathcal{M}_j(S)$, for any $\epsilon > 0$, there exist $h \in \mathcal{E}_{(S,j)}^+,\ u \in \mathcal{M}_j(S)$, both depending on S and ϵ, satisfying: $v = h + u$ and $\overline{I}_j u(S) \leq \epsilon$.

Remark 4.6. If $f \in \mathcal{E}_{(S,j)} + \mathcal{M}_j(S) - \mathcal{M}_j(S)$ then, from the last result of Proposition 4.5, for any $\epsilon > 0$, f can be written as $f = h + v - u$ with $h \in \mathcal{E}_{(S,j)}, u, v \in \mathcal{M}_j(S)$, and $\overline{I}_j u(S),\ \overline{I}_j v(S) \leq \epsilon$. See Theorem 4.15.

Corollary 4.7 (see proof in Appendix D)**.** *Fix a node (S,j) such that $(L_{(S,j)})$ holds and $u, v \in \mathcal{M}_j(S), h \in \mathcal{E}_{(S,j)}$:*

(1) $\overline{I}_j(u+\alpha v)(S) = \overline{\sigma}_j(u+\alpha v)(S) = \overline{\sigma}_j u(S) + \alpha\overline{\sigma}_j v(S) = \overline{I}_j u(S) + \alpha \overline{I}_j v(S)$, *whenever $\alpha \geq 0$.*
(2) $\underline{\sigma}_j(h+v-u)(S) = \overline{\sigma}_j(h+v-u)(S) = I_j h(S) + \overline{I}_j v(S) - \overline{I}_j u(S)$.

Therefore, if property (L_j)-a.e. holds then, $\mathcal{E}_j + \mathcal{M}_j - \mathcal{M}_j \subset \mathcal{L}_{\overline{\sigma}_j}$ and

$$\int_j (h+v-u) \doteq \int_j h + \int_j v - \int_j u. \tag{4.2}$$

The remaining of the section introduces the key sets $\mathcal{L}_j$ which we prove satisfy $\mathcal{L}_j \subseteq \mathcal{L}_{\overline{\sigma}_j}^+$ and so $\mathcal{L}_j - \mathcal{L}_j \subseteq \mathcal{L}_{\overline{\sigma}_j}$. Under an strengthening of property $(L_{(S,j)})$ we

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

also show that $\mathcal{L}_j = \mathcal{L}^+_{\overline{\sigma}_j}$ (see Theorem 4.15). We pursue some results involving $\mathcal{L}_j$ and, in that way gain some generality.

Definition 4.8 (Positive Cone of Integrable Functions)**.**

$$\mathcal{L}_j \equiv \{f : \mathcal{S} \to [0,\infty] : \text{ the following holds a.e. in the variable } S \in \mathcal{S} :$$
$$[\text{ for all } \epsilon > 0\ \exists\, u, v \in \mathcal{M}_j(S) \text{ such that } f = v - u,\ a.e. \text{ on } \mathcal{S}_{(S,j)} \text{ and } \overline{I}_j u(S) \leq \epsilon]\}.$$

Notice that $\mathcal{L}_j$ is a positive cone that contains the non-negative null functions. To check the latter claim let $g \in P$ be a null function, it follows from item (5) of Proposition 3.2 that $\overline{I}_j(|g|) = 0$ a.e. i.e. $[g = 0\ a.e. \text{ on } \mathcal{S}_{(S,j)}]$ holds a.e. in the variable S. This shows $g \in \mathcal{L}_j$.

The set $\mathcal{L}^K_j$, introduced next, will be shown to be the set of integrable functions in Theorem 4.15.

Definition 4.9.

$$\mathcal{L}^K_j \equiv \{f : \mathcal{S} \to [-\infty,\infty] : \text{ the following holds a.e. in the variable } S \in \mathcal{S} :$$
$$[\text{for all } \epsilon > 0\ \exists\, u, v \in \mathcal{M}_j(S),\ h \in \mathcal{E}_{(S,j)}, f = (h + v - u)\ a.e. \text{ on } \mathcal{S}_{(S,j)} \text{ and}$$
$$\overline{I}_j u(S) \leq \epsilon,\ \overline{I}_j v(S) \leq \epsilon]\}.$$

Corollary 4.10 (see proof in Appendix D)**.** *Assume that property* (L_j)*-a.e. holds, then,*

$$\mathcal{L}_j \subseteq \mathcal{L}^+_{\overline{\sigma}_j} \text{ and } \mathcal{L}^K_j \subseteq \mathcal{L}_{\overline{\sigma}_j}.$$

From $\mathcal{L}_j \subseteq \mathcal{L}^+_{\overline{\sigma}_j}$, it follows that $\mathcal{L}_j - \mathcal{L}_j \subseteq \mathcal{L}_{\overline{\sigma}_j}$ given that the latter is a vector space. As we pointed out, $\mathcal{L}_j$ is a positive cone so $\alpha\mathcal{L}_j + \beta\mathcal{L}_j \subseteq \mathcal{L}_{\overline{\sigma}_j}$, $\alpha, \beta \in \mathbb{R}$ follows.

We take the opportunity to also introduce analogous sets $\mathcal{L}^G_j, \mathcal{L}^{K,G}_j$; both definitions have a global characteristic to them. These sets are further used in Theorem 4.15 and Proposition E.2.

Definition 4.11 (Global Versions of $\mathcal{L}_j$ and $\mathcal{L}^K_j$)**.**

$$\mathcal{L}^G_j \equiv \{f : \mathcal{S} \to [0,\infty] : \text{ for each } \epsilon > 0\ \exists\, u, v \in \mathcal{M}_j \text{ such that the following} \tag{4.3}$$
$$\text{holds a.e. in } S\text{: } [f = v - u\ a.e. \text{ on } \mathcal{S}_{(S,j)} \text{ and } \overline{I}_j u(S) \leq \epsilon,\ \overline{I}_j v(S) < \infty\}.$$
$$\mathcal{L}^{K,G}_j \equiv \{f : \mathcal{S} \to [-\infty,\infty] : \text{ for each } \epsilon > 0\ \exists\, u, v, \in \mathcal{M}_j,\ h \in \mathcal{E}_j \text{ such that}$$
$$\text{the following holds a.e. in } S : [f = h + v - u\ a.e. \text{ on } \mathcal{S}_{(S,j)} \text{ and } \overline{I}_j u(S), \overline{I}_j v(S) \leq \epsilon]\}.$$

The key role of König's condition is that it enforces consistency between $\overline{I}$ and $\overline{\sigma}$ on non-negative functions. The following proposition is a nodewise version of a result in [16].

Proposition 4.12 (see proof in Appendix D)**.** *Fix a node* (S,j), $j \geq 0$. *The following assertions are equivalent:*

(1) $(K_{(S,j)})$.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

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(2) $\overline{I}_j f(S) = \overline{\sigma}_j f(S)$ *for every* $f \in P$.
(3) $\overline{I}_j f^-(S) = \overline{\sigma}_j f^-(S)$ *for every* $f \in \mathcal{E}_{(S,j)}$.

As a direct corollary to the proof of Proposition 4.12 we derive an a.e. version of the implication (1) $\implies$ (2) above (see Corollary 4.14). The a.e. notion involved is a key assumption in Theorem 4.15 and it is introduced next.

Definition 4.13 (($\overline{\sigma}_j = \overline{I}_j$) a.e. Uniformly)**.** For a fixed $j \geq 0$, we will say that ($\overline{\sigma}_j = \overline{I}_j$) holds a.e. uniformly over P if there exists a null set $\mathcal{K} \subseteq \mathcal{S}$ such that the following holds:

$$[\overline{\sigma}_j f(S) = \overline{I}_j(f)(S),\ \forall f \in P \text{ and } \forall\ S \in \mathcal{K}^c].$$

Corollary 4.14. *If* (K_j)*-a.e. is in force, then* ($\overline{\sigma}_j = \overline{I}_j$) *holds a.e. uniformly over* P*, i.e., there is a global null set* $\mathcal{K}'$ *(independent of* f*) such that* $\overline{\sigma}_j f(S) = \overline{I}_j f(S)$ *for every* $f \in P$ *and* $S \notin \mathcal{K}'$

The proof of Theorem 4.15 below follows the steps of an analogous result from [16] (Satz 2.4) but taking care of the a.e. condition in the definition of conditional integrable functions. The theorem relies on the sets $\mathcal{L}_j^G$ and $\mathcal{L}_j^{K,G}$ which were introduced in Definition 4.11. Since one can easily check that $\mathcal{L}_j^{K,G}$ coincides with the space $\mathcal{R}_j$ of functions which can be replicated by generalized portfolios, Theorem 2.14 is a special case of the following result.

Theorem 4.15 (Characterization of Integrable Functions)**.** *Assume* (L_j)*-a.e. holds and that* ($\overline{\sigma}_j = \overline{I}_j$) *holds a.e. uniformly over* P*. Then:*

$$\mathcal{L}^+_{\overline{\sigma}_j} = \mathcal{L}_j = \mathcal{L}_j^G. \tag{4.4}$$

$$\mathcal{L}_{\overline{\sigma}_j} = \mathcal{L}_j^K = \mathcal{L}_j^{K,G}. \tag{4.5}$$

Proof. The second equalities, in both displays above, are proven in Proposition E.2 and are reproduced here for emphasis. The inclusions $\mathcal{L}_j \subseteq \mathcal{L}^+_{\overline{\sigma}_j}$ and $\mathcal{L}_j^K \subseteq \mathcal{L}_{\overline{\sigma}_j}$ are established in Corollary 4.10. All these results only require the validity of (L_j)-a.e., the hypothesis that ($\overline{\sigma}_j = \overline{I}_j$) holds a.e. uniformly over P is only used to establish the remaining two inclusions.

To prove the inclusion $\mathcal{L}^+_{\overline{\sigma}_j} \subseteq \mathcal{L}_j$ in (4.4), let $f \in \mathcal{L}^+_{\overline{\sigma}_j}$ and $\epsilon > 0$. For $m \geq 1$, choose $\epsilon_m > 0$ such that $\epsilon = \sum_{m\geq 1} \epsilon_m$. Define also $N_f = \{S \in \mathcal{S} : \overline{\sigma}_j f(S) - \underline{\sigma}_j f(S) \neq 0\}$, which is a null set given that $f \in \mathcal{L}_{\overline{\sigma}_j}$. Let N_K denote the null set related to our assumption that ($\overline{\sigma}_j = \overline{I}_j$) holds a.e. uniformly over P (see Corollary 4.14). Finally let N_L denote the null set related to our assumption that (L_j)-a.e. holds.

Until we establish the mentioned inclusion, in what follows, we reserve the notation S to mean that $S \in N_f^c \cap N_K^c \cap N_L^c$ and show that for each such S there are sequences of functions $(h_m)_{m\geq 1}$ in $\mathcal{M}_j(S)$ and $(f_m)_{m\geq 1}$ in $\mathcal{L}^+_{\overline{\sigma}_j}$ with $\overline{\sigma}_j f_m(S) \leq \epsilon_m$, such that $f_m = h_m - f_{m-1}$ on $\mathcal{S}_{(S,j)}$.

