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HAMILTONICITY OF RECTANGULAR GRID GRAPHS (MESHES) WITH AN L-SHAPED HOLE

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Abstract. Finding the Hamiltonian cycles in graphs is a well-known problem. Although, Hamiltonicity of grid graphs has been studied in the literature, there are few results on Hamiltonicity of grid graphs that have holes. In this paper, we study Hamiltonicity of rectangular grid graphs (meshes) with an L-shaped hole, and give a linear-time algorithm. The holes in meshes correspond to the faulty nodes.

1. Introduction

The Hamiltonian cycle problem is one of the most important problems in computer science and mathematics. In this problem, the goal is to find a cycle that passes through every vertex of a graph, exactly once. There are various studies and results regarding the Hamiltonian cycle problem in graphs [3]. A grid graph is a subgraph of the infinite grid, where the vertices have integer coordinates, and there is an edge between two vertices if their Euclidean distance is 1, see Fig 1(a) and 1(b).

One application of the Hamiltonian cycle problem in grid graphs is in the exploration problem. Specifically, the off-line exploration problem involves a mobile robot that needs to visit every cell in a known cellular room and return to the starting point, while minimizing the number of cells that are visited multiple times by the robot. This exploration problem can be translated into finding a tour in a grid graph that visits all the vertices, where each vertex corresponds to a cell in the environment. Thus, exploring the cellular room without revisiting any cell is equivalent to finding a Hamiltonian cycle in the corresponding grid graph. In this context, the environment is divided into cells, each represented by a vertex in the grid graph, and two vertices are considered adjacent if their corresponding cells share a common edge [10, 15].

Itai *et al.* [12] showed that the Hamiltonian cycle problem is NP-complete for general grid graphs. However, they also proposed a linear-time algorithm for finding Hamiltonian paths in rectangular grid graphs. Chen *et al.* [5] improved Itai's algorithm and proposed a parallel algorithm for solving the problem on mesh architectures. Zamfirescu and Zamfirescu [25] provided sufficient conditions for a

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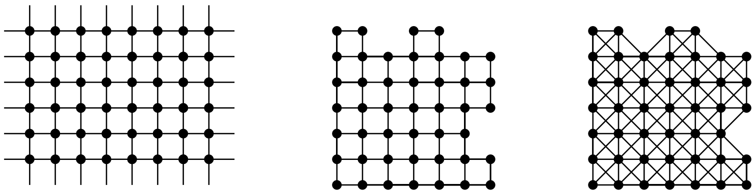

FIGURE 1. (a) An infinite grid, (b) a grid graph, and (c) a supergrid graph.

grid graph to be Hamiltonian. A graph is called Hamiltonian if it has a Hamiltonian cycle. Afrati [1] presented a linear-time algorithm for finding Hamiltonian cycles in staircase grid graphs. Umans and Lenhart [24] introduced an $O(n^4)$-time algorithm for finding Hamiltonian cycles in solid grid graphs with 2-factors. They posed the question of whether a polynomial-time algorithm exists for finding Hamiltonian cycles in grid graphs with specific types of holes. Salman [22, 23] identified classes of alphabet grid graphs that have Hamiltonian cycles. Alphabet grid graphs are a special type of grid graphs with shapes resembling alphabet characters. Keshavarz-Kohjerdi and Bagheri [13, 16, 17] presented linear-time algorithms for finding Hamiltonian paths and cycles in rectangular grid graphs with rectangular holes. Nishat and Whitesides [20] studied the reconfiguration of Hamiltonian cycles in L-shaped grid graphs.

Islam *et al.* [11] proved that finding Hamiltonian cycles is NP-complete for hexagonal grid graphs. Reay and Zamfirescu [21] and Gordon et al. [6] explored the Hamiltonian cycle problem in triangular grid graphs. Arkin *et al.* [2] established complexity results for the Hamiltonicity of various classes of square, triangular, and hexagonal grids. Hou and Lynch [7] investigated the Hamiltonian cycle problem in grid graphs of semiregular tessellations and proved its NP-completeness. Hung *et al.* [9] demonstrated that the Hamiltonian cycle and path problems in general supergrid graphs are NP-complete. In supergrid graphs, besides of the edges in grid graphs, we also have edges between vertices of Euclidean distance $\sqrt{2}$, see Fig. 1(c). They also showed that linear-convex supergrid graphs always contain Hamiltonian cycles [8]. Keshavarz-Kohjerdi and Bagheri studied the Hamiltonicity and Hamiltonian-connectivity of solid supergrid graphs [18, 19].

As mentioned, there are various results on the Hamiltonicity of solid grid graphs, but only few results for grid graphs that have holes. In this paper, we study the Hamiltonicity of rectangular grid graphs with an L-shaped hole, and give a linear-time algorithm.

The structure of the paper is as follows. Section 2 provides the necessary preliminaries and presents some relevant known results about grid graphs. The algorithm is given in Section 3. The conclusion and future works are given in Section 4.

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2. Preliminaries

In this section, we review the definitions and the results that we need throughout the paper. These definitions and results have been previously established in [12, 13, 18].

A *grid graph*, denoted by $G_g = (V(G_g), E(G_g))$, is a subgraph of the infinite grid where the vertices have integer coordinates. In this graph, two vertices are connected by an edge if their Euclidean distance is equal to one; as illustrated in Fig. 2(a). Let $v \in V(G_g)$, we use the notation v_x to represent the x-coordinate and v_y to represent the y-coordinate of v, respectively. We say that two vertices u and v are *adjacent* if there exists an edge between them, i.e., $(u, v) \in E(G_g)$, and we denote it by $u \sim v$. Additionally, we say that two edges $e_1 = (v_1, u_1)$ and $e_2 = (v_2, u_2)$ are *parallel* if $v_1 \sim v_2$ and $u_1 \sim u_2$. It is clear that every vertex in G_g, such as v, is adjacent to at most four neighbouring vertices; the down neighbour $D(v) = (v_x, v_y - 1)$, the left neighbour $L(v) = (v_x - 1, v_y)$, the up neighbour $U(v) = (v_x, v_y + 1)$, and the right neighbour $R(v) = (v_x + 1, v_y)$. The degree of a vertex v is equal to the number of adjacent vertices of v. We denote the degree of vertex v by $d(v)$.

