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ON L_p KY FAN DETERMINANT INEQUALITIES

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Abstract. This paper establishes an extension of Ky Fan's determinant inequality when the usual matrix addition is replaced by the power mean of positive definite matrices. We further explore variants of this newly derived L_p Ky Fan's inequality, extending a determinant difference inequality formulated by Yuan and Leng [J. Aust. Math. Soc., 83(1), 2007].

1. Introduction

In the context of positive definite $n \times n$ matrices A and B, the foundational determinant inequality can be expressed as follows:

$$|A + B| \geq |A| + |B|.$$

An enhanced version of this determinant inequality is given by the Minkowski determinant inequality [8, p.510]

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}. \tag{1.1}$$

Over the course of several decades, the Minkowski determinant inequality has undergone substantial generalization within scholarly literature, as documented in works such as [1, 3–5, 8–11, 13–16]. A notable advancement of inequality (1.1) is the Ky Fan's determinant inequality (see [5] or [13, p.687]). It describes how the determinant of a block matrix is bounded by products of determinants involving its principal submatrices and Schur complements. This inequality is a fundamental result in the study of matrix analysis, finding applications across numerous disciplines, including optimization theory, statistics, and mathematical physics. By revealing connections between determinants and the structure of matrices through Schur complements, Ky Fan's determinant inequality plays a pivotal role in understanding the interplay between different parts of a matrix and their collective impact on its determinant value. Specifically, the Ky Fan's determinant inequality states as

$$\left(\frac{|A+B|}{|(A+B)_k|}\right)^{\frac{1}{n-k}} \geq \left(\frac{|A|}{|A_k|}\right)^{\frac{1}{n-k}} + \left(\frac{|B|}{|B_k|}\right)^{\frac{1}{n-k}}, \tag{1.2}$$

where M_k denotes the k-th leading principal submatrix of matrix M.

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In 2007, Yuan and Leng [15] proved an elegant extension of Ky Fan's determinant inequality: For positive definite $n \times n$ matrices A and B, and non-negative real numbers a and b such that $A \geq aI_n$ and $B \geq bI_n$, the following inequality holds:

$$\left(\frac{|A+B|}{|(A+B)_k|} - |(a+b)I_{n-k}|\right)^{\frac{1}{n-k}} \geq \left(\frac{|A|}{|A_k|} - |aI_{n-k}|\right)^{\frac{1}{n-k}} + \left(\frac{|B|}{|B_k|} - |bI_{n-k}|\right)^{\frac{1}{n-k}}. \tag{1.3}$$

Positive definite matrices occupy a central position among matrix operations, distinguished by their unique properties and crucial role in diverse fields. Their significance extends beyond mere algebraic manipulation, as evidenced by fascinating behavior when raised to powers and subsequently combined. For $n \times n$ positive definite matrices A and B, this behavior leads to the emergence of a captivating operator, denoted as $+_p$, for $p \in \mathbb{R}$. When $p > 1$, this operator defines the power mean, $A +_p B$, of A and B as follows:

$$A +_p B = (A^p + B^p)^{1/p}.$$

Building upon the intriguing behavior of positive definite matrices under the power mean operator (as explored in works including [7] and [16, Section 1.2]), this paper delves into their combined influence through the lens of the power mean operator. This operator bridges the gap between matrix algebra and the inherent positive definiteness of these matrices, offering valuable insights into their joint characteristics. Motivated by these observations, we investigate the existence of a Ky Fan's determinant inequality for the power mean of positive definite matrices.

We aim to establish the L_p versions of inequalities (1.2) and (1.3) specifically for $p > 1$. Our efforts focus on achieving the following:

Theorem 1.1. *Let A and B be two symmetric, positive definite matrices of order n and let A_k and B_k be the k-th leading principal submatrix of A and B, respectively. If $p > 1$, then,*

$$\left(\frac{|A +_p B|}{|(A +_p B)_k|}\right)^{\frac{p}{n-k}} \geq \left(\frac{|A|}{|A_k|}\right)^{\frac{p}{n-k}} + \left(\frac{|B|}{|B_k|}\right)^{\frac{p}{n-k}}, \tag{1.4}$$

with equality if and only if $A = cB$ for some $c > 0$.