Rename $f_0 = f$; from our choice of S, $0 \leq \overline{I}_j f_0(S) = \overline{\sigma}_j f_0(S) < \infty$ (Lemma D.1 applies to our particular S providing $\overline{\sigma}_j f_0(S) < \infty$). By definition of $\overline{I}_j$ there

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

exists $h_1 \in \mathcal{M}_j(S)$ with

$$f_0 \leq h_1 \; on \mathcal{S}_{(S,j)} \text{ and } \overline{I}_j h_1(S) \leq \overline{I}_j f_0(S) + \epsilon_1. \tag{4.6}$$

Define $f_1 \equiv h_1 - f_0 \geq 0$ on each $\mathcal{S}_{(S,j)}$, and $f_1(\hat{S}) \equiv 0$ whenever $\hat{S} \in N_f \cup N_K \cup N_L$. This defines f_1 on $\mathcal{S}$. For the set of trajectories S under consideration: $\overline{\sigma}_j h_1(S) - \underline{\sigma}_j h_1(S) = 0$ which follows from the fact that $\overline{\sigma}_j h_1(S) = \overline{I}_j h_1(S) < \infty$ from Proposition 4.5. And also $\overline{\sigma}_j f_0(S) - \underline{\sigma}_j f_0(S) = 0$. Then, applying Lemma D.1 to f_0 and h_1 we obtain $\overline{\sigma}_j f_1(S) - \underline{\sigma}_j f_1(S) = 0$ and so this equality holds for all $S \in N_f^c \cap N_K^c \cap N_L^c$, in particular a.e. Noticing that $f_1 \geq 0$ we have then shown that $f_1 \in \mathcal{L}^+_{\overline{\sigma}_j}$. By means of Lemma D.1, the validity of $\overline{\sigma}_j = \overline{I}_j$ a.e. and uniformly over P, and (4.6) we derive:

$$\overline{\sigma}_j f_1(S) = \overline{\sigma}_j h_1(S) - \overline{\sigma}_j f_0(S) \leq \overline{I}_j f_0(S) + \epsilon_1 - \overline{I}_j f_0(S) = \epsilon_1.$$

For $m \geq 2$ we can then proceed inductively. Whenever $f_{m-1} \in \mathcal{L}^+_{\overline{\sigma}_j}$ has been constructed satisfying: $\overline{\sigma}_j f_{m-1}(S) - \underline{\sigma}_j f_{m-1}(S) = 0$ and $\overline{\sigma}_j f_{m-1}(S) \leq \epsilon_{m-1}$; it then follows from $0 \leq \overline{I}_j f_{m-1}(S) = \overline{\sigma}_j f_{m-1}(S) < \infty$ that there exist $h_m \in \mathcal{M}_j(S)$ with

$$f_{m-1} \leq h_m \;\text{ on }\; \mathcal{S}_{(S,j)}, \;\text{ and }\; \overline{I}_j h_m(S) \leq \overline{I}_j f_{m-1}(S) + \epsilon_m. \tag{4.7}$$

We can then define

$$f_m \equiv h_m - f_{m-1} \;\text{ on each }\; \mathcal{S}_{(S,j)}, \;\text{ and }\; f_1(\hat{S}) \equiv 0 \text{ whenever } \hat{S} \in N_f \cup N_K \cup N_L, \tag{4.8}$$

this defines f_m on $\mathcal{S}$. As we have argued for the case of f_1, it follows that $\overline{\sigma}_j f_m(S) - \underline{\sigma}_j f_m(S) = 0$ for all S under consideration, $f_m \in \mathcal{L}^+_{\overline{\sigma}_j}$ and $\overline{\sigma}_j f_m(S) \leq \epsilon_m$.

Observing that $\sum_{m \geq 1} f_m \geq 0$ and

$$\overline{\sigma}_j[\sum_{m \geq 1} f_m](S) = \overline{I}_j[\sum_{m \geq 1} f_m](S) \leq \sum_{m \geq 1} \overline{I}_j f_m(S) < \infty,$$

then by Proposition 3.3 item (8), $\sum_{m \geq 1} f_m < \infty$ *a.e. on* $\mathcal{S}_{(S,j)}$, from where $\lim_{m \to \infty} f_m = 0$ *a.e. on* $\mathcal{S}_{(S,j)}$.

On the other hand, from (4.8), $h_m = f_{m-1} + f_m$ (if $f_{m-1}(\hat{S}) = \infty$ for some $\hat{S} \in \mathcal{S}$, it must be, from (4.7) that $h_m(\hat{S}) = \infty$). Then, using Lemma D.1, for $m \geq 2$,

$$\overline{I}_j h_m(S) = \overline{I}_j(f_m + f_{m-1})(S) = \overline{\sigma}_j(f_m + f_{m-1})(S) = \overline{\sigma}_j f_{m-1}(S) + \overline{\sigma}_j f_m(S) < \epsilon_{m-1} + \epsilon_m.$$

Set $u = \sum_{m \geq 1} h_{2m}$ and $v = \sum_{m \geq 0} h_{2m+1}$, then $u, v \in \mathcal{M}_j(S)$, since

$$\|u\|_j(S) = \overline{I}_j u(S) \leq \overline{I}_j[\sum_{m \geq 1} h_{2m}](S) < \sum_{m \geq 1} (\epsilon_{2m-1} + \epsilon_{2m}) = \epsilon,$$

and

$$\overline{I}_j v(S) \leq \sum_{m \geq 0} \overline{I}_j h_{2m+1}(S) < \overline{I}_j f + \epsilon_1 + \sum_{m \geq 1} (\epsilon_{2m} + \epsilon_{2m+1}) = \overline{I}_j f + \epsilon < \infty.$$

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
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Thus, on $\mathcal{S}_{(S,j)}$, for $n \geq 1$

$$\sum_{m=1}^{n} (-1)^{m-1} h_m = \sum_{m=1}^{n} ((-1)^{m-1} f_{m-1} - (-1)^m f_m) = f - (-1)^n f_n,$$

from where

$$f = \sum_{m=1}^{2n} (-1)^{m-1} h_m + f_{2n} = \sum_{m=0}^{n-1} h_{2m+1} - \sum_{m=1}^{n} h_{2m} + f_{2n}.$$

Finally, taking the limit $n \to \infty$, it follows that $f = v - u$ *a.e. on* $\mathcal{S}_{(S,j)}$ with $\overline{I}u(S) \leq \epsilon$, $u, v \in \mathcal{M}_j(S)$ and these properties hold a.e. in S given that $N_f \cup N_K \cup N_L$ is a null set. It then follows that $f \in \mathcal{L}_j$.

In order to establish $\mathcal{L}_{\overline{\sigma}_j} \subseteq \mathcal{L}_j^K$ in (4.5), let $f \in \mathcal{L}_{\overline{\sigma}_j}$ and so $N_f \equiv \{S \in \mathcal{S} : \overline{\sigma}_j f(S) - \underline{\sigma}_j f(S) \neq 0\}$ is a null set. For the remaining of the proof we will consider $S \in N_1^c \cap N_K^c \cap N_L$ (where we rely on notation introduced in the preceding part of the proof). In particular $-\infty < \overline{\sigma}_j f(S) < \infty$; therefore, there exist $\tilde{h} \in \mathcal{E}_{(S,j)}, \tilde{w} \in \mathcal{M}_j(S)$ such that $f \leq \tilde{h} + \tilde{w}$ on $\mathcal{S}_{(S,j)}$, with $I_j\tilde{h}(S) + \overline{I}_j\tilde{w}(S) \leq \overline{\sigma}_j f(S) + \epsilon$. Set $\tilde{f} \equiv \tilde{h} + \tilde{w} - f$ on each $\mathcal{S}_{(S,j)}$, with $S \in N_1^c \cap N_K^c \cap N_L$, and $\tilde{f} = 0$ otherwise. By Proposition 3.4 item (4) $\overline{\sigma}_j\tilde{h}(S) + \overline{\sigma}_j(-\tilde{h})(S) = 0$, then reasoning as in the case of f_m before, $\overline{\sigma}_j\tilde{f}(S) - \underline{\sigma}_j\tilde{f}(S) = 0$, in particular $\tilde{f} \in \mathcal{L}^+_{\overline{\sigma}_j}$.

So, by (4.4) $\tilde{f} = \tilde{v} - \tilde{u}$ *a.e. on* $\mathcal{S}_{(S,j)}$, with $\tilde{u}, \tilde{v} \in \mathcal{M}_j(S)$ and $\|\tilde{u}\|_j(S) < \epsilon$. Consequently $f = \tilde{h} + \tilde{w} + \tilde{u} - \tilde{v}$ *a.e. on* $\mathcal{S}_{(S,j)}$.

Then, by Proposition 4.5, $\tilde{w}+\tilde{u} = \tilde{h}_1+\tilde{v}_1$, $\tilde{v} = \tilde{h}_2+\tilde{v}_2$ with $\tilde{h}_1, \tilde{h}_2 \in \mathcal{E}^+_{(S,j)}$, $\tilde{v}_1, \tilde{v}_2 \in \mathcal{M}_{(S,j)}$, such that $\overline{I}_j\tilde{v}_1(S), \overline{I}_j\tilde{v}_2(S) < \epsilon$, and

$$f = \tilde{h} + \tilde{h}_1 - \tilde{h}_2 + \tilde{v}_1 - \tilde{v}_2 \ \textit{a.e. on } \mathcal{S}_{(S,j)},$$

notice that $h \equiv (\tilde{h} + \tilde{h}_1 - \tilde{h}_2) \in \mathcal{E}_{(S,j)}$ and so, we have established $f \in \mathcal{L}_j^K$. □

The next theorem provides a norm-closure characterization of $\mathcal{L}_{\overline{\sigma_j}}$.

Theorem 4.16. *Assume that* (L_j)*-a.e. holds. Then,*

a) $\overline{\sigma}_j$ *is linear, continuous and positive, in an a.e. manner, on* $\mathcal{L}_{\overline{\sigma}_j}$.

Furthermore, assume that $(\overline{\sigma}_j = \overline{I}_j)$ *holds a.e. uniformly over* P. *Then,*

(1) $f \in \mathcal{L}_{\overline{\sigma}_j}$, *if and only if there is a sequence* $\{h_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{E}_j$ *such that*

$$\lim_{n\to\infty} \|f - h_n\|_j = 0 \text{ a.e.}$$

(2) $f \in \mathcal{L}^+_{\overline{\sigma}_j}$, *if and only if there is a sequence* $\{h_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{E}_j^+$ *such that*

$$\lim_{n\to\infty} \|f - h_n\|_j = 0 \text{ a.e.}$$

Proof. Linearity in item *a*) follows from Proposition 4.1, continuity and positivity from Corollary 3.5. The statements are qualified by a.e. given our hypothesis (L_j)-a.e.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

For item (1), take $f \in \mathcal{L}_{\overline{\sigma}_j}$, then, by Theorem 4.15, for any $n \geq 1$ there exist $h_n \in \mathcal{E}_j$, $u_n, v_n \in \mathcal{M}_j$ such that $f = h_n + v_n - u_n$ *a.e. on* $\mathcal{S}_{(S,j)}$, and $\overline{I}_j u_n(S), \overline{I}_j v_n(S) < \frac{1}{n}$ for almost every S. Observe that $f - h_n = \mathbf{1}_{A^c}[v_n - u_n] + \mathbf{1}_A[f - h_n]$ on $\mathcal{S}_{(S,j)}$ holds, where $A = \{f \neq h_n + v_n - u_n\}$ is a conditional null set on (S, j) for almost every $S \in \mathcal{S}$. This implies that

$$|f - h_n| \leq \mathbf{1}_{A^c}|v_n - u_n| + \mathbf{1}_A|f - h_n| \quad \text{from where} \quad \|f - h_n\|_j \leq \tfrac{2}{n} \text{ a.e..}$$

Here, Proposition 3.2 item (3) was used because $\mathbf{1}_{A^c}|v_n - u_n| = |v_n - u_n|$ a.e. on $\mathcal{S}_{(S,j)}$. Hence, $\lim_{n\to\infty} \|f - h_n\|_j = 0$ a.e. The converse implication is an immediate consequence of Theorem 4.3.

The proof of item (2) is similar: Any $f \in \mathcal{L}^+_{\overline{\sigma}_j}$ can be written as $f = v_n - u_n$ *a.e. on* $\mathcal{S}_{(S,j)}$, $u_n, v_n \in \mathcal{M}_j$, with $\overline{I}_j u_n(S) < \frac{1}{n}$ for almost every S. Also, from Proposition 4.5, $v_n = h_n + \tilde{v}_n$ with $h_n \in \mathcal{E}^+_j$, $\tilde{v}_n \in \mathcal{M}_j$ and $\overline{I}_j \tilde{v}_n(S) < \frac{1}{n}$. Then, the argument for item (1) can be repeated almost verbatim. □

The classical Beppo-Levi and monotone convergence theorems hold in $\mathcal{L}_{\overline{\sigma}_j}$.