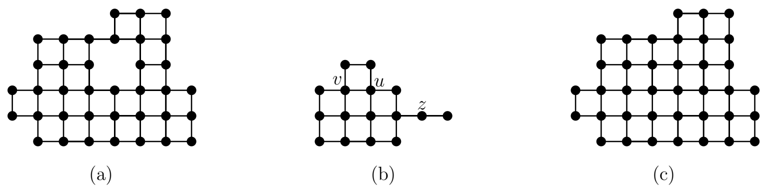

Figure 2. (a) A grid graph with a hole, (b) definitions of the cut vertex and the vertex cut, and (c) a solid grid graph.

Let G_g be a connected graph and V' be a subset of the vertex set $V(G_g)$. V' is a *vertex cut* of G if removing V' from G_g, denoted by $G_g \backslash V'$, results in a disconnected graph. A vertex v of G_g is considered a *cut vertex* of G_g if the singleton set $\{v\}$ forms a vertex cut of G_g. For instance, in Fig. 2(b), the vertex z is a cut vertex since removing it results in a disconnected graph. In Fig. 2(b), the set $\{v, u\}$ is a vertex cut.

A *solid grid graph* is a grid graph in which all the internal faces are unit squares (see Fig. 2(c)). A grid graph G_g is referred to as a *rectangular grid graph* denoted by $R(m, n)$ (or simply G_R) if its vertex set $V(G_g)$ includes all the vertices v of the infinite grid, where $1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n$. Fig. 3(a) illustrates a rectangular grid graph $R(5, 4)$. $R(m, n)$ is a *k-rectangle* if either $m = k$ or $n = k$. A rectangular grid graph $G_R = R(m, n)$ is characterized by four corners; the top-left, the top-right, the bottom-left, and the bottom-right corners. In this paper, we establish the convention that the top-left, top-right, bottom-left, and bottom-right corners of G_R are positioned at coordinates $(1, 1)$, $(m, 1)$, $(1, n)$, and (m, n), respectively (see Fig. 3(a)).

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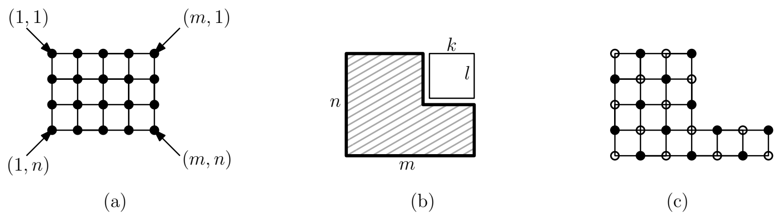

FIGURE 3. (a) An example of a rectangular grid graph, (b) defining parameters of an L-shaped grid graph, and (c) an example of an L-shaped grid graph

Grid graphs are bipartite graphs, which means they are two-colorable. Therefore, we can color the vertices of them using two colors, say black and white. If $v_x + v_y$ is even, then the vertex v is colored white, otherwise, it is colored black. It is evident that every cycle (or path) in G_g alternates between black and white vertices. We denote the set of black and white vertices by V_B and V_W, respectively. The number of black and white vertices in graph G_g may be different. The color that is assigned to the majority of vertices is called the majority color, while the other color is referred to as the minority color.

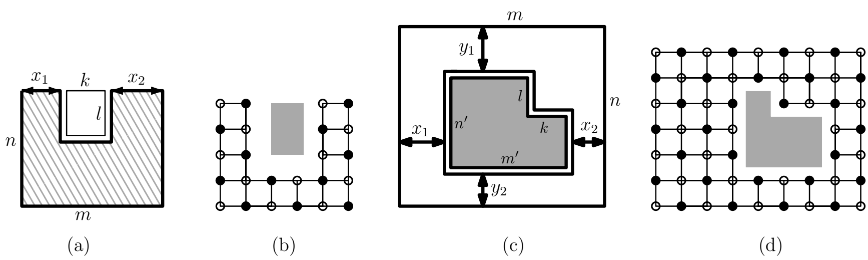

FIGURE 4. (a) Defining parameters of a C-shaped grid graph, (b) an example of a C-shaped grid graph, (c) defining parameters of a rectangular grid graph with an L-shaped hole (denoted by R_L), and (d) an example of R_L.

A *rectangular grid graph with a rectangular hole* refers to a rectangular grid graph $R(m, n)$ from which a rectangular grid subgraph $R(k, l)$ is removed, where $k, l \geqslant 1$ and $m, n > 1$. When $R(m, n)$ shares exactly two adjacent sides with $R(k, l)$ (as shown in Fig. 3(b) and 3(c)), we obtain an L-shaped grid graph denoted by $L(m, n; k, l)$ (or simply G_L). If $R(m, n)$ shares exactly one border side with $R(k, l)$ (as depicted in Fig. 4(a) and 4(b)), we obtain a C-shaped grid graph denoted by $C(m, n; k, l; x_1)$ (or simply G_C). It should be noted that $x_1 \geqslant 1$, $x_2 = m-k-x_1 \geqslant 1$, and $n - l \geqslant 1$. If G_g is an L-shaped or a C-shaped grid graph, the number of vertices in G_g can be calculated as $mn - kl$.