Theorem 1.2. *Let A and B denote two symmetric, positive definite matrices of order n, with A_k and B_k representing the k-th leading principal submatrices of A and B respectively. Given $1 < p \leq n-k$ and non-negative real numbers α and β such that $A > \alpha I_n$ and $B > \beta I_n$, the following inequality holds:*

$$\left(\frac{|A +_p B|}{|(A +_p B)_k|} - |(\alpha^p + \beta^p)^{\frac{1}{p}} I_{n-k}|\right)^{\frac{p}{n-k}} \geq \left(\frac{|A|}{|A_k|} - |\alpha I_{n-k}|\right)^{\frac{p}{n-k}} + \left(\frac{|B|}{|B_k|} - |\beta I_{n-k}|\right)^{\frac{p}{n-k}}. \tag{1.5}$$

Equality holds in (1.5) if and only if either $p = n-k$ or $\alpha^{-1}A = \beta^{-1}B$.

Theorem 1.3. *Let A and B be two symmetric, positive definite matrices of order n. Let A_k and B_k denote the k-th leading principal submatrix of A and B, respectively.*

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If $p > n-k$ and α and β are positive real numbers such that $0 < A < \alpha I_n$ and $0 < B < \beta I_n$, then

$$\left(|(\alpha^p+\beta^p)^{\frac{1}{p}} I_{n-k}| - \frac{|A+_p B|}{|(A+_p B)_k|}\right)^{\frac{p}{n-k}} \leq \left(|\alpha I_{n-k}| - \frac{|A|}{|A_k|}\right)^{\frac{p}{n-k}} + \left(|\beta I_{n-k}| - \frac{|B|}{|B_k|}\right)^{\frac{p}{n-k}}. \tag{1.6}$$

Equality holds in (1.6) if and only if $\alpha^{-1}A = \beta^{-1}B$.

As not all positive definite matrices inherently secure the positive definiteness of their Schur complements, the imposition of symmetry becomes pivotal in assuring this favorable attribute. Therefore, we enforce symmetry on the matrices referenced in the aforementioned theorems.

2. Preliminaries and auxiliary results

Let $M_n(\mathbb{R})$ denote the set of real $n \times n$ matrices, and let I_n denote the $n \times n$ identity matrix. Given an A taken from $M_n(\mathbb{R})$, the notation $A > 0$ ($A \geq 0$) is used to indicate that matrix A is positive definite (positive semi-definite). Thus, the relation $A > B$ ($A \geq B$) is synonymous with $A - B > 0$ ($A - B \geq 0$). The operation A^T denotes the transpose of matrix A. An $n \times n$ matrix is said to be symmetric if $A^T = A$.

Denoted by $\mathcal{H}^n$ the subset of symmetric, positive definite matrices of $M_n(\mathbb{R})$, and by $\mathcal{A}$ a convex subset of $\mathcal{H}^n$. A function $\phi : \mathcal{A} \to \mathcal{H}^n$ is said to be matrix concave if

$$\phi(\lambda A + \bar{\lambda} B) \geq \lambda\phi(A) + \bar{\lambda}\phi(B),$$

for $\lambda \in [0,1]$, where $\bar{\lambda} = 1 - \lambda$.

Suppose that A, B are symmetric, positive definite $n \times n$ matrices and $\alpha, \beta \geq 0$. For $p > 0$, the p-sum, $\alpha \cdot A +_p \beta \cdot B$, of A and B is defined as a positive definite matrix:

$$\alpha \cdot A +_p \beta \cdot B = [\alpha A^p + \beta B^p]^{1/p}.$$

Especially, for $\alpha > 0$,

$$\alpha \cdot A = \alpha^{1/p} A.$$

Note that if $p = 1$, then $\alpha \cdot A +_p \beta \cdot B$ reduces to the usual matrix addition $\alpha A + \beta B$.