Theorem 4.17 (Beppo-Levi)**.** *Assume (L_j)-a.e. holds and let $\{f_n\}_{n\geq 1} \subset \mathcal{L}_{\overline{\sigma}_j}$ such that $\sum_{n\geq 1} \|f_n\|_j(S) < \infty$ a.e. Define*

$$N \equiv \{S \in \mathcal{S} | \sum_{n=1}^{\infty} f_n(S) \text{ does not converge in } \mathbb{R}\}, \quad f(S) \equiv \begin{cases} \sum_{n=1}^{\infty} f_n(S), & S \in \mathcal{S} \setminus N, \\ 0, & S \in N \end{cases}$$

Then, N is a conditional null set on (S, j) for a.e. $S \in \mathcal{S}$ and $\lim_{k\to 0} \|f - \sum_{n=1}^{k} f_k\|_j = 0$ a.e. In particular, $f \in \mathcal{L}_{\overline{\sigma}_j}$ and

$$\int_j \sum_{n\geq 1} f_n \doteq \sum_{n\geq 1} \int_j f_n.$$

Proof. Denote by $\tilde{N}$ a global null set such that $\sum_{n\geq 1} \|f_n\|_j(S) < \infty$ and $(L_{(S,j)})$ holds for every $S \notin \tilde{N}$. Applying the generalized Beppo-Levi theorem [17], p. 260, to the elementary integral $I_{(S,j)}$, we observe that N is a conditional null set on (S, j) and that $\lim_{k\to\infty} \|f - \sum_{n=1}^{k} f_k\|_j(S) = 0$. This shows the first two claims. The 'in particular'-part then is an immediate consequence of Theorem 4.3 and the linearity of the integral on $\mathcal{L}_{\overline{\sigma}_j}$. □

Theorem 4.18 (Monotone Convergence Theorem)**.** *Assume that the following two statements hold: (L_j)-a.e. and $(\overline{\sigma}_j = \overline{I}_j)$ a.e. uniformly over P. Let $\{f_n\}_{n\geq 1} \subseteq \mathcal{L}_{\overline{\sigma}_j}$, $f_n \nearrow f \in Q$. If $\lim_{n\to\infty} \int_j f_n \leq C < \infty$ a.e. (for some constant C), then $\|f - f_n\|_j \to 0$ a.e., $f \in \mathcal{L}_{\overline{\sigma}_j}$ and $\int_j f \doteq \lim_{n\to\infty} \int_j f_n$.*

Proof. Define for $n \geq 1$, $g_n \equiv f_n - f_{n-1} \geq 0$, with $f_0 \equiv 0$, then for a.e. $S \in \mathcal{S}$

$$\sum_{n=1}^{m} \|g_n\|_j = \sum_{n=1}^{m} \overline{\sigma}_j g_n = \overline{\sigma}_j f_{m+1} = \int_j f_{m+1} \leq C < \infty \; a.e. \text{ on } \mathcal{S}_{(S,j)}.$$

The result then follows from Theorem 4.17. □

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Appendix A. On the role of zero-neutral nodes

In this appendix, we discuss the importance of the concept of zero-neutral nodes for the elementary integrals $I_{(S,j)}$.

We rely on the following simple Lemma (see Lemma 1 in [13]) that gives a basic procedure to construct particularly useful trajectories that move in a *contrarian* way, at each node (S^n, i), to a collection of bets $F_i(S^n)$.

Lemma A.1. *Assume every node in $\mathcal{S}$ is 0-neutral, $n_0 \geq 0$, and let $F = \{F_i\}_{i \geq n_0}$ be a sequence of non-anticipative functions and $\epsilon > 0$. Then, for any $S \in \mathcal{S}$ there exists a sequence of trajectories $\{S^n\}_{n \geq n_0}$ with $S^{n_0} = S$ such that for every $n > n_0$, $S^n \in \mathcal{S}_{(S^{n-1}, n-1)} \subset \mathcal{S}_{(S,n_0)}$ and*

$$F_i(S^n)\Delta_i S^n < \frac{\epsilon}{2^{i+1}},\ n_0 \leq i \leq n-1.$$

In particular,

$$\sum_{i=n_0}^{n-1} F_i(S^n)\ \Delta_i S^n < \sum_{i=n_0}^{n-1} \frac{\epsilon}{2^{i+1}}.$$

Remark A.2. If at any point in the construction, a node (S^n, n) is an arbitrage node of type I we could choose, without loss of generality, $S^{n+1} \in \mathcal{S}_{(S^n,n)}$ such that $\Delta_n S^{n+1} = S^{n+1}_{n+1} - S^{n+1}_n = 0$.

Corollary A.3. *Assume every node in $\mathcal{S}$ is 0-neutral and let F, G be sequences of non-anticipative functions. The following holds at an arbitrary node (S, j):*

- *If $\Pi^{V,F}_{j,n} \geq 0$ on $\mathcal{S}_{(S,j)}$ then $\Pi^{V,F}_{j,k}(S) \geq 0$ on $\mathcal{S}_{(S,j)}$ for all $j \leq k \leq n$, and so $V(S) \geq 0$ as well. In particular, this is valid for $\Pi^{V,F}_{j,n} \in \mathcal{E}_j$.*
- *If $\Pi^{U,G}_{j,m} \leq \Pi^{V,F}_{j,n}$ then $U(S) \leq V(S)$. Consequently I_j is well defined i.e. if $f = \Pi^{V^f,H^f}_{j,n^f}, g = \Pi^{V^g,H^g}_{j,n^g} \in \mathcal{E}_j$ and $f|_{\mathcal{S}_{(S,j)}} = g|_{\mathcal{S}_{(S,j)}}$ then $I_j f(S) = V^f(S) = V^g(S) = I_j g(S)$ (see also* [13, Proposition 4]*).*

Proof. We only argue explicitly for the first item as the second item follows from the first. Assume that for some $j \leq k \leq n$ there exist $\tilde{S} \in \mathcal{S}_{(S,j)}$ such that $\Pi^{V,F}_{j,k}(\tilde{S}) = \delta < 0$. From Lemma A.1, for $n_0 = k$, $\tilde{S}$, and $\epsilon = -\frac{\delta}{2}$ there exists $S^n \in \mathcal{S}_{(\tilde{S},k)}$, such that $\sum_{i=k}^{n-1} F_i(S^n)\Delta_i S^n < -\frac{\delta}{2}$, it then follows the contradiction $0 \leq V(\tilde{S}) + \sum_{i=j}^{n-1} F_i(S^n)\Delta_i S^n \leq \frac{\delta}{2} < 0$. ☐

The previous corollary implies the following: Fix a node (S, j). If $f = g$ (on $\mathcal{S}_{(S,j)}$) for some $f = \Pi^{V^f,H^f}_{j,n^f}, g = \Pi^{V^g,H^g}_{j,n^g} \in \mathcal{E}_{(S,j)}$, then $I_{(S,j)} f = V_f(S) = V_g(S) = I_{(S,j)} f$. Hence, $I_{(S,j)}$ is well-defined. Repeating the same argument with '$\leq$' instead of '$=$' yields the isotony of $I_{(S,j)}$. The linearity of the map $\Pi^{V,H}_{j,n} \to V$ is obvious (once it has been shown to be well-defined).

Hence, $I_{(S,j)}$ is a well-defined, linear, isotone operator for every node (S, j), if every node is zero-neutral.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina — Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Remark A.4. Suppose a node (S,j) fails to be zero-neutral. Let us assume without loss of generality that there is an $\epsilon > 0$ such that $\tilde{S}_{j+1} - S_j > \epsilon$ for every $\tilde{S} \in \mathcal{S}_{(S,j)}$. Consider $\Pi^{V,F}_{j,n} \in \mathcal{E}_{(S,j)}$ with $n = j+1$, $F_j = 1$, $F_i = 0$ for $i > j$ and $V = -\epsilon/2$. Then, $\Pi^{V,F}_{j,n} \geq \epsilon/2$ on $\mathcal{S}_{(S,j)}$. Hence, $I_{(S,j)}$ cannot be a well-defined, linear, and isotone operator.

Appendix B. Proofs for Subsection 3.1

In this appendix, we provide the proofs for the results stated in Subsection 3.1.

Proof of Proposition 3.1. We only provide the proof of countable subadditivity.

We may assume that $\sum_{k\geq 1} \overline{I}_j g_k(S) < \infty$, which leads to $\overline{I}_j g_k(S) < \infty$ for $k \geq 1$. Therefore, for a fixed $\epsilon > 0$ and for any $k \geq 1$, by definition of $\overline{I}_j$ there exist $H^{m,k} \in \mathcal{H}_{(S,j)}$ and $V^{m,k} \in \mathbb{R}$, $m \geq 1$, such that:

$$g_k(\tilde{S}) \leq \sum_{m=1}^{\infty} [V^{m,k} + \sum_{i=j}^{n_m-1} H^{m,k}_i(\tilde{S})\Delta_i\tilde{S}] \quad \forall\ \tilde{S} \in \mathcal{S}_{(S,j)},$$

with

$$V^{m,k} + \sum_{i=j}^{n_m-1} H^{m,k}_i(\tilde{S})\Delta_i\tilde{S} \geq 0 \quad \forall\tilde{S} \in \mathcal{S}_{(S,j)}, \quad n \geq j \quad \text{and}$$

$$\sum_{m=1}^{\infty} V^{m,k} \leq \overline{I}_j g_k(S) + \frac{\epsilon}{2^k}.$$

Then

$$\sum_{k=1}^{\infty} g_k(\tilde{S}) \leq \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} [V^{m,k} + \sum_{i=j}^{n_m-1} H^{m,k}_i(\tilde{S})\Delta_i\tilde{S}] \quad \forall\tilde{S} \in \mathcal{S}_{(S,j)},$$

noticing that the double sum of non-negative terms can be reordered into a single sum, we can then deduce that

$$\overline{I}_j g(S) \leq \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} V^{m,k} \leq \sum_{k=1}^{\infty} \overline{I}_j g_k(S) + \epsilon.$$

□

Proof of Proposition 3.2. (1) Assume $\|g\|_j(S) = 0$, consider $A = \{\tilde{S} \in \mathcal{S}_{(S,j)} : g(\tilde{S}) \neq 0\}$.

From $\mathbf{1}_A(\tilde{S}) \leq \sum_{k\geq 1} |g(\tilde{S})|$ it follows that $\|\mathbf{1}_A\|_j(S) \leq \sum_{k\geq 1} \|g\|_j(S) = 0$. Therefore, A is a conditionally null set at (S,j) and so $g(\tilde{S}) = 0$ holds conditionally a.e. at (S,j).

For the converse of (1), by assumption, there exists $B \subseteq \mathcal{S}_{(S,j)}$ such that $\|\mathbf{1}_B\|_j(S) = 0$ and $g(\tilde{S}) = 0$ $\forall\tilde{S} \in \mathcal{S}_{(S,j)} \setminus B$. Given that $|g(\tilde{S})| \leq \sum_{k\geq 1} \mathbf{1}_B(\tilde{S})$ $\forall\tilde{S} \in \mathcal{S}_{(S,j)}$ we obtain $\|g\|_j(S) = 0$.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

(2) Let $A \equiv \{\tilde{S} \in \mathcal{S}_{(S,j)} : g(\tilde{S}) = \infty\}$, then $n\, \mathbf{1}_A(\tilde{S}) \leq |g(\tilde{S})|$, $\forall \tilde{S} \in \mathcal{S}_{(S,j)}$, $\forall n \geq 1$, thus

$$n\, \|\mathbf{1}_A\|_j(S) \leq \|g\|_j(S) \quad \text{and so} \quad \|\mathbf{1}_A\|_j(S) = 0.$$

(3) Let $N \equiv \{\tilde{S} \in \mathcal{S}_{(S,j)} : |f(\tilde{S})| > |g(\tilde{S})|\}$.

Then $|f|(\tilde{S}) \leq |g|(\tilde{S}) + \sum_{m\geq 1} \mathbf{1}_N(\tilde{S})$ for $\tilde{S} \in \mathcal{S}_{(S,j)}$. Therefore:

$$\|f\|_j(S) = \overline{I}_j|f|(S) \leq \overline{I}_j|g|(S) + \overline{I}_j(\sum_{m\geq 1} \mathbf{1}_N)(S) = \overline{I}_j|g|(S) = \|g\|_j(S).$$

(4) Follows from the countable subadditivity of $\overline{I}_j$.

(5) Assume $f \leq \sum_{m\geq 1} f_m$, with $f_m = \Pi^{V^m,H^m}_{j,n_m} \in \mathcal{E}^+_j, n \geq j,\ m \geq 1$.

Since also $f_m \in \mathcal{E}^+_k, n \geq j,\ m \geq 1$, it results that for any $S \in \mathcal{S}$

$$\overline{I}_k f(S) \leq \sum_{m\geq 1} \left(V^m(S) + \sum_{i=j}^{k-1} H^m_i(S) \Delta_i S \right) = \sum_{m\geq 1} I_k f_m(S).$$

Since $I_k f_m \in \mathcal{E}^+_j$, $m \geq 1$, and $V^m(S) = V^m(S_0, ..., S_j)$, it follows that $\overline{I}_j[\overline{I}_k f](S) \leq \sum_{m\geq 0} V^m(S)$ and consequently

$$\overline{I}_j[\overline{I}_k f] \leq \overline{I}_j f. \tag{B.1}$$

Assume now that $g \in Q$ is conditionally null at $\mathcal{S}_{(S,j)}$ i.e. $\overline{I}_j(|g|)(S) = 0$. It then follows from (B.1) and and item (1) that $\overline{I}_k(|g|) = 0$ a.e. on $\mathcal{S}_{(S,j)}$.