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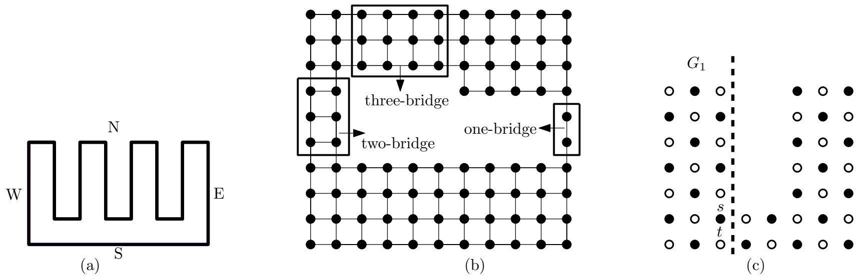

FIGURE 5. (a) A rectangular grid graph with four sides N, W, S, and E, (b) examples of one, two, and three-bridges and (c) an example of forbidden condition $\mathcal{FC}3$.

The number of vertices of a grid graph G_g is defined as the order of G_g and denoted by $|G_g|$. A grid graph G_g has an even order if $|V_B| = |V_W|$. Conversely, a grid graph G_g has an odd order if $||V_B| - |V_W|| = 1$.

A *rectangular grid graph with an L-shaped hole* is a rectangular grid graph $R(m,n)$ such that an L-shaped grid subgraph $L(m',n';k,l)$ is removed from it, where $m,n > 3$, $m',n' > 1$, and $k,l \geq 1$. Let $R_L(m,n;m',n';k,l;x_1,y_1)$ be a rectangular grid graph $R(m,n)$ with an L-shaped grid subgraph $L(m^{'},n^{'};k,l)$ as its hole; as shown in Fig. 4(c) and 4(d). Let $x_2 = m - x_1 - m'$ and $y_2 = n - y_1 - n'$. In this paper, we assume that x_1, x_2, y_1, and y_2 are greater than zero, i.e. the hole has no common border with $R(m,n)$. In the following for simplicity, we use R_L interchangeable with $R_L(m,n;m',n';k,l;x_1,y_1)$.

Let s and t be two specified vertices of G_g. We say (G_g,s,t) is *color-compatible*, if G_g is even-ordered and s and t have different colors or G_g is odd-ordered and s and t have the majority color. It is evident that the color-compatibility of (G_g,s,t) is a necessary condition for the existence of a Hamiltonian path in G_g between vertices s and t [12].

The literature [12] has already provided the necessary and sufficient conditions for a rectangular grid graph to be Hamiltonian. In this paper, we present several established results and offer redefined versions of some of them.

Lemma 2.1. [12] *A rectangular grid graph G_R, is Hamiltonian if and only if it does not meet the conditions $\mathcal{FC}1$ and $\mathcal{FC}2$ as defined below.*

$\mathcal{FC}1$ :: $|V_B| \neq |V_W|$.

$\mathcal{FC}2$ :: *G_R contains a cut vertex.*

Lemma 2.2. [13] *In a rectangular grid graph G_R, we can always find a Hamiltonian cycle that contains all the boundary edges of the four sides N, W, S, and E of G_R as shown in Fig. 5(a), except at most one side of G_R which includes boundary edges every other one.*

Clearly, if either $\mathcal{FC}1$ or $\mathcal{FC}2$ holds for any grid graph G_g, it implies that the graph is not Hamiltonian. In any bipartite graph, the vertices of any cycle alternate

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between black and white colors, resulting in $|V_B| = |V_W|$. It is also a well-known fact that any Hamiltonian graph does not contain a cut vertex [4]. Therefore, in the following, we assume that $\mathcal{FC}1$ and $\mathcal{FC}2$ do not hold, implying that the grid graph satisfies the necessary conditions for being Hamiltonian.

A *one-bridge* is a one-rectangle subgraph in graph G_g, that all the vertices of it have degree two in G_g. A *two-bridge* is a two-rectangle subgraph in graph G_g, that all the vertices of it have degree three in G_g. A *three-bridge* is a three-rectangle subgraph in graph G_g, that the vertices of it with y-coordinates y or $y+2$ (resp. x-coordinates x or $x+2$) have degree three in G_g, and its other vertices have degree four in G_g. Assume the top-left vertex of the three-rectangle has coordinate (x, y). Fig. 5(b) illustrates examples of one-bridge, two-bridge, and three-bridge subgraphs.

Lemma 2.3. [16] *A C-shaped grid graph G_C is Hamiltonian if and only if it does not satisfy conditions $\mathcal{FC}1$–$\mathcal{FC}3$. Whereas condition $\mathcal{FC}3$ is defined as follows:*

- $\mathcal{FC}3$ **::** *Let G_1 be a grid subgraph of G_C that is connected to $G_C \setminus G_1$ by a two-bridge (see Fig. 5(c)). Let two vertices s and t be the connecting vertices of G_1 to the two-bridges. And (G_1, s, t) is not color-compatible.*

Theorem 2.4. [12, 14] *Let G_g be a rectangular or a C-shaped grid graph. The Hamiltonian cycle of G_g can be constructed in linear time.*

In [17], Keshavarz-Kohjerd and Bagheri provided the necessary and sufficient conditions for the existence of Hamiltonian cycles in rectangular grid graphs with rectangular holes. In the following, we utilize their results to construct Hamiltonian cycles in rectangular grid graphs with an L-shaped hole. The two forbidden conditions $\mathcal{FC}4$ and $\mathcal{FC}5$ are defined as follows:

- $\mathcal{FC}4$ **::** Let G_1 be a grid subgraph of G_g that is connected to $G_g \setminus G_1$ by two one-bridges (see Fig. 6(a) and 6(b)). Assume two vertices s and t are the connecting vertices of G_1 to the one-bridges. And (G_1, s, t) is not color-compatible.
- $\mathcal{FC}5$**:** Let G_1 be a grid subgraph of G_g that is connected to $G_g \setminus G_1$ by exactly one one-bridge and one three-bridge. Let w be the connecting vertex of the one-bridge to G_1 and u, v, and z be the connecting vertices of the three-bridge to G_1, where $d(z) = 4$ (see Fig. 6(c) and 6(d)). Let $s, t \in G_1$ such that $s \sim w$ and $t \sim z$, and (G_1, s, t) is color-compatible.