Lemma 2.1. *Suppose $p > 1$, $\lambda \in [0,1]$, and $A, B \in \mathcal{H}^n$. Then*

$$\lambda \cdot A +_p \bar{\lambda} \cdot B \geq \lambda A + \bar{\lambda} B. \tag{2.1}$$

Proof. It is well known that M^s is matrix concave for $M \in \mathcal{H}^n$ and $0 < s < 1$; see, e.g., [2]. Denoting $\bar{A} = A^p$ and $\bar{B} = B^p$, we have

$$\begin{aligned}\lambda \cdot A +_p \bar{\lambda} \cdot B &= \left(\lambda \bar{A} + \bar{\lambda}\bar{B}\right)^{1/p} \\ &\geq \lambda \bar{A}^{1/p} + \bar{\lambda}\bar{B}^{1/p} \\ &= \lambda A + \bar{\lambda} B.\end{aligned}$$

$\square$

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Let $A \in \mathcal{H}^n$. If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is of order k and invertible, then the Schur complement, $\mathcal{S}(A)$, of A_{11} in A, is defined as

$$\mathcal{S}(A) = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

The following facts about the Schur complement can be found in many literatures; see, e.g., [6, 8, 10, 13, 14].

Lemma 2.2. *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *and* $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ *be symmetric, positive definite matrices such that* A_{11} *and* B_{11} *are* $k \times k$ *invertible matrices. Then*

(i) $|A| = |A_{11}||\mathcal{S}(A)|$;
(ii) *If* $A \geq B$, *then* $\mathcal{S}(A) \geq \mathcal{S}(B)$;
(iii) *we have*

$$\mathcal{S}(A + B) \geq \mathcal{S}(A) + \mathcal{S}(B), \tag{2.2}$$

with equality if and only if $A_{21}A_{11}^{-1} = B_{21}B_{11}^{-1}$.

It is useful to present inequality (2.2) as the concavity of the Schur complement $\mathcal{S}(\cdot)$ (see, for example, [13, E.7.g]); that is to say, for $A, B \in \mathcal{H}^n$ and $\lambda \in [0, 1]$,

$$\mathcal{S}(\lambda A + \bar{\lambda} B) \geq \lambda \mathcal{S}(A) + \bar{\lambda} \mathcal{S}(B). \tag{2.3}$$

Lemma 2.3. *Let* $\vec{a} = (a_1, \cdots, a_m)$ *and* $\vec{b} = (b_1, \cdots, b_m)$ *represent two* m*-tuples of positive real numbers. Suppose that* q *is a real number such that* $a_1^q \geq \sum_{j=2}^m a_j^q$ *and* $b_1^q \geq \sum_{j=2}^m b_j^q$. *If* $q \geq 1$, *then,*

$$\left((a_1 + b_1)^q - \sum_{j=2}^m (a_j + b_j)^q \right)^{1/q} \geq \left(a_1^q - \sum_{j=2}^m a_j^q \right)^{1/q} + \left(b_1^q - \sum_{j=2}^m b_j^q \right)^{1/q}. \tag{2.4}$$

If $0 < q < 1$, *then inequality (2.4) is reversed. Moreover, equalities hold in these inequalities if and only if either* $q = 1$ *or* $\vec{a} = \nu \vec{b}$, *where* ν *is a positive constant.*

A proof of Lemma 2.3 can be found in [12]. The inequality (2.4) was first proved by Bellman [3, p.38], which plays a crucial role in establishing (1.3).

Building upon the multiple triangle inequality on $\mathbb{R}^n$, a multiple version of Bellman's inequality (2.4) exists, extending its applicability to multiple sets of vectors (see [9, Proposition 2.6] for details).