□

Proof of Proposition 3.3. (1) follows from the definitions and (2) is clear taking $f_0 = 0$.
For (3), taking $f_0 = f$ and $f_m = 0$ for $m \geq 1$ the result follows.
To establish (4), notice that $f \leq g$, *a.e.* on $\mathcal{S}_{(S,j)}$ allows to write $f(\tilde{S}) \leq g(\tilde{S}) + \infty\, \mathbf{1}_N(\tilde{S})$ for all $\tilde{S} \in \mathcal{S}_{(S,j)}$ with $N \subseteq \mathcal{S}_{(S,j)}$ and $\overline{I}_j \mathbf{1}_N(S) = 0$. Therefore $\overline{\sigma}_j f(S) \leq \overline{\sigma}_j g(S) + \overline{\sigma}_j(\infty\, \mathbf{1}_N)(S) \leq \overline{\sigma}_j g(S) + \overline{I}_j(\infty\, \mathbf{1}_N)(S) \leq \overline{\sigma}_j g(S)$.
(5) Assume $gf(\tilde{S}) \leq \sum_{m\geq 0} \Pi^{V^m,H^m}_{j,n_m}(\tilde{S})$, $\tilde{S} \in \mathcal{S}_{(S,j)}$, with $\Pi^{V^0,H^0}_{j,n_0} \in \mathcal{E}_{(S,j)}$ and, for $m \geq 1$, $\Pi^{V^m,H^m}_{j,n_m} \in \mathcal{E}^+_{(S,j)}$. For each $\tilde{S} \in \mathcal{S}_{(S,j)}$ and $m \geq 0$ define

$$U^m(\tilde{S}) = \frac{V^m}{g(S)}, \quad \text{and} \quad G^m_i(\tilde{S}) = \frac{H^m_i(\tilde{S})}{g(S)}, \text{ for } i \geq j.$$

It follows that $f(\tilde{S}) \leq \sum \Pi^{U^m,G^m}_{j,n_m}(\tilde{S})$, $\tilde{S} \in \mathcal{S}_{(S,j)}$ with $\Pi^{U^0,G^0}_{j,n_0} \in \mathcal{E}_{(S,j)}$, and for $m \geq 1$, $\Pi^{U^m,G^m}_{j,n_m} \in \mathcal{E}^+_{(S,j)}$ $n \geq j$. Thus

$$\overline{\sigma}_j f(S) \leq \frac{\overline{\sigma}_j[gf](S)}{g(S)}.$$

The reverse inequality follows similarly.
(6) Observe that $f \leq |f-g| + g$ (as $f(\hat{S}) = \infty$ implies $|f-g|(\hat{S}) = \infty$), from where, (6) holds.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

(7) Since $\overline{\sigma}_j 0 \leq 0$ assume that there exist at least some $S \in \mathcal{S}$ satisfying $\overline{\sigma}_j 0(S) < 0$. Therefore, there exist $\tilde{f}_0 = \Pi_{j,\tilde{n}_0}^{\tilde{V}^0,\tilde{H}^0} \in \mathcal{E}_{(S,j)}$ and, for $m \geq 1$, $\tilde{f}_m = \Pi_{j,\tilde{n}_m}^{\tilde{V}^m,\tilde{H}^m} \in \mathcal{E}^+_{(S,j)}$, such that $0 \leq \sum_{m\geq 0} \tilde{f}_m(\tilde{S})$ for any $\tilde{S} \in \mathcal{S}_{(S,j)}$ and $\sum_{m\geq 0} \tilde{V}^m = r < 0$.

Consider now $f \in Q$, if there exists $f_0 = \Pi_{j,n_0}^{V^0,H^0} \in \mathcal{E}_{(S,j)}$ and $f_m = \Pi_{j,n_m}^{V^m,H^m} \in \mathcal{E}^+_{(S,j)}$ for $m \geq 1$, such that $f(\tilde{S}) \leq \sum_{m\geq 0} f_m(\tilde{S})$ for any $\tilde{S} \in \mathcal{S}_{(S,j)}$ with $\sum_{m\geq 0} V^m$ finite, then (since also $f(\tilde{S}) \leq \sum_{m\geq 0} [f_m(\tilde{S}) + \gamma\, \tilde{f}_m(\tilde{S})]$, for any given $\gamma > 0$):

$$\overline{\sigma}_j f(S) \leq \sum_{m\geq 0} (V^m + \gamma\, \tilde{V}^m) = \sum_{m\geq 0} V^m + \gamma\, r,$$

thus $\overline{\sigma}_j f(S) = -\infty$. On the other hand if such family of functions f_m does not exist, $\overline{\sigma}_j f(S) = \infty$.

(8) Let $A = \{\hat{S} \in \mathcal{S}_{(S,j)} : f(\hat{S}) = \infty\}$ and assume that $\overline{\sigma}_j f(S) < \infty$. There exist $f_0 = \Pi_{j,n_0}^{V^0,H^0} \in \mathcal{E}_{(S,j)}$, and for $m \geq 1$, $f_m = \Pi_{j,n_m}^{V^m,H^m} \in \mathcal{E}^+_{(S,j)}$ $n \geq j$, such that $f(\tilde{S}) \leq \sum_{m\geq 0} f_m(\tilde{S})$ for any $\tilde{S} \in \mathcal{S}_{(S,j)}$ and

$$\sum_{m\geq 0} V^m < \overline{\sigma}_j f(S) + 1.$$

For $\hat{S} \in A$ results that $\infty = f(\hat{S}) \leq \sum_{m\geq 1} f_m(\hat{S})$, because $|f_0| < \infty$. Thus, for any $n > 0$ and $\tilde{S} \in \mathcal{S}_{(S,j)}$, it follows that

$$n\mathbf{1}_A(\tilde{S}) \leq \sum_{m\geq 1} f_m(\tilde{S}).$$

From which

$$n\|\mathbf{1}_A\|_j(S) \leq \sum_{m\geq 1} V^m < \infty.$$

(9) From hypothesis $0 = \overline{\sigma}_j 0 \leq \overline{\sigma}_j f(S) + \overline{\sigma}_j(-f)(S)$, the two terms in the right hand side being finite by hypothesis, we can then conclude that $-\infty < \overline{\sigma}_j(-f)(S), -\infty < \overline{\sigma}_j f(S)$. The last statement follows from item (8). □

Proof of Proposition 3.4. (2) follows from (1) as follows.

Let $f \in \mathcal{E}_{(S,j)}$ and $f_m = \Pi_{j,n_m}^{V^m,H^m} \in \mathcal{E}^+_{(S,j)}$ such that $f \leq \sum_{m\geq 1} f_m$ on $\mathcal{S}_{(S,j)}$. Then $0 \leq -f + \sum_{m\geq 1} f_m$ on $\mathcal{S}_{(S,j)}$, thus (taking $f_0 \equiv -f \in \mathcal{E}_{(S,j)}$), by the definition $\overline{\sigma}_j$, $0 = \overline{\sigma}_j(0)(S) \leq I_j f_0(S) + \sum_{m\geq 0} V^m(S)$, which leads to $I_j f(S) \leq \sum_{m\geq 1} V^m(S)$ as required.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Assume now that continuity from below holds at (S,j), $f \in \mathcal{E}_{(S,j)}$ and $f^+ \leq \sum_{m\geq 1} f_m$ on $\mathcal{S}_{(S,j)}$, and $f_m = \Pi^{V^m,H^m}_{j,n_m} \in \mathcal{E}^+_{(S,j)}$. Since $f \leq f^+$, then

$$I_j f(S) \leq \sum_{m\geq 1} V^m(S), \quad \text{which implies} \quad I_j f(S) \leq \overline{I}_j f^+(S).$$

Let us now show that (3) implies (4). We follow the proof in [16, Behauptung 1.9], but here its (*) condition in page 449 is not needed.

Fix $f_0 \in \mathcal{E}_{(S,j)}$ and for $m \geq 1$, $f_m = \Pi^{V^m,H^m}_{j,n_m} in \mathcal{E}^+_{(S,j)}$ such that $f \leq \sum_{m\geq 0} f_m$ on $\mathcal{S}_{(S,j)}$, then from linearity of I_j and (3),

$$I_j f(S) - I_j f_0(S) = I_j(f - f_0)(S) \leq \overline{I}_j (f - f_0)^+(S).$$

Moreover, since $(f - f_0)^+ \leq \sum_{m\geq 1} f_m$ on $\mathcal{S}_{(S,j)}$ it follows that $\overline{I}_j(f - f_0)^+(S) \leq \sum_{m\geq 1} V^m(S)$. Therefore $I_j f(S) \leq I_j f_0(S) + \sum_{m\geq 0} V^m(S)$, so $I_j f(S) \leq \overline{\sigma}_j f(S)$, thus $I_j f(S) = \overline{\sigma}_j f(S)$ from Proposition 3.3 item (2). Consequently (4) holds, since $\underline{\sigma}_j f(S) = -\overline{\sigma}_j(-f)(S) = I_j f(S)$.

From (4) $\overline{\sigma}_j 0(S) = I_j 0(S) = 0$.

Let us now see that $(L_{(S,j)})$ implies (5). Fix $f \in \mathcal{E}_{(S,j)}$ Then by $(L_{(S,j)})$, $I_j f(S) \leq \overline{I}_j f^+(S) \leq \overline{I}_j |f|(S) = \|f\|_j(S)$. The same analysis gives $-I_j f(S) = I_j[-f](S) \leq \|f\|_j(S)$, and so $-\|f\|_j(S) \leq I_j f(S) \leq \|f\|_j(S)$.

(6) follows from (5), since for $f \in \mathcal{E}^+_{(S,j)}$, $0 \leq I_j f(S) = |I_j f(S)| \leq \overline{I}_j |f|(S) = \overline{I}_j f(S) \leq I_j f(S)$, where we relied on item (4) of Proposition 3.1 for the last inequality.

□

Proof of Corollary 3.5. For item (1), $0 \leq \overline{\sigma}_j f(S) + \overline{\sigma}_j(-f)(S)$ follows from Proposition 3.3, item (1), and $\overline{\sigma}_j 0(S) = 0$. Furthermore, if $|\overline{\sigma}_j(-f)(S)| < \infty$, it follows that $\underline{\sigma}_j f(S) = -\overline{\sigma}_j(-f)(S) \leq \overline{\sigma}_j f(S)$. While if $\overline{\sigma}_j(-f)(S) = \infty$, then $\underline{\sigma}_j f(S) = -\infty \leq \overline{\sigma}_j f(S)$. Finally if $\overline{\sigma}_j(-f)(S) = -\infty$, then $0 \leq \overline{\sigma}_j f(S) + \overline{\sigma}_j(-f)(S)$ implies $\overline{\sigma}_j f(S) = \infty = -\overline{\sigma}_j(-f)(S) = \underline{\sigma}_j f(S)$.

For item (2), from $-|f| \leq f \leq |f|$ it follows that $\overline{\sigma}_j(-|f|) \leq \overline{\sigma}_j f \leq \overline{\sigma}_j |f|$. From item (1): $-\overline{\sigma}_j |f|(S) \leq \overline{\sigma}_j(-|f|)(S)$ and so $|\overline{\sigma}_j f(S)| \leq \overline{\sigma}_j |f|(S)$.

For item (3); from Proposition 3.3 item (2): $0 \leq \overline{\sigma}_j |f|(S) \leq \overline{I}_j |f|(S) = 0$ (where the last equality holds by hypothesis). Then $\overline{\sigma}_j f(S) = 0$ by item (2), the latter applied to $-f$ gives $\underline{\sigma}_j f(S) = 0$. Moreover, since also $|f| = 0$ *a.e. on* $\mathcal{S}_{(S,j)}$ the first chain of equalities hold. For the remaining statements, it is enough to observe that $0 \leq f^-, f^+ \leq |f|$ which gives $f^-, f^+ = 0$ *a.e. on* $\mathcal{S}_{(S,j)}$ and so the same previous reasoning that we used for f applies to f^+ and f^- as well.

For item (4), from Proposition 3.3 item (4), it follows that $0 = \underline{\sigma}_j 0(S) \leq \underline{\sigma}_j g(S)$.