Theorem 2.5. [17] *For any grid graph G_g to be Hamiltonian, the forbidden conditions $\mathcal{FC}1$–$\mathcal{FC}5$ should not hold.*

Note that the conditions $\mathcal{FC}2$ and $\mathcal{FC}3$ do not occur for grid graphs with holes.

Corollary 2.6. *A rectangular grid graph with an L-shaped hole has a Hamiltonian cycle if the conditions $\mathcal{FC}1$, $\mathcal{FC}4$, and $\mathcal{FC}5$ are not satisfied.*

3. The algorithm

In this section, we present an algorithm for finding a Hamiltonian cycle in a rectangular grid graph with an L-shaped hole, denoted by R_L. This algorithm is

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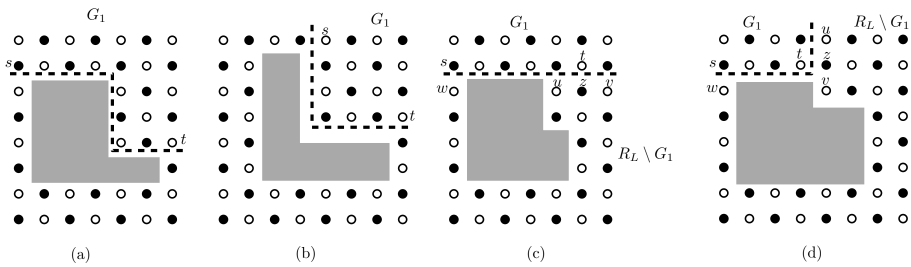

FIGURE 6. Examples of forbidden conditions $\mathcal{FC}4$ and $\mathcal{FC}5$ in R_L.

based on a divide-and-conquer approach. If any of the forbidden conditions $\mathcal{FC}1$, $\mathcal{FC}4$, and $\mathcal{FC}5$ holds for R_L, then it is not Hamiltonian. So, in the following we assume that these forbidden conditions do not hold for R_L. Initially, the graph is divided into several subgraphs, and then a Hamiltonian cycle is obtained in each subgraph. Finally, by combining the Hamiltonian cycles of the subgraphs, a Hamiltonian cycle in the original graph is obtained. We will now explain the details of each step of the algorithm.

To begin, we partition R_L into at most five grid subgraphs, $G_1 = R(m_1, n_1)$, $G_2 = R(m_2, n_2)$, $G_3 = R(m_3, n_3)$, $G_4 = R(m_4, n_4)$, and $G_5 = R_L \setminus (G_1 \cup G_2 \cup G_3 \cup G_4)$ by making two vertical and two horizontal cuts. Where $m_1 = r_1$, $n_1 = n$, $m_2 = m - r_2 + 1$, $n_2 = n$, $m_3 = m - m_1 - m_2$, $n_3 = r_3$, $m_4 = m - m_1 - m_2$, $n_4 = n - r_4 + 1$, and r_1 to r_4 are defined as follows:

$$r_1 = \begin{cases} x_1 - 1; & \text{if } (x_1 \mod 2) = 1 \\ x_1 - 2; & \text{otherwise} \end{cases}$$

$$r_2 = \begin{cases} x_1 + m' + 2; & \text{if } (x_1 + m' + 1 \mod 2) = (m \mod 2) \\ x_1 + m' + 3; & \text{otherwise} \end{cases}$$

$$r_3 = \begin{cases} y_1 - 1; & \text{if } (y_1 \mod 2) = 1 \\ y_1 - 2; & \text{otherwise} \end{cases}$$

$$r_4 = \begin{cases} y_1 + n' + 2; & \text{if } (y_1 + n' + 1 \mod 2) = (n \mod 2) \\ y_1 + n' + 3; & \text{otherwise} \end{cases}$$

Here, r_1 represents the right-most column of G_1, r_2 represents the left-most column of G_2, r_3 represents the bottom-most row of G_3, r_4 represents the top-most row of G_4. Note that if x_1, x_2, y_1 and y_2 in R_L be 1 or 2, then G_1, G_2, G_3, and G_4 are empty, respectively. See Fig. 7, for a visual illustration. Notice that, $G_5 = R_L(m - m_1 - m_2, n - n_3 - n_4; m', n'; k, l; x_1 - m_1, y_1 - n_3)$, $m_3 = m_4$, and $m_3 > 2$. A simple verification indicates that m_1, m_2, n_3, and n_4 are even. Therefore

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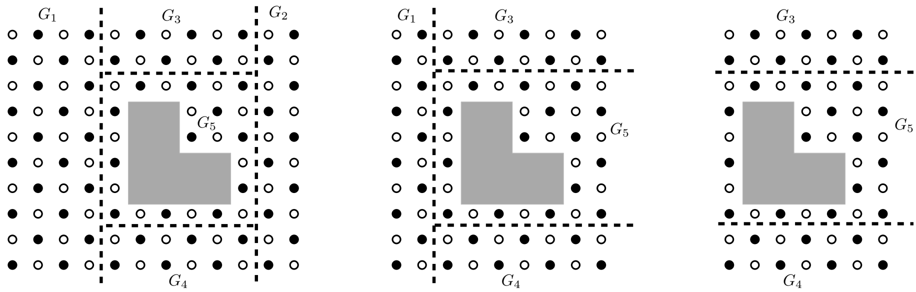

FIGURE 7. Dividing R_L into several subgraphs.

for each subgraph G_i, where $1 \leqslant i \leqslant 5$, we have $|V_B(G_i)| = |V_W(G_i)|$. Since $|V_B(G_i)| = |V_W(G_i)|$, $1 \leqslant i \leqslant 4$, and $n, m_3, m_4 > 1$, therefore, based on Lemma 2.1, it follows that G_1 to G_4 have Hamiltonian cycles. Furthermore, according to the algorithm presented in [5], a Hamiltonian cycle can be constructed in G_1 to G_4.