Lemma 2.4. *For* $i = 1, 2, \cdots, m$, *let* $\vec{a}_i = (a_{i1}, a_{i2}, \cdots, a_{il})$ *denote* l*-tuples of non-negative real numbers. Suppose that* q *is a real number such that* $a_{i1}^q \geq \sum_{j=2}^l a_{ij}^q$ *for each* $i = 1, 2, \cdots, m$. *If* $q \geq 1$, *then*

$$\left(\left(\sum_{i=1}^m a_{i1} \right)^q - \sum_{j=2}^l \left(\sum_{i=1}^m a_{ij} \right)^q \right)^{1/q} \geq \sum_{i=1}^m \left(a_{i1}^q - \sum_{j=2}^l a_{ij}^q \right)^{1/q}. \tag{2.5}$$

Equality holds in (2.5) if and only if either $q = 1$ *or the sets* $\{\vec{a}_i\}_{i=1}^m$ *are proportional.*

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The following result, which is an equivalent formulation of the multiple Minkowski inequality, is highly beneficial for our analysis.

Lemma 2.5. *Let a_i and b_i be non-negative real numbers such that $a_i \geq b_i$ for all $i = 1, 2, \cdots, m$. If $0 < q < 1$, then*

$$\left(\left(\sum_{i=1}^{m} a_i\right)^q - \left(\sum_{i=1}^{m} b_i\right)^q\right)^{1/q} \leq \sum_{i=1}^{m} \left(a_i^q - b_i^q\right)^{1/q}. \tag{2.6}$$

Equality holds in (2.6) if and only if $a_i = \nu b_i$ for some $\nu > 0$ and all $i = 1, 2, \cdots, m$.

Proof. Noting that for non-negative x_i, y_i with $i = 1, 2, \cdots, m$ and $r > 1$, the multiple Minkowski inequality can be expressed as

$$\left[\sum_{i=1}^{m} (x_i + y_i)^r\right]^{1/r} \leq \left(\sum_{i=1}^{m} x_i^r\right)^{1/r} + \left(\sum_{i=1}^{m} y_i^r\right)^{1/r}, \tag{2.7}$$

with equality if and only if the vectors $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_m)$ are proportional.

Setting $r = 1/q$ and $x_i = b_i^q$, $y_i = a_i^q - b_i^q$ in (2.7), we obtain

$$\left(\sum_{i=1}^{m} a_i\right)^q \leq \left(\sum_{i=1}^{m} b_i\right)^q + \left[\sum_{i=1}^{m} (a_i^q - b_i^q)^{1/q}\right]^q,$$

which yields the desired inequality. By the equality case of the multiple Minkowski inequality (2.7), equality holds in (2.6) if and only if $b_i^q = \mu(a_i^q - b_i^q)$ for some $\mu > 0$, which implies $a_i = \nu b_i$ for some $\nu > 0$ and all $i = 1, 2, \cdots, m$. □

3. The L_p Ky Fan's determinant inequality and its variants

Lemma 3.1. *Suppose $p > 1$ and $\lambda \in [0, 1]$. If $M, N \in \mathcal{H}^n$ are symmetric, positive definite matrices such that $|\mathcal{S}(M)| = |\mathcal{S}(N)| = 1$, then*

$$|\mathcal{S}(\lambda \cdot M +_p \bar{\lambda} \cdot N)| \geq 1. \tag{3.1}$$

For $\lambda \in (0, 1)$, equality holds if and only if $M = N$.