□

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Appendix C. Proofs for Subsections 3.2 and 3.3

C.1. **Establishing Property ($L_{(S,j)}$).** In this subsection, we provide the ramifications of the proof of Theorem 3.10 and the proof of Corollary 3.11. The key idea is to reduce the statements step by step to the case of a complete trajectory set and no portfolio restrictions.

Lemma C.1. *Fix a node (S,j). If $(L_{(S,j)})$ is satisfied in the case of no portfolio restrictions, then so it is for any set of portfolio restrictions satisfying $(H.1)$–$(H.4)$.*

Proof. Let $\overline{I}_{(S,j)}$ denote the operator in (2.4) with a set of portfolio restrictions satisfying $(H.1)$–$(H.4)$, and write $\overline{I}'_{(S,j)}$ for the corresponding operator without portfolio restrictions. Then, clearly $\overline{I}'_{(S,j)}f \leq \overline{I}_{(S,j)}f$, because the minimization in the former expression runs over a potentially larger set. Hence, $(L_{(S,j)})$ without portfolio restrictions implies $(L_{(S,j)})$ with portfolio restrictions. □

Lemma C.2. *Suppose the RFP (as per Definition 3.7) is satisfied at a node (S,j). Then $(L_{(S,j)})$ holds with respect to the original trajectory set $\mathcal{S}$, if $(L_{(S,j)})$ holds with respect to its completion $\overline{\mathcal{S}}$*

Proof. Let $f_0 \in \mathcal{E}_{(S,j)}$, $f_m \in \mathcal{E}^+_{(S,j)}$, $m \geq 1$, such that $\sum_{m=0}^{\infty} f_m(\tilde{S}) \geq 0$ for every $\tilde{S} \in \mathcal{S}_{(S,j)}$ and $\sum_{m=0}^{\infty} I_{(S,j)} f_m < \infty$. Let $\bar{S} \in \overline{\mathcal{S}_{(S,j)}}$ and $\{S^n\}_{n\geq 0}$ be a sequence as in (3.2). Note that the f_m's are also elementary functions with respect to the completion $\overline{\mathcal{S}_{(S,j)}}$ and $f_m(\bar{S}) = \lim_{n\to\infty} f_m(S^n)$. Indeed, if $f_m = \Pi^{V^m,H^m}_{j,k_m}$, then $f_m(S^n)$ becomes constant for $n \geq k_m$. Note also that $I_{(S,j)} f_m = V^m$ in $\mathcal{S}_{(S,j)}$ and in $\overline{\mathcal{S}_{(S,j)}}$. In view of Proposition 3.4, we need to show that $\sum_{m=0}^{\infty} V^m \geq 0$. By RFP,

$$\sum_{m=0}^{\infty} f_m(\bar{S}) = \sum_{m=0}^{\infty} \limsup_{n\to\infty} f_m(S^n) \geq \limsup_{n\to\infty} \sum_{m=0}^{\infty} f_m(S^n) \geq 0.$$

Hence, the inequality $\sum_{m=0}^{\infty} f_m(\tilde{S}) \geq 0$ extends from $\tilde{S} \in \mathcal{S}_{(S,j)}$ to $\tilde{S} \in \overline{\mathcal{S}_{(S,j)}}$. As $(L_{(S,j)})$ is satisfied in $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}}_{(S,j)} = \overline{\mathcal{S}_{(S,j)}}$, Proposition 3.4 implies $\sum_{m=0}^{\infty} V^m \geq 0$ and finishes the proof. □

The next lemma connects the global version of RFP to the local one.

Lemma C.3. *Suppose there are no portfolio restrictions. If $\mathcal{S}$ satisfies GRFP, then RFP holds at any node (S,j).*

Proof. Fix an arbitrary node (S,j), $j \geq 0$ and take $\overline{S} \in \overline{\mathcal{S}_{(S,j)}}$; therefore, $\overline{S} \in \overline{\mathcal{S}}$ as well. Consider $g_m(\hat{S}) = U^m(S) + \sum_{i=j}^{n_m-1} G^m_i(\hat{S})\, \Delta_i\hat{S}$, $m \geq 0$, $\hat{S} \in \mathcal{S}_{(S,j)}$, such that $g_0 \in \mathcal{E}_{(S,j)}, g_m \in \mathcal{E}^+_{(S,j)}$, $m \geq 1$, and $\sum_{m\geq 1} U^m(S) < \infty$.

Define, for all $m \geq 0$ and $\tilde{S} \in \mathcal{S}$: (i) $H^m_i(\tilde{S}) = 0$ whenever $0 \leq i < j$, (ii) let $H^m_i = G^m_i$ on $\mathcal{S}_{(S,j)}$ and $H^m_i = 0$ otherwise, whenever $j \leq i$. Also set the constants $V^m \equiv U^m(S)$ for all $m \geq 0$. Define $f_m(\tilde{S}) \equiv V^m + \sum_{i=0}^{n_m-1} H^m_i(\tilde{S})\, \Delta_i\tilde{S}$ for $\tilde{S} \in \mathcal{S}$ and notice $f_m \in \mathcal{E}^+$, $m \geq 1$, $f_0 \in \mathcal{E}$ and $\sum_{m\geq 1} V^m < \infty$.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Given that $\mathcal{S}$ satisfies the RFP and $\overline{S} \in \overline{\mathcal{S}}$, for any $n \geq 0$ there exists a sequence $\{S^n\}_{n\geq 0} \in \mathcal{S}$ such that $\overline{S} = \lim_{n\to\infty} S^n$, satisfying

$$\sum_{m\geq 0} \limsup_{n\to\infty} \sum_{i=0}^{n_m-1} H_i^m(S^n)\ \Delta_i S^n \geq \limsup_{n\to\infty} \sum_{m\geq 0} \sum_{i=0}^{n_m-1} H_i^m(S^n)\ \Delta_i S^n. \tag{C.1}$$

Define now $\tilde{S}^n \equiv S^{j+n}$ for $n \geq 0$, notice that since $S^{j+n} \in \mathcal{S}_{(S,j)}$ then $\tilde{S}^n \in \mathcal{S}_{(S,j)}$, and $\overline{S} = \lim_{n\to\infty} \tilde{S}^n$, this follows because $\overline{S}_n = S^n_n = S^{n+j}_n = \tilde{S}^n_n$.

Using (C.1), we compute as follows

$$\begin{aligned}
&\sum_{m\geq 0} \limsup_{n\to\infty} \sum_{i=j}^{n_m-1} G_i^m(\tilde{S}^n)\ \Delta_i \tilde{S}^n = \sum_{m\geq 0} \limsup_{n\to\infty} \sum_{i=j}^{n_m-1} G_i^m(S^{n+j})\ \Delta_i S^{j+n} \\
=\ &\sum_{m\geq 0} \limsup_{n\to\infty} \sum_{i=0}^{n_m-1} H_i^m(S^{n+j})\ \Delta_i S^{n+j} = \sum_{m\geq 0} \limsup_{n\to\infty} \sum_{i=0}^{n_m-1} H_i^m(S^n)\ \Delta_i S^n \\
\geq\ &\limsup_{n\to\infty} \sum_{m\geq 0} \sum_{i=0}^{n_m-1} H_i^m(S^n)\ \Delta_i S^n = \limsup_{n\to\infty} \sum_{m\geq 0} \sum_{i=0}^{n_m-1} H_i^m(S^{j+n})\ \Delta_i S^{j+n} \\
=\ &\limsup_{n\to\infty} \sum_{m\geq 0} \sum_{i=j}^{n_m-1} G_i^m(\tilde{S}^n)\ \Delta_i \tilde{S}^n.
\end{aligned}$$

This completes the proof. □

Proof of Corollary 3.11. Item (2) is a direct consequence of Theorem 3.10, because the model is assumed to have no-arbitrage nodes of type II. Item (1) can also be reduced to Theorem 3.10. First note that initial node $(S, 0)$ is not an arbitrage node of type II by assumption, and hence (L) holds under the assumptions of Theorem 3.10. In view of the latter theorem, it remains to show that

$$\mathcal{N}_{II} = \{S \in \mathcal{S} :\ (S, j) \text{ is an arbitrage node of type II for some } j \geq 1\}$$

is a global null set, if there are no portfolio restrictions. Define $f^{\pm,m}$, $m \geq 1$ via

$$f^{\pm,m}(S) = \sum_{j=0}^{m-1} \pm \mathbf{1}_{\{(S,j) \text{ is arbitrage node of type II and } \pm(\tilde{S}_{j+1}-S_j)>0 \text{ for every } \tilde{S}\in(S,j)\}} \cdot (S_{j+1}-S_j),$$

for every $S \in \mathcal{S}$. Then, $f_m = f^{+,m} + f^{-,m} \in \mathcal{E}_0^+$ for every $m \geq 1$ and $\sum_{m\geq 1} f_m = \infty \mathbf{1}_{\mathcal{N}_{II}}$. Hence, $\mathcal{N}_{II}$ is a global null set. □

Remark C.4. Under assumptions (H.1)–(H.4) it cannot be guaranteed that the functions f_m constructed in the previous proof belong to $\mathcal{E}_0$, see Example 3, where a similar issue arises at type I arbitrage nodes.

C.2. **Establishing Property ($K_{(S,j)}$).** In this subsection, we explain the ramifications of the proof of Theorem 3.12 and give the proof of Corollary 3.13.

The following lemma on the accumulation of portfolios is a variant of Lemma 3 in [13] and can be proved as in [13].

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Lemma C.5 (Aggregation Lemma)**.** *Suppose all nodes are 0-neutral. For any $m \geq 0$, let $H^m = \{H^m_i\}_{i \geq j}$, be sequences of non-anticipative functions on $\mathcal{S}$, and V^m functions defined on $\mathcal{S}$, depending for each S only on $S_0, ..., S_j$, $j \geq 0$ fixed. We fix a node (S, j) and assume:*

$$\Pi^{V^m, H^m}_{j,n}(\tilde{S}) = V^m(S) + \sum_{i=j}^{n_m - 1} H^m_i(\tilde{S}) \Delta_i \tilde{S} \geq 0, \ \tilde{S} \in \mathcal{S}_{(S,j)}, \ n_m \geq j \text{ and}$$

$\sum_{m \geq 1} V^m(S) < \infty$. *Define, for any $\tilde{S} \in \mathcal{S}_{(S,j)}$ and $k \geq j$:*

$$H_k(\tilde{S}) \equiv \begin{cases} \sum_{m \geq 1} H^m_k(\tilde{S}), & \text{whenever this series is convergent in } \mathbb{R} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any $\hat{S} \in \mathcal{S}_{(S,j)}$ satisfying

$$[\text{if } (\hat{S}, p) \text{ is an arbitrage node, } j \leq p, \text{ then it is of type I and } \hat{S}_{p+1} = \hat{S}_p],$$

the following holds for all k s.t. $j \leq k$:

$$\sum_{m=1}^{\infty} [H^m_k(\hat{S}) \Delta_k \hat{S}] = H_k(\hat{S}) \Delta_k \hat{S}$$

and

$$\sum_{m \geq 1} H^m_k(\hat{S}) \text{ converges whenever } (\hat{S}, k) \text{ is an up-down node.}$$

Proof of Corollary 3.13. Under both sets of assumption (1) or (2), the reversed Fatou property is satisfied and the trajectory set has no-arbitrage nodes of type II. Hence, by Theorem 3.10, $(L_{(S,j)})$ holds at every node. It remains to apply Theorem 3.12. □

Appendix D. Auxiliary Results and Proofs for Section 4

The next lemma considers the linear property of the integral at a point S where the necessary integrability conditions hold. This shows that the said property is local in the specified sense.