If G_5 does not satisfy the forbidden conditions $\mathcal{FC}4$ and $\mathcal{FC}5$, then its Hamiltonian cycle is constructed following the patterns given in Fig. 9 to 11. Which pattern to be used, depends on the dimensions of G_5. In the following, by omitting similar cases, we only consider the following distinct cases:

(1) Both m and n are even and [(k is even) or (both k and l are odd)].
(2) Both m and n are odd and [(k is even) or (both k and l are odd)].
(3) m is odd and n is even.

Then Hamiltonian cycles in the subgraphs are combined by using parallel edges. Let G_1 and G_2 be two subgraphs of R_L, and $\mathcal{HC}_1$ (resp. $\mathcal{HC}_2$) be a Hamiltonian cycle of G_1 (resp. G_2). Assume $e_1 = (v_1, u_1) \in \mathcal{HC}_1$ and $e'_1 = (v_2, u_2) \in \mathcal{HC}_1$ be two parallel edges. We can merge $\mathcal{HC}_1$ and $\mathcal{HC}_2$ into one cycle, as a Hamiltonian cycle of $G_1 \cup G_2$, by removing e_1 and e'_1, and adding two edges (v_1, v_2) and (u_1, u_2), see Fig. 8. This is called the merge operation, and denoted by $\oplus$. In the following lemma, we demonstrate how Hamiltonian cycles in the subgraphs G_1 to G_5 can be combined.

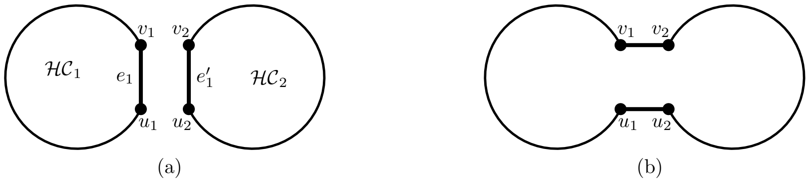

FIGURE 8. The merge operation.

Lemma 3.1. *Let G_1, G_2, G_3, G_4, and G_5 be a partition of R_L, as previously defined. Let $\mathcal{HC}_1$, $\mathcal{HC}_2$, $\mathcal{HC}_3$, and $\mathcal{HC}_4$ represent the Hamiltonian cycles in G_1,*

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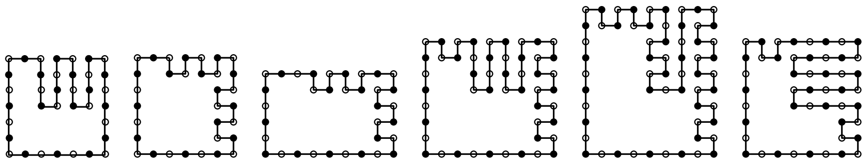

Figure 9. A Hamiltonian cycle in G_5, in the case where $x_1 = 1$ and $y_2 = 1$, in G_5.

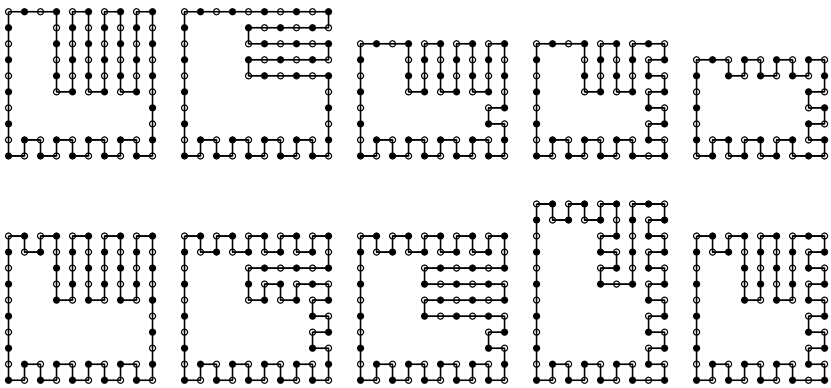

Figure 10. A Hamiltonian cycle in G_5, in the case where $x_1 = 1$ and $y_2 = 2$, in G_5.

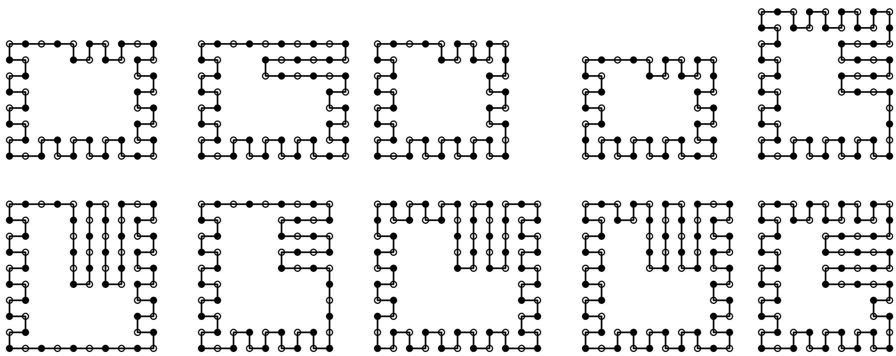

Figure 11. A Hamiltonian cycle in G_5, in the case where $x_1 = 2$, in G_5.

G_2, G_3, and G_4, respectively. If G_5 does not satisfy the conditions $\mathcal{FC}4$ and $\mathcal{FC}5$, then the Hamiltonian cycle of R_L, i.e. $\mathcal{HC}(R_L)$, can be constructed by merging $\mathcal{HC}_1$, $\mathcal{HC}_2$, $\mathcal{HC}_3$, $\mathcal{HC}_4$, and the Hamiltonian cycle in G_5 ($\mathcal{HC}_5$).