Proof. Recalling that for any $A \in \mathcal{H}^n$, the Schur complement $\mathcal{S}(A)$ of A possesses positive definiteness. Consider $M, N \in \mathcal{H}^n$. Employing (2.1), (2.3), and the Minkowski determinant inequality (1.1) with n substituted by $n - k$, we obtain

$$\begin{aligned} |\mathcal{S}(\lambda \cdot M +_p \bar{\lambda} \cdot N)| &\geq |\mathcal{S}(\lambda M + \bar{\lambda} N)| \\ &\geq |\lambda \mathcal{S}(M) + \bar{\lambda} \mathcal{S}(N)| \\ &\geq \left(\lambda |\mathcal{S}(M)|^{\frac{1}{n-k}} + \bar{\lambda} |\mathcal{S}(N)|^{\frac{1}{n-k}}\right)^{n-k} \\ &= 1. \end{aligned}$$

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To establish the equality case of (3.1), we begin by setting $M = N$. This leads to

$$\begin{aligned}|\mathcal{S}(\lambda \cdot M +_p \bar{\lambda} \cdot M)| &= \left|\mathcal{S}\left((\lambda M^p + \bar{\lambda} M^p)^{\frac{1}{p}}\right)\right| \\ &= |\mathcal{S}(M)| \\ &= 1.\end{aligned}$$

Next, we assume that $|\mathcal{S}(\lambda \cdot M +_p \bar{\lambda} \cdot N)| = 1$. From (2.1), we infer that this implies $|\lambda \mathcal{S}(M) + \bar{\lambda}\mathcal{S}(N)| = 1$. Invoking the equality case of inequality (1.1), we deduce that $\mathcal{S}(M) = c\mathcal{S}(N)$, for some $c > 0$. However, since $|\mathcal{S}(M)| = |\mathcal{S}(N)|$, it follows that $c = 1$ and thus $M = N$. This establishes that $|\mathcal{S}(\lambda \cdot M +_p \bar{\lambda} \cdot N)| = 1$ implies $M = N$, completing the proof. □

Theorem 3.2. *Suppose $p > 1$. If $A, B \in \mathcal{H}^n$ are two symmetric, positive definite matrices, then*

$$\left(\frac{|A +_p B|}{|(A +_p B)_k|}\right)^{\frac{p}{n-k}} \geq \left(\frac{|A|}{|A_k|}\right)^{\frac{p}{n-k}} + \left(\frac{|B|}{|B_k|}\right)^{\frac{p}{n-k}}, \tag{3.2}$$

with equality if and only if $A = cB$ for some $c > 0$.

Proof. In view of Lemma 2.2, it suffices to show that

$$|\mathcal{S}(A +_p B)|^{\frac{p}{n-k}} \geq |\mathcal{S}(A)|^{\frac{p}{n-k}} + |\mathcal{S}(B)|^{\frac{p}{n-k}}. \tag{3.3}$$

Setting

$$\begin{aligned}M &= \frac{1}{|\mathcal{S}(A)|^{\frac{p}{n-k}}} \cdot A = \frac{1}{|\mathcal{S}(A)|^{\frac{1}{n-k}}} A; \\ N &= \frac{1}{|\mathcal{S}(B)|^{\frac{p}{n-k}}} \cdot B = \frac{1}{|\mathcal{S}(B)|^{\frac{1}{n-k}}} B; \\ \lambda &= \frac{|\mathcal{S}(A)|^{\frac{p}{n-k}}}{|\mathcal{S}(A)|^{\frac{p}{n-k}} + |\mathcal{S}(B)|^{\frac{p}{n-k}}}\end{aligned}$$

in Lemma 3.1, we have

$$\left|\mathcal{S}\left(\lambda \cdot (|\mathcal{S}(A)|^{\frac{p}{k-n}} \cdot A) +_p \bar{\lambda} \cdot (|\mathcal{S}(B)|^{\frac{p}{k-n}} \cdot B)\right)\right| \geq 1.$$

Applying the fact that $c \cdot A = c^{1/p} A$ $(c > 0)$ twice, we obtain

$$|\mathcal{S}(\mu A +_p \mu B)| \geq 1,$$

where $\mu = \left(|\mathcal{S}(A)|^{\frac{p}{n-k}} + |\mathcal{S}(B)|^{\frac{p}{n-k}}\right)^{-1/p}$. Equivalently,

$$\left|\mathcal{S}\left(\mu\,(A^p + B^p)^{\frac{1}{p}}\right)\right| \geq 1.$$

Since the