Lemma D.1. *Let $f, g \in Q$ and consider a fixed node (S, j). If: $\overline{\sigma}_j f(S) - \underline{\sigma}_j f(S) = 0 = \overline{\sigma}_j g(S) - \underline{\sigma}_j g(S)$, then all the involved quantities are finite and*

$$(a) \quad \overline{\sigma}_j(cf)(S) = c\overline{\sigma}_j f(S) = c\underline{\sigma}_j f(S) = \underline{\sigma}_j(cf)(S) \quad \forall c \in \mathbb{R},$$

Furthermore if property $(L_{(S,j)})$ holds:

$$(b) \quad \overline{\sigma}_j(f+g)(S) = \overline{\sigma}_j f(S) + \overline{\sigma}_j g(S) = \underline{\sigma}_j f(S) + \underline{\sigma}_j g(S) = \underline{\sigma}_j(f+g)(S).$$

Proof. The finiteness claims follow from our conventions on computing with $\pm\infty$ (introduced right before Definition 2.6). We then see that the hypotheses imply that $\overline{\sigma}_j f(S) = \underline{\sigma}_j f(S)$ and $\overline{\sigma}_j g(S) = \underline{\sigma}_j g(S)$.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

For (a), if $c = 0$ or $c = -1$ the result is clear. For $c > 0$ it follows from item (5) of Proposition 3.3, from where, if $c < 0$

$$\overline{\sigma}_j(cf)(S) = \overline{\sigma}_j(-c(-f))(S) = -c\overline{\sigma}_j(-f)(S) = c\overline{\sigma}_j f(S).$$

(b) holds from

$$\overline{\sigma}_j f(S)+\overline{\sigma}_j g(S) = \underline{\sigma}_j f(S)+\underline{\sigma}_j g(S) \leq \underline{\sigma}_j[f+g](S) \leq \overline{\sigma}_j[f+g](S) \leq \overline{\sigma}_j f(S)+\overline{\sigma}_j g(S),$$

where we relied on Corollary 3.5, item (1), for the second inequality. □

Remark D.2. For $f \in Q$ and $S \in \mathcal{S}$ fixed, $-\infty < \overline{\sigma}_j f(S) = \underline{\sigma}_j f(S) < \infty$ is equivalent to $\overline{\sigma}_j f(S) - \underline{\sigma}_j f(S) = 0$. This will be used implicitly several times to apply Lemma D.1 or to justify that $f \in \mathcal{L}_{\overline{\sigma}_j}$.

Proof of Proposition 4.1. Assertion (1) follows from Proposition 3.4 item (4). For assertion (2), notice that Proposition 3.3 item (4) implies $\overline{\sigma}_j f(S) = \overline{\sigma}_j g(S)$ and this holds a.e. (in S). The same reasoning applied to $-f, -g$ gives the validity of $\underline{\sigma}_j f(S) = \underline{\sigma}_j g(S)$ a.e. Given that $\overline{\sigma}_j f(S) - \underline{\sigma}_j f(S) = 0$ holds a.e, it follows that $\overline{\sigma}_j g(S) - \underline{\sigma}_j g(S) = 0$ holds a.e. and $g \in \mathcal{L}_{\overline{\sigma}_j}$ is then established. Notice that $\int_j f \equiv \overline{\sigma}_j f \doteq \overline{\sigma}_j g \equiv \int_j g$ which completes the proof of (2).

For (3), $f \in \mathcal{L}_{\overline{\sigma}_j}$ gives $\overline{\sigma}_j f - \underline{\sigma}_j f \doteq 0$ then, (a) from Lemma D.1 implies $\overline{\sigma}_j(cf) - \underline{\sigma}_j(cf) \doteq 0$. Hence $cf \in \mathcal{L}_{\overline{\sigma}_j}$. For $c \geq 0$ $\int_j cf = c\int_j f$ follows from Proposition 3.3 item (5) and our standing assumption, on the other hand if $c \leq 0$: $\int_j cf = \overline{\sigma}_j cf = |c|\ \overline{\sigma}_j(-f) = -c\int_j(-f) = c\underline{\sigma}_j f \doteq c\overline{\sigma}_j f = c\int_j f$.

Finally, to establish (4), $f, g \in \mathcal{L}_{\overline{\sigma}_j}$ gives $\overline{\sigma}_j f - \underline{\sigma}_j f \doteq 0$ and $\overline{\sigma}_j g - \underline{\sigma}_j g \doteq 0$ then, (b) from Lemma D.1 implies $\overline{\sigma}_j(f+g) - \underline{\sigma}_j(f+g) \doteq 0$ and so $(f+g) \in \mathcal{L}_{\overline{\sigma}_j}$. Moreover, the same result also gives $\int_j(f+g) = \overline{\sigma}_j(f+g) \doteq \overline{\sigma}_j f + \overline{\sigma}_j g = \int_j f + \int_j g$. □

Proof of Proposition 4.5*:* We argue as follows, for fixed $N \geq 0$

$$\sum_{m=1}^{N} \underline{\sigma}_j f_m(S) \leq \underline{\sigma}_j(\sum_{m=1}^{N} f_m)(S) \leq \underline{\sigma}_j v(S) \leq \overline{\sigma}_j v(S) \leq \overline{I}_j v(S) \leq \sum_{m\geq 1} \overline{I}_j f_m(S) \leq \sum_{m\geq 1} I_j f_m(S), \tag{D.1}$$

where the first inequality holds by superadditivity of $\underline{\sigma}_j$, the second one by isotony, the third one from Corollary 3.5 item (1) and the validity of $(L_{(S,j)})$ and the rest by Proposition 3.1.

From (D.1) we can conclude (4.1): item (4) in Proposition 3.4, available as $(L_{(S,j)})$ holds, gives $\sum_{m=1}^{N} I_j f_m(S) = \sum_{m=1}^{N} \underline{\sigma}_j f_m(S)$, then (4.1) follows by taking $N \to \infty$.

From (4.1) and (L_j)-a.e., if $v \in \mathcal{M}_j$ then $0 \leq \underline{\sigma}_j v = \overline{\sigma}_j v < \infty$ *a.e.*, thus $\overline{\sigma}_j v - \underline{\sigma}_j v = 0$ *a.e.* and, being non-negative, $v \in \mathcal{L}^+_{\overline{\sigma}_j}$, consequently last equality in (4.1) and Proposition 3.4 item (4) (both results available a.e. as per property (L_j)-a.e.) imply $\int_j v \doteq \sum_m \int_j f_m$.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Finally, since $v = \sum_{m=1}^{\infty} f_m$ and $\sum_{m=1}^{\infty} I_j f_m(S) < \infty$, then for a given $\epsilon > 0$, there exist m_ν such that $\sum_{m>m_v}^{\infty} I_j f_m(S) \leq \epsilon$, then $v = h + u$, where $h = \sum_{m=1}^{m_v} f_m \in \mathcal{E}^{+}_{(S,j)}$ and $u = \sum_{m>m_v}^{\infty} f_m$ satisfies $\overline{I}_j u(S) \leq \sum_{m>m_v}^{\infty} I_j f_m(S) \leq \epsilon$, hence $u \in \mathcal{M}_j(S)$. □

Proof of Corollary 4.7*:* (4.1) in Proposition 4.5 implies the first and last equalities in (1) given that $u,\ v$,
$u + \alpha v \in \mathcal{M}_j(S)$. The second equality in (1) follows from Lemma D.1 by taking $f \equiv u$, $g \equiv v$; this result is applicable given that by Proposition 4.5 and hypothesis $0 \leq \underline{\sigma}_j u(S) = \overline{\sigma}_j u(S) < \infty$ so $\overline{\sigma}_j u(S) - \underline{\sigma}_j u(S) = 0$ (similarly for v).

To derive (2), by Proposition 3.4 item (4) $\overline{\sigma}_j h(S) = \underline{\sigma}_j h(S)$, and (4.1) in Proposition 4.5 gives $\overline{\sigma}_j u(S) = \underline{\sigma}_j u(S)$ and $\overline{\sigma}_j v(S) = \underline{\sigma}_j v(S)$. Since in the three cases the involved values are finite, h, v, u satisfy the hypothesis of Lemma D.1, from where $\overline{\sigma}_j (h+v)(S) = \underline{\sigma}_j (h+v)(S)$, so for the same reason $h + v$ satisfies the referred hypothesis. Consequently

$$\overline{\sigma}_j (h+v-u)(S) = \overline{\sigma}_j (h+v)(S) - \overline{\sigma}_j u(S) = \overline{\sigma}_j h(S) + \overline{\sigma}_j v(S) - \overline{\sigma}_j u(S).$$

From where, by Proposition 3.4 item (4) and (4.1) in Proposition 4.5 again,

$$\overline{\sigma}_j (h+v-u)(S) = I_j h(S) + \overline{I}_j v(S) - \overline{I}_j u(S) = \underline{\sigma}_j h(S) + \underline{\sigma}_j v(S) - \underline{\sigma}_j u(S).$$

And another use of Lemma D.1 gives the first equality in (2).

$\mathcal{E}_j + \mathcal{M}_j - \mathcal{M}_j \subseteq \mathcal{L}_{\overline{\sigma}_j}$ as well as (4.2) follow directly from (2) (which is available a.e. given our assumption that (L_j)-a.e. holds) just proven and finiteness of the involved values. □

Proof of Corollary 4.10*:* For $f \in \mathcal{L}_j$, property (L_j)-a.e. permits to find a set $A \subseteq \mathcal{S}$, with null complement, such that $f = v - u$ a.e. on $\mathcal{S}_{(S,j)}$ as well as $(L_{(S,j)})$ both hold for all $S \in A$. Let $S \in A$, by Proposition 3.3 item (4) and Corollary 4.7 item (2) it follows that

$$\overline{\sigma}_j f(S) = \overline{\sigma}_j [v-u](S) = \underline{\sigma}_j [v-u](S) = \underline{\sigma}_j f(S).$$

Moreover since $u, v \in \mathcal{M}_j(S)$, again by Corollary 4.7 item (2),

$$|\overline{\sigma}_j [v-u](S)| = |\underline{\sigma}_j [v-u](S)| = |\overline{I}_j v(S) - \overline{I}_j u(S)| < \infty.$$

Thus $\overline{\sigma}_j f - \underline{\sigma}_j f = 0$ *a.e.* so, f being non-negative by definition of $\mathcal{L}_j$, $f \in \mathcal{L}^{+}_{\overline{\sigma}_j}$.

$\underline{\sigma}_j f \dot{=} \overline{\sigma}_j f$ for $f \in \mathcal{L}^K_j$ follows from Corollary 4.7. In particular, $\mathcal{L}^K_j \subseteq \mathcal{L}_{\overline{\sigma}_j}$ then follows.

To establish $\mathcal{L}_j - \mathcal{L}_j \subseteq \mathcal{L}^K_j$, consider $f = f_1 - f_2 \in \mathcal{L}_j - \mathcal{L}_j$. Let $S \in \mathcal{S}$ be such that for all $\epsilon > 0$ there exist u, v, u', v' in $\mathcal{M}_j(S)$, such that: $f_1 = (v-u)$, $f_2 = (v'-u')$ a.e. on $\mathcal{S}_{(S,j)}$ with $\overline{I}_j u(S) \leq \epsilon/2, \overline{I}_j u'(S) \leq \epsilon/2$. Given that $\overline{I}_j v(S) < \infty, \overline{I}_j v'(S) < \infty$ as well, we can assume (by relying on (2) from Proposition 3.2) that u, v, u', v' are finite on the set where the equalities $f_1 = (v-u)$, $f_2 = (v'-u')$ take place a.e. on $\mathcal{S}_{(S,j)}$.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Writing $v = \sum_k v_k,\ v' = \sum_k v'_k,\ v'_k, v_k \in \mathcal{E}_j^+$, we can then find n_1, n_2 such that $\overline{I}_j(\sum_{k>n_1} v_k(S)) \le \epsilon/2$ and $\overline{I}_j(\sum_{k>n_2} v'_k(S)) \le \epsilon/2$. Let $h_1 \equiv \sum_{k=1}^{n_1} v_k$, $h_2 \equiv \sum_{k=1}^{n_2} v'_k$,

From last assertion of Proposition 4.5 we can find $h, h' \in \mathcal{E}_j^+$, $v_1, v'_1 \in \mathcal{M}_j(S)$ such that $v = h + v_1, v' = h' + v'_1$ and $\overline{I}_j v_1(S), \overline{I}_j v'_1(S) \le \epsilon/2$, therefore $f = \tilde{v} - \tilde{u} + h$ a.e. on $\mathcal{S}_{(S,j)}$ where $\tilde{v} \equiv v_1 + u'$, $\tilde{u} \equiv v'_1 + u$ and $h \equiv h_1 - h_2 \in \mathcal{E}_j$. Given that the above properties hold a.e. in the originally chosen $S \in \mathcal{S}$ we have established $f \in \mathcal{L}_j^K$. ☐

Proof of Proposition 4.12: [the proof of (1) ⇒ (2) is from [16, Behauptung 1.8] but in the conditional setting]