Proof. Let $v_1 = (r_1 + 1, r_4 - 1)$, $u_1 = (r_1 + 1, r_4 - 2)$, $v_2 = (r_2 - 1, r_3 + 1)$, $u_2 = (r_2 - 1, r_3 + 2)$, $v_3 = (r_1 + 1, r_3 + 1)$, $u_3 = (r_1 + 2, r_3 + 1)$, $v_4 = (r_2 - 1, r_4 - 1)$,

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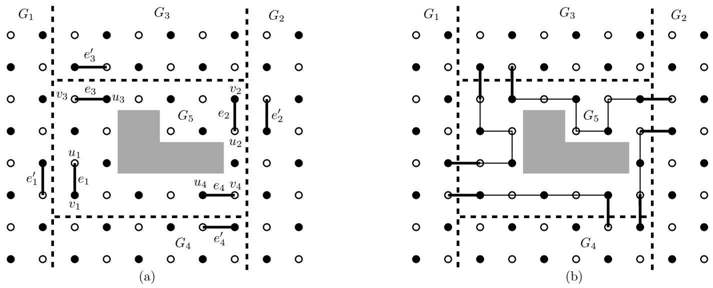

FIGURE 12. Merging $\mathcal{HC}_1$, $\mathcal{HC}_2$, $\mathcal{HC}_3$, $\mathcal{HC}_4$, and $\mathcal{HC}_5$.

and $u_4 = (r_2 - 2, r_4 - 1)$. Since $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$ in G_5, the edges $e_1 = (v_1, u_1)$, $e_2 = (v_2, u_2)$, $e_3 = (v_3, u_3)$, and $e_4 = (v_4, u_4)$ are included in any Hamiltonian cycle $\mathcal{HC}_5$ of G_5 (see Fig. 12(a)). Let e'_1, e'_2, e'_3, and e'_4 be the edges of G_1, G_2, G_3, and G_4 that are parallel to the edges e_1, e_2, e_3, and e_4, respectively. According to Lemma 2.2, it is always possible to make Hamiltonian cycles $\mathcal{HC}_1$, $\mathcal{HC}_2$, $\mathcal{HC}_3$, and $\mathcal{HC}_4$ such that it includes the edges e'_1, e'_2, e'_3, and e'_4, respectively. If the subgraphs G_1, G_2, G_3, and G_4 are not empty, then we can merge their Hamiltonian cycles by the Hamiltonian cycle $\mathcal{HC}_5$ of G_5, using the parallel edges e_i and e'_i, $1 \leqslant i \leqslant 4$ (see Fig. 12(b)). □

Now, consider the case where G_5 satisfies $\mathcal{FC}4$ or $\mathcal{FC}5$.

Lemma 3.2. *If G_5 satisfies $\mathcal{FC}4$ or $\mathcal{FC}5$, then one of the following conditions occurs for R_L.*

- $\mathcal{C}1$ **::** *n is even, m and x_1 are odd, and either*
  - (a) *x_2 is odd, y_2 is even, or*
  - (b) *$k = 1$ and x_2, l, and y_1 are even.*
- $\mathcal{C}2$ **::** *m, n, and y_2 are even, l is odd, and either*
  - (a) *k, x_2, and y_1 are odd and x_1 is even, or*
  - (b) *k, x_2, and y_1 are even, x_1 is odd, and $l = 1$.*

Proof. We have assumed that $\mathcal{FC}4$ and $\mathcal{FC}5$ do not hold for the given grid graph R_L. After dividing R_L into G_1 to G_5, in G_5, the variables x_1, x_2, y_1, and y_2 take values of either 1 or 2. If $\mathcal{FC}4$ holds for G_5, then there are two possible cases:

Case 1. In R_L, x_1, x_2, and m are odd, and y_2 is even. This case corresponds to $\mathcal{C}1$(a).

Case 2. In R_L, k, l, y_1, and x_2 are odd. This case corresponds to $\mathcal{C}2$(a).

If $\mathcal{FC}5$ holds for G_5, then x_2 is even, x_1 is odd, and there are two possible cases:

Case 1. In R_L, $k = 1$, l is even, m is odd, and $y_2 + (n' - l)$ is even. This case corresponds to $\mathcal{C}1$(b).

Case 2. In R_L, $l = 1$, k is even, x_1 is odd, y_1 and $x_1 + (m' - k)$ are even. This case corresponds to $\mathcal{C}2$(b). □

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In the following, we explain how to construct a Hamiltonian cycle in R_L in the case where one of the conditions $\mathcal{C}1$ or $\mathcal{C}2$ occurs. We consider two cases.

Case I:: $x_1 = 1$, $x_2 = 1$, $y_1 = 1$, or $y_2 = 1$. This case is investigated in Lemma 3.3.

Case II:: $x_1 > 1$, $x_2 > 1$, $y_1 > 1$, and $y_2 > 1$. This case is investigated in Lemma 3.4.

Lemma 3.3. *Assume that R_L satisfies condition $\mathcal{C}1$ or $\mathcal{C}2$, and Case I holds. If R_L does not satisfy conditions $\mathcal{FC}4$ and $\mathcal{FC}5$, then R_L has a Hamiltonian cycle.*

Proof. In this case, by modifying the values of r_1, r_2, or r_3, we transform the subgraph G_5 into a state that has a Hamiltonian cycle.

Let $\mathcal{C}1(a)$ holds, then x_1 and x_2 are odd. Here, we have $x_1 \geqslant 5$ or $x_2 \geqslant 5$. Because if both x_1 and x_2 are less than 5, then either condition $\mathcal{FC}4$ or $\mathcal{FC}5$ is satisfied for R_L. If $x_1 \geqslant 5$, we modify r_1 as $r_1 = x_1 - 2$. If $x_2 \geqslant 5$ and $x_1 = 1$, then if $(l > 1)$ or $(l = 1$ and y_1 is odd), we modify r_2 as $r_2 = x_1 + m' + 3$. Otherwise, we modify r_1 and r_3 as $r_1 = x_1 - 2$ and $r_3 = y_1 - 1$, respectively. Let $\mathcal{C}1(b)$ holds, then x_2 is even and $x_2 > 2$. Because if $x_2 = 2$, then either condition $\mathcal{FC}4$ or $\mathcal{FC}5$ is satisfied for R_L. In this case, we modify r_2 as $r_2 = x_1 + m' + 2$. Since n is even, $|V_B(G_i)| = |V_W(G_i)|$, where $i = 1$ or 2. Also, since n_3 is even or $m - m_1 - m_2$ is even, $|V_B(G_3)| = |V_W(G_3)|$.

Let $\mathcal{C}2(a)$ holds, then x_1 is even, and y_1 and x_2 are odd. We have $x_2 \geqslant 5$ or $y_1 \geqslant 5$. Because if both x_2 and y_1 are less than 5, then either condition $\mathcal{FC}4$ or $\mathcal{FC}5$ is satisfied for R_L. If $x_2 \geqslant 5$, then we modify r_2 as $r_2 = x_1 + m' + 3$. If $x_2 = 1$ and $y_1 \geqslant 5$, then we modify r_3 as $r_3 = y_1 - 2$. Let $\mathcal{C}2(b)$ holds, then x_1 is odd, and y_1, y_2, and x_2 are even. Here, $y_1 > 2$ and $x_1 = 1$. Because if $y_1 = 2$, then the condition $\mathcal{FC}5$ is satisfied for R_L. In this case, we modify r_3 as $r_3 = y_1 - 1$. Since n and m are even, $|V_B(G_i)| = |V_W(G_i)|$, where $i = 2$ or 3.

A simple check reveals that G_5 has a Hamiltonian cycle, and its Hamiltonian cycle is one of the patterns given in Fig. 9 to 11. Combining the Hamiltonian cycle $\mathcal{HC}_5$ of G_5 with the Hamiltonian cycles $\mathcal{HC}_1$, $\mathcal{HC}_2$, $\mathcal{HC}_3$, and $\mathcal{HC}_4$ of G_1, G_2, G_3, and G_4, respectively, are done according to Lemma 3.1. ☐

Lemma 3.4. *Assume that R_L satisfies condition $\mathcal{C}1$ or $\mathcal{C}2$, and Case II holds. If R_L does not satisfy conditions $\mathcal{FC}4$ and $\mathcal{FC}5$, then R_L has a Hamiltonian cycle.*

Proof. Based on the value of y_2, we consider the following two cases.

Case 1. y_2 is even. We divide R_L into two connected components, $G_1 = C(n, x; n', m' - k; y_1)$ and $G_2 = C(n, m - x; n' - l, k; y_1 + l)$, by a vertical cut at $x + m' - k$; as illustrated in Fig. 13(a). Consider the subgraph G_1. First, let y_1 is even. Since n, y_2, and y_1 are even, we observe that $|V_B(G_1)| = |V_W(G_1)|$. Now, let y_1 is odd, then k and l are odd. A simple check shows that m' is odd and $m' - k$ is even. Since n and $m' - k$ are even, we have $|V_B(G_1)| = |V_W(G_1)|$. Since $|V_B(R_L)| = |V_W(R_L)|$, we can deduce that $|V_B(G_2)| = |V_W(G_2)|$. So, $\mathcal{FC}1$ does not hold. We assumed that Case II holds, so $\mathcal{FC}2$ dos not hold. Since $[(x_1 > 2)$ or $(x_1 = 2$ and y_2 is even)] and $[(x_2 > 2)$ or $(x_2 = 2$ and y_2 is even)], the condition

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$\mathcal{FC}3$ is not met for G_1 and G_2. Based on Lemma 2.3, it can be concluded that both G_1 and G_2 have a Hamiltonian cycle. According to the algorithm described in [14], a Hamiltonian cycle is constructed in G_1 and G_2. Finally, the Hamiltonian cycles are combined using two parallel edges, $e_1 = (v_1, u_1)$ and $e'_1 = (v_2, u_2)$, resulting in a Hamiltonian cycle in R_L (see Fig. 13(b)). Let $v_1 = (x, n)$, $u_1 = (x, n-1)$, $v_2 = (x+1, n)$, and $u_2 = (x+1, n-1)$. Since $d(v_1) = 2$ (in G_1) and $d(v_2) = 2$ (in G_2), the edges $e_1 = (v_1, u_1)$ and $e'_1 = (v_2, u_2)$ are in any Hamiltonian cycle of G_1 and G_2, respectively.

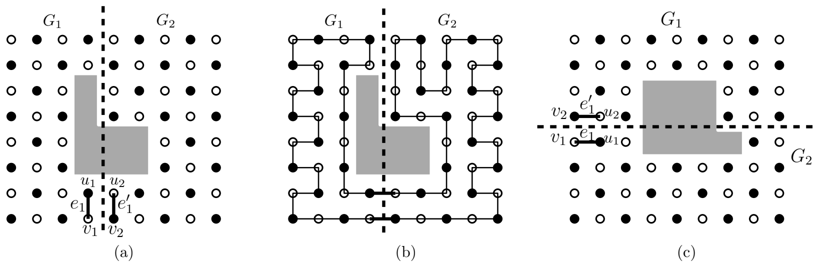

FIGURE 13. Combining Hamiltonian cycles in G_1 and G_2.