Proof. (1) ⇒ (2). By Proposition 3.3 item (2), it is enough to prove that $\overline{I}_j f \le \overline{\sigma}_j f$ for $f \in P$. Fix $S \in \mathcal{S}$ and $f_0 \in \mathcal{E}_{(S,j)}$ and for $m \ge 1$, $f_m \in \mathcal{E}^+_{(S,j)}$, such that $f \le \sum_{m\ge 0} f_m$ on $\mathcal{S}_{(S,j)}$. It is enough to prove that $\overline{I}_j f(S) \le \sum_{m\ge 0} I_j f_m(S)$, we can then assume $\sum_{m\ge 0} I_j f_m(S) < \infty$. For $n > 0$ it holds that

$$f \le [\sum_{m=0}^{n} f_m]^+ + \sum_{m\ge n+1} f_m \text{ on } \mathcal{S}_{(S,j)}, \text{ and } \overline{I}_j f(S) \le \overline{I}_j[\sum_{m=0}^{n} f_m]^+(S) + \overline{I}_j[\sum_{m\ge n+1} f_m](S).$$

Since $\sum_{m=0}^{n} f_m \in \mathcal{E}_{(S,j)}$, by property (1)

$$\overline{I}_j[\sum_{m=0}^{n} f_m]^+(S) = \sum_{m=0}^{n} I_j f_m(S) + \overline{I}_j[\sum_{m=0}^{n} f_m]^-(S).$$

On the other hand, from $\sum_{m\ge 0} f_m \ge f \ge 0$ it follows that

$$(\sum_{m=0}^{n} f_m)^- = (-\sum_{m=0}^{n} f_m)^+ \le (\sum_{m\ge n+1} f_m)^+ = \sum_{m\ge n+1} f_m.$$

Summarizing we have

$$\overline{I}_j f(S) \le \sum_{m=0}^{n} I_j f_m(S) + 2\overline{I}_j[\sum_{m\ge n+1} f_m](S) \le \sum_{m=0}^{n} I_j f_m(S) + 2\sum_{m\ge n+1} I_j f_m(S),$$

where countable subadditivity of $\overline{I}_j$ and Proposition 3.4 item (6) were used. Taking limits for $n \to \infty$, (2) follows.

The implication (2) ⇒ (3) is trivial. It remains to prove the implication (3) ⇒ (1). Then, for fixed $f \in \mathcal{E}_j$ we may assume that $\overline{I}_j f^+ < \infty$. Consider $f_m \in \mathcal{D}_j^+$ such that $f^+ \le \sum_{m\ge 1} f_m$. As $f_0 \equiv -f \in \mathcal{E}_j$, it follows that $f^- \le \sum_{m\ge 0} f_m$ thus: $\overline{\sigma}_j f^- \le -I_j(f) + \overline{I}_j f^+$. Since by (3) $\overline{\sigma}_j f^- = \overline{I}_j f^-$, one inequality of (1) is obtained. The other inequality follows by applying the said inequality to $-f$. ☐

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Appendix E. Global extensions

The following result will give us the tools to extend functions defined by local conditions (i.e. in terms of the conditional spaces $\mathcal{S}_{(S,j)}$) to functions defined directly on $\mathcal{S}$ (i.e. *globally* defined).

A subset $\mathcal{C} \subseteq \mathcal{S}$ will be called *admissible* if it satisfies: whenever $S^1, S^2 \in \mathcal{C}$ then $\mathcal{S}_{(S^1,j)} \cap \mathcal{S}_{(S^2,j)} = \emptyset$. For a fixed $j \geq 0$ and a given admissible subset $\mathcal{C} \subseteq \mathcal{S}$ we will use the notation $w(\cdot)$ for a choice function of the following type:

$w : \widetilde{\mathcal{C}} \equiv \cup_{S\in\mathcal{C}} \mathcal{S}_{(S,j)} \to \mathcal{C}$ such that if we let $S \equiv w(\tilde{S})$ we have $\tilde{S} \in \mathcal{S}_{(S,j)}$.
Notice that $\mathcal{S}_{(w(\tilde{S}),j)} = \mathcal{S}_{(\tilde{S},j)}$ for any $\tilde{S} \in \widetilde{\mathcal{C}}$, and that $w(\widetilde{\mathcal{C}}) = \mathcal{C}$.

Lemma E.1. *Fix $j \geq 0$ and let $\mathcal{C}$ be admissible as defined above. Let $(h^S)_{S\in\mathcal{C}}$ and $(v^S)_{S\in\mathcal{C}}$ be families of functions where $h^S \in \mathcal{E}_{(S,j)}$, $v^S \in \mathcal{M}_j(S)$ for $S \in \mathcal{C}$ and set $\widetilde{\mathcal{C}} \equiv \cup_{S\in\mathcal{C}} \mathcal{S}_{(S,j)}$. Then, there exist $h \in \mathcal{E}_j$, $v \in \cap_{S\in\widetilde{\mathcal{C}}} \mathcal{M}_j(S)$ such that the following holds for all $S \in \widetilde{\mathcal{C}}$:*

$$h = h^{w(S)} \ \text{ on } \mathcal{S}_{(S,j)} \quad \text{ and } \quad I_j h(S) = I_j h^{w(S)}(S), \tag{E.1}$$

$$v = v^{w(S)} \ \text{ on } \mathcal{S}_{(S,j)} \quad \text{ and } \quad \overline{I}_j v(S) = \overline{I}_j v^{w(S)}(S). \tag{E.2}$$

Proof. We will prove (E.2), the proof of (E.1) is similar. For each $S \in \mathcal{C}$, $v^S = \sum_{m=1}^{\infty} f_m^S$, $f_m^S \in \mathcal{E}_j^+$, so by Remark 2.12 $f_m^S|_{\mathcal{S}_{(S,j)}} \in \mathcal{E}^+_{(S,j)}$. Keeping in mind that $\mathcal{C} \subseteq \widetilde{\mathcal{C}}$, define for $m \geq 1$:

$$f_m(S) = \begin{cases} f_m^{w(S)}(S) & if \quad S \in \widetilde{\mathcal{C}} \\ 0 & otherwise \end{cases} \quad \text{and} \quad v \equiv \sum_{m=1}^{\infty} f_m.$$

Then, for $S \in \widetilde{\mathcal{C}}$, $f_m|_{\mathcal{S}_{(S,j)}} = f_m^{w(S)}|_{\mathcal{S}_{(S,j)}} \in \mathcal{E}^+_{(S,j)}$ and so $v = v^{w(S)}$ on $\mathcal{S}_{(S,j)}$. On the other hand, if $S \notin \widetilde{\mathcal{C}}$ we have $f_m|_{\mathcal{S}_{(S,j)}} = 0 \in \mathcal{E}^+_{(S,j)}$, therefore, $f_m \in \mathcal{E}_j^+$.

Moreover, if $S \in \widetilde{\mathcal{C}}$, from Proposition 4.5

$$\overline{I}_j v(S) = \sum_{m=1}^{\infty} I_j f_m(S) = \sum_{m=1}^{\infty} I_j f_m^{w(S)}(S) = \overline{I}_j v^{w(S)}(S) = \overline{I}_j v^{w(S)}(w(S)) < \infty,$$

where we used Corollary A.3 for the second equality and the fact that $\overline{I} v^{w(S)}$ is constant on $\mathcal{S}_{(w(S),j)}$, and $S \in \mathcal{S}_{(w(S),j)}$ for the last equality. The last inequality follows from the assumption $v^S \in \mathcal{M}_j(S)$ for any $S \in \mathcal{C}$. We then conclude that $v \in \mathcal{M}_j(S)$ for each $S \in \widetilde{\mathcal{C}}$. □

The next proposition establishes global formulations for the sets $\mathcal{L}_j$ and $\mathcal{L}_j^K$. The result is used in Theorem 4.15 to obtain that the global formulation applies to integrable functions as well. The sets $\mathcal{L}_j^G, \mathcal{L}_j^{K,G}$, appearing below, were introduced in Definition 4.3.

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

Proposition E.2.

$$\mathcal{L}_j = \mathcal{L}_j^G,\ \mathcal{L}_j^K = \mathcal{L}_j^{K,G}.$$

Proof. We only prove $\mathcal{L}_j = \mathcal{L}_j^G$, the proof of $\mathcal{L}_j^K = \mathcal{L}_j^{K,G}$ is analogous.

Let $f \in \mathcal{L}_j^G$ and $\epsilon > 0$, then there exist $u, v \in \mathcal{M}_j$ such that for *a.e.* $S \in \mathcal{S}$ it holds that $f = v - u$ *a.e.* on $\mathcal{S}_{(S,j)}$ and $\overline{I}u(S) \leq \epsilon$, $\overline{I}v(S) < \infty$. Thus, for the referred $S \in \mathcal{S}$, u, v (the same for any S) verify the conditions which gives that $f \in \mathcal{L}_j$.

Let now $f \in \mathcal{L}_j$ then, there exists $\hat{\mathcal{S}} \subset \mathcal{S}$, with $\hat{\mathcal{S}}^c$ a null set, such that for each $S \in \hat{\mathcal{S}}$ we have: for any $\epsilon > 0$ there exist $u^S, v^S \in \mathcal{M}_j(S)$ satisfying

$$f = v^S - u^S \ \ a.e. \text{ on } \mathcal{S}_{(S,j)} \ \text{ and } \ \overline{I}_j u^S(S) < \epsilon.$$

We can construct $\mathcal{C} \subset \hat{\mathcal{S}}$ that satisfies $\tilde{\mathcal{C}} \equiv \cup_{S\in\mathcal{C}}\mathcal{S}_{(S,j)} = \cup_{S\in\hat{\mathcal{S}}}\mathcal{S}_{(S,j)}$ and $\mathcal{S}_{(S^1,j)} \cap \mathcal{S}_{(S^2,j)} = \emptyset$ for $S^1, S^2 \in \mathcal{C}$. Thus, the families of functions $(h^S)_{S\in\mathcal{C}}$, $(u^S)_{S\in\mathcal{C}}$ and $(v^S)_{S\in\mathcal{C}}$ satisfy the corresponding hypothesis of Lemma E.1, which gives the existence of $u, v \in \cap_{S\in\tilde{\mathcal{C}}}\mathcal{M}_j(S)$ such that for any $S \in \tilde{\mathcal{C}}$ (where $w(S)$ below is as introduced in Lemma E.1)

$$f = v^{w(S)} - u^{w(S)} = v - u \text{ a.e. on } \mathcal{S}_{(S,j)} \text{ and}$$

$$\overline{I}u(S) = \overline{I}u^{w(S)}(S) \leq \epsilon,\ \overline{I}v(S) = \overline{I}v^{w(S)}(S) < \infty.$$

Notice that $\hat{\mathcal{S}} \subseteq \cup_{S\in\hat{\mathcal{S}}}\mathcal{S}_{(S,j)}$, it then follows that $\tilde{\mathcal{C}}^c = \cap_{S\in\hat{\mathcal{S}}}\mathcal{S}^c_{(S,j)}$ is a null set; therefore, $u, v \in \mathcal{M}_j$ which allows us to conclude $f \in \mathcal{L}_j^G$. □

Appendix F. Additional examples

The next two examples illustrate the role of the reversed Fatou property (RFP). The first of them (closely related to the up-branch in Example 1) shows how Leinert's condition fails, because RFP is not valid. The second one provides a trajectorially incomplete trajectory space, in which RFP is valid and, then, implies Leinert's condition.

Example 5. In this first example $(S, 0)$ is an arbitrage node of type I but we have failure of $(L_{(S,0)})$. This is explained by the failure of the RFP (see Definition 3.7).

Let $\mathcal{S}$ the trajectory space composed by the following trajectories, see Figure 3:

$$S^n,\ n \in \mathbb{N}: \quad S_i^n = 1,\ \ 0 \leq i < n; \qquad S_i^n = 2\ \ i \geq n.$$

Here $\overline{\sigma}_0 0 = -\infty$, and we will show that the reverse Fatou condition fails (which is what should happen by Theorem 3.10).

We note that $\overline{\mathcal{S}}\backslash\mathcal{S} = \{\overline{S} : \overline{S}_i \equiv 1,\ i \geq 0\}$, and we will show that the RFP fails for the portfolios: $V^m = 0$, $H_m^m(S) = 1$ if $S_i = 1$ for $0 \leq i \leq m$, $H_i^m(S) = 0$ otherwise, therefore $n_m = m + 1$. Then $f_m(S^n) = (S_{n+1}^n - S_n^n) = 1$ if $m = n$ and $f_m(S^n) = 0$

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4351.