Case 2. y_2 is odd. In this case, only condition $\mathcal{C}1$(b) occurs. Clearly, y_1 and l are even. We divide R_L into two connected components, $G_1 = C(m, y; m' - k, l; x_1)$ and $G_2 = C(m, n-y; m', n'-l; x_1)$, by a horizontal cut at $y = y_1 + l$; as illustrated in Fig. 13(c). Since y_1 and y are even, we conclude that $|V_B(G_1)| = |V_W(G_1)|$. Since $|V_B(R_L)| = |V_W(R_L)|$, we can deduce that $|V_B(G_2)| = |V_W(G_2)|$. So, $\mathcal{FC}1$ does not hold. We assumed that Case II holds, so $\mathcal{FC}2$ dos not hold. Since y and $n-y$ are even, the condition $\mathcal{FC}3$ is not met for G_1 and G_2. A Hamiltonian cycle in R_L can be constructed similarly to Case 1. Here, let $v_1 = (1, y+1)$, $u_1 = (2, y+1)$, $v_2 = (1, y)$, and $u_2 = (2, y)$. Since $d(v_1) = 2$ (in G_1) and $d(v_2) = 2$ (in G_2), edges $e_1 = (v_1, u_1)$ and $e'_1 = (v_2, u_2)$ are in any Hamiltonian cycle of G_1 and G_2, respectively. □

Theorem 3.5. *A rectangular grid graph $R(m, n)$ with an L-shaped hole $L(m', n'; k, l)$ is Hamiltonian if and only if none of the forbidden conditions $\mathcal{FC}1$, $\mathcal{FC}4$, and $\mathcal{FC}5$ hold.*

Algorithm 1 shows the pseudo code of the algorithm. In the pseudo code, by $\oplus$ we mean the merge operation.

Theorem 3.6. *A Hamiltonian cycle for a rectangular grid graph with an L-shaped hole, can be constructed in linear time.*

Proof. To compute a Hamiltonian cycle for R_L, first we divide R_L into at most five subgraphs G_1, G_2, G_3, G_4, and G_5. This partitioning is done in $O(1)$ time. Then we check if G_5 satisfies the forbidden conditions $\mathcal{FC}4$ and $\mathcal{FC}5$. This can be done in $O(1)$ time. If G_5 does not satisfy $\mathcal{FC}4$ and $\mathcal{FC}5$, we compute a Hamiltonian cycle for G_1, G_2, G_3, and G_4 in linear time, according to Theorem 2.4. A Hamiltonian

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Algorithm 1 HamCycle (R_L)

Input: a rectangular grid graph $R(m,n)$ with an L-shaped hole $L(m',n';k,l)$
Output: a Hamiltonian cycle of R_L
if any of the conditions $\mathcal{FC}1$, $\mathcal{FC}4$, and $\mathcal{FC}5$ holds for R_L **then**
  report R_L is not Hamiltonian, and exit.
else
  Partitioning R_L into at most five grid subgraphs G_1 to G_5 by making two vertical cuts and two horizontal cuts.
  if none of the conditions $\mathcal{FC}4$ or $\mathcal{FC}5$ are satisfied for G_5 **then**
    Let $\mathcal{HC}_1$, $\mathcal{HC}_2$, $\mathcal{HC}_3$, $\mathcal{HC}_4$, and $\mathcal{HC}_5$ be the Hamiltonian cycles in G_1, G_2, G_3, G_4, and G_5, respectively.
    return $\mathcal{HC}(R_L) = (\mathcal{HC}_1 \oplus (\mathcal{HC}_2 \oplus (\mathcal{HC}_3 \oplus (\mathcal{HC}_4 \oplus \mathcal{HC}_5))))$
  end if
  if any of the conditions $\mathcal{FC}4$ or $\mathcal{FC}5$ holds for G_5 **then**
    if $x_1 = 1$, $x_2 = 1$, $y_1 = 1$, or $y_2 = 1$ **then**
      Modify the partitioning of R_L According to Lemma 3.3.
      Let $\mathcal{HC}_1$, $\mathcal{HC}_2$, $\mathcal{HC}_3$, $\mathcal{HC}_4$, and $\mathcal{HC}_5$ be the Hamiltonian cycles in G_1, G_2, G_3, G_4, and G_5, respectively.
      return $\mathcal{HC}(R_L) = (\mathcal{HC}_1 \oplus (\mathcal{HC}_2 \oplus (\mathcal{HC}_3 \oplus (\mathcal{HC}_4 \oplus \mathcal{HC}_5))))$
    end if
    if x_1, x_2, y_1, and y_2 are greater than 1 **then**
      Partitioning R_L into two C-shaped grid subgraphs G_1 and G_2 by making a vertical cut (or a horizontal cut).
      Let $\mathcal{HC}_1$ and $\mathcal{HC}_2$ be the Hamiltonian cycles in G_1 and G_2, respectively.
      return $\mathcal{HC}(R_L) = \mathcal{HC}_1 \oplus \mathcal{HC}_2$
    end if
  end if
end if

cycle for G_5 is computed according to the patterns given in Fig. 9 to 11, which can be done in linear time. Combining the Hamiltonian cycles of G_1, G_2, G_3, G_4, and G_5 is done in $O(1)$ time. On the other hand, if G_5 satisfies $\mathcal{FC}4$ or $\mathcal{FC}5$, then either we modify the partitioning or do a new partitioning and divide R_L into two C-shaped grid subgraphs G_1 and G_2. This can be done in $O(1)$ time. Finding Hamiltonian cycles of the C-shaped subgraphs are done in linear time, according to Theorem 2.4. Hamiltonian cycles of the other subgraphs are also computed in linear time, as mentioned before. Combining the computed Hamiltonian cycles of the subgraphs is done in $O(1)$ time. Putting all together yields a linear-time algorithm. ☐

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4. Conclusion and future works

In this paper, we considered Hamiltonicity of rectangular grid graphs with an L-shaped hole. A linear-time algorithm was presented for the problem. Although, Hamiltonicity of grid graphs has been studied in the literature, there are few results on Hamiltonicity of grid graphs that have holes. As a future work, we can study Hamiltonicity of grid graphs with holes of other shapes. Also we can consider grid graphs with more than one holes.

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