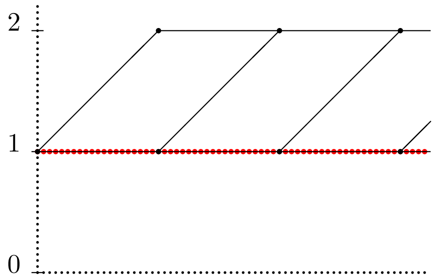

Figure 3. Trajectory set for Example 5.

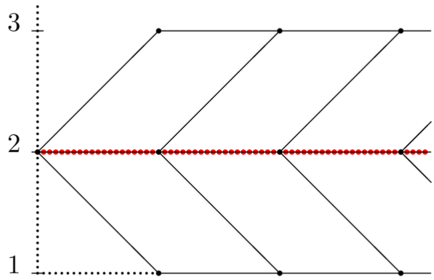

Figure 4. Trajectory set for Example 6.

if $m \neq n$. Therefore $\sum_{m\geq 1} f_m(S) = 1$ for all $S \in \mathcal{S}$, whereas $\sum_{m\geq 1} f_m(\bar{S}) = 0$. Thus, for every sequence $\{S^{(k)}\}_{k\in\mathbb{N}}$ in $\mathcal{S}$ satisfying $\lim_{k\to\infty} S^{(k)} = \bar{S}$, we obtain

$$\limsup_{k\to\infty} \sum_{m\geq 1} f_m(S^{(k)}) = 1 > 0 = \sum_{m\geq 1} f_m(\bar{S}) = \sum_{m\geq 1} \limsup_{k\to\infty} f_m(S^{(k)}),$$

i.e. RFP fails.

Example 6. In this example the trajectory space has no-arbitrage nodes of type II. We will show that the RFP holds, and it then follows from Theorem 3.10 that $(L_{(S,j)})$ holds at all nodes. The point of this example is that $\mathcal{S}$ is not complete.

Consider the example with $\mathcal{S}$ given in the Figure 4. We will verify that Leinert's condition holds at every node by an application of Theorem 3.10 by checking that the reversed Fatou condition holds for this example. The trajectories are

$$S^{u,n},\ n\in\mathbb{N}: \quad S^{u,n}_i = 2,\ \ 0\leq i<n; \qquad S^{u,n}_i = 3\ \ i\geq n,\ \text{ and}$$

$$S^{d,n},\ n\in\mathbb{N}: \quad S^{d,n}_i = 2,\ \ 0\leq i<n; \qquad S^{d,n}_i = 1\ \ i\geq n.$$

We note that $\overline{\mathcal{S}}\setminus\mathcal{S} = \{\overline{S} : \overline{S}_i = 2,\ i\geq 0\}$, we will consider the sequence $S^n \equiv (2,2,\ldots,2,*,*,*,\ldots)$, where the last appearing 2 is the nth coordinate of the

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

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sequence and the entry value $*$ will be equal to 3 or 1 (i.e. $S^n_i = 3$ for all $i \geq n+1$ or $S^n_i = 1$ for all $i \geq n+1$), which value to be chosen, 3 or 1, will depend on n as well as on a given sequence $f_m \in \mathcal{E}^+,\ \ m \geq 1$ and $f_0 \in \mathcal{E}$. The sequence S^n to be constructed is a variation from the usual contrarian trajectory construction, now, a contrarian move is used to define S^n but discarded in S^{n+1} given that in fact we keep the flat trajectory (which, of course, also acts as a contrarian trajectory) converging to $\overline{S}$. The construction of S^n is completed next.

Given $f_0 \in \mathcal{E},\ f_m \in \mathcal{D}^+\ m \geq 1$ with $\sum_{m\geq 1} V^m < \infty$; notice that $H^m_i(S^n)(S^n_{i+1} - S^n_i) = 0$ if $i \neq n$, so $\sum_{m\geq 1} H^m_i(S^n)(S^n_{i+1} - S^n_i) = 0$ if $i \neq n$ and $f_m(S^n) = V^m$ if $n \geq n_m$.

Set $H_i(S) \equiv H^0_i(S) + \sum_{m\geq 1} H^m_i(S)$ which exists by an application lemma C.5 (in fact we only need to apply the lemma along $\overline{S}$, i.e. we only need $H_i(\overline{S}) = \sum_{m\geq 0} H^m_i(\overline{S})$). Depending if $H_n(\overline{S}) \leq 0$ choose $S^n_i = * = 1$ for $i \geq n+1$ or if $H_n(\overline{S}) > 0$ choose $S^n_i = * = 3$ for $i \geq n+1$. (e.g. if $H_3(\overline{S}) \leq 0$, $S^3 \equiv S^{d,4} = (2,2,2,2,1,1,1,\ldots)$, if $H_4(\overline{S}) > 0$, $S^4 \equiv S^{u,5} = (2,2,2,2,2,3,3,3,\ldots)$ etc). Therefore, using the standard Fatou's lemma,

$$\sum_{m\geq 0} f_m(S^n) = \sum_{m\geq 0}[V^m + \sum_{i=0}^{n_m-1} H^m_i(S^n)\Delta_i S^n] \leq \tag{F.1}$$

$$\liminf_{p\to\infty}[\sum_{m\geq 0} V^m + \sum_{i=0}^{p}\sum_{m\geq 0} H^m_i(S^n)\Delta_i S^n] =$$

$$\liminf_{p\to\infty}[\sum_{m\geq 0} V^m + \mathbf{1}_{p\geq n}\sum_{m\geq 0} H^m_n(S^n)(S^n_{n+1} - S^n_n))] =$$

$$\sum_{m\geq 0} V^m + H_n(S^n)(S^n_{n+1} - S^n_n) \leq \sum_{m\geq 0} V^m.$$

Inequality (F.1), combined with the fact that $\limsup_{n\to\infty} f_m(S^n) = V^m$, allows us to check the reversed Fatou condition directly:

$$\limsup_{n\to\infty}\sum_{m\geq 0} f_m(S^n) \leq \sum_{m\geq 0} V^m = \sum_{m\geq 0}\limsup_{n\to\infty} f_m(S^n).$$

Then, Theorem 3.10 is applicable to this example.

The final example illustrates how integrable functions may arise in the limit as time goes to infinity.

Example 7. As mentioned in Remark 2.15 the space of integrable functions in our setting coincides with a classical L^1-space for binary trees. In this example, we provide some explicit examples of integrable functions for a trinomial model. We make no claim of generality but provide the examples as illustration.

The first split is binomial and with $s_0 = \frac{1}{2}$ and S_n dyadic numbers in $[0,1]$, as

$$S^0_n = \frac{1}{2^{n+1}};\quad S^1_n = \frac{2^{n+1}-1}{2^{n+1}},\ n \geq 0.$$

Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
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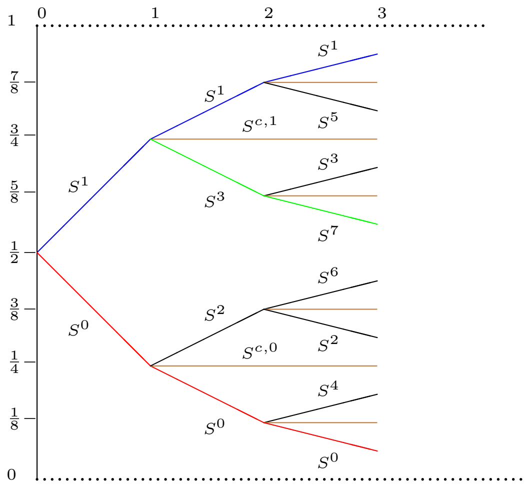

FIGURE 5. Trajectory set for Example 6.

$$S_0^2 = \frac{1}{2},\ S_1^2 = \frac{1}{4},\ S_n^2 = \frac{2^{n-1}+1}{2^{n+1}}, \quad n \geq 2.$$

$$S_0^3 = \frac{1}{2},\ S_1^3 = \frac{3}{4},\ S_n^3 = \frac{3\ 2^{n-1}-1}{2^{n+1}}, \quad n \geq 2,$$

and we expect it is clear how the construction continues. Instead of providing a detailed formal description we illustrate the trajectory set in Figure 5 so that the full trajectory set can be easily envisioned. The model has the trajectories of a binomial model, adding also constant trajectories at each node, except at the initial one. It follows that $|S_n - S_{n-1}| \leq \frac{1}{2^{n+1}}, n \geq 1$, so the limit $\lim_{n\to\infty} S_n = S_T$ exists.

For any $c, K \in \mathbb{R}$ with $0 < K \leq 2\ |c|$, and $a = (a_i)_{i\geq 0}$, $a_i : \mathcal{S} \to \mathbb{R}$ non-anticipative functions such that $|a_i| \leq K$ (i.e. a_i is constant in each subspace $\mathcal{S}_{(S,i)}$). Also assume that $a \in \mathcal{H}$. Define $h_n(S) = c + \sum_{i=0}^{n-1} a_i(S)\Delta_i S$, $S \in \mathcal{S}$. Knowing that $-\frac{1}{2^{i+2}} \leq \Delta_i S \leq \frac{1}{2^{i+2}}$ it follows that for $c > 0$ and any $S \in \mathcal{S}$,

$$h_n(S) \geq c - \sum_{i=0}^{n-1} \frac{|a_i(S)|}{2^{i+2}} \geq c - \frac{1}{4}\sum_{i=0}^{n-1} \frac{K}{2^i} = c - \frac{K}{2}(1 - \frac{1}{2^n}) > c - \frac{K}{2} \geq 0.$$

In a similar way, for $c < 0$, $h_n \leq 0$. Then $h_n \in \mathcal{D}^+$ or $\mathcal{D}^-$, so $h_n \in \mathcal{E} \subset \mathcal{L}_{\bar{\sigma}}$. For any $S \in \mathcal{S}$ and $m > n$

$$|h_m(S) - h_n(S)| \leq \sum_{i=n}^{m-1} |a_i(S)|\ |\Delta_i S| \leq K \sum_{i=n}^{m-1} \frac{1}{2^{i+2}} \leq \frac{K}{4^n} \sum_{i=0}^{m-n-1} \frac{1}{2^{i+2}} < \frac{K}{2^n},$$

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consequently the sequence $(h_n)_{n\geq 0}$ converges uniformly. Define

$$h(S) = c + \sum_{i=0}^{\infty} a_i(S)\Delta_i S, \quad S \in \mathcal{S}. \tag{F.2}$$

The partial sums are $h_n(S)$, then $(h_n)_{n\geq 0}$ converges to h in $\|.\|$ (the norm defined on $\mathcal{S}$), so, by Theorem 4.3, $h \in \mathcal{L}_{\bar{\sigma}}$. We also comment that a European call or put with strike 1/2 is also integrable.

Noticing that the portfolios h_n can be seen as piecewise constant functions of S_T, we remark that they can be represented in terms of Haar-like expansions. We just introduce the first four of them for illustration.

$h_0(S) = c$, so $h_0(S) = c\phi_{00}(S)$ where $\phi_{00} = \mathbf{1}_{\mathcal{S}}$.

$$h_1(S) - h_0(S) = \tfrac{a_0}{4}\begin{cases} 1, & S \in \mathcal{S}_{(S^1,1)} \\ -1 & S \in \mathcal{S}_{(S^0,1)} \end{cases} = \tfrac{a_0}{4}\psi_{00}(S)$$

$$h_2(S) - h_1(S) = \begin{cases} \tfrac{a_1(S^1)}{8}\psi_{11}(S) \\ \\ \tfrac{a_1(S^0)}{8}\psi_{10}(S) \end{cases}, \text{ where}$$

$$\psi_{11}(S) = \begin{cases} 1, & S \in \mathcal{S}_{(S^1,2)} \\ 0, & S = S^{c,1} \\ -1, & S \in \mathcal{S}_{(S^3,2)} \\ 0, & S \in \mathcal{S}_{(S^0,1)} \end{cases} ; \quad \psi_{10}(S) = \begin{cases} 0, & S \in \mathcal{S}_{(S^1,1)} \\ 1, & S \in \mathcal{S}_{(S^2,2)} \\ 0, & S = S^{c,0} \\ -1, & S \in \mathcal{S}_{(S^0,2)} \end{cases} .$$

So (F.2) can be seen as an expansion of the above introduced Haar-type functions.

Examples of functions in $\mathcal{M}$ (see Definition 4.4) are also similarly obtained.

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Submitted: July 25, 2023
Accepted: July 31, 2024
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Revista de la Unión Matemática Argentina **Accepted article · Early view version**

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Submitted: July 25, 2023
Accepted: July 31, 2024
Published (early view): August 28, 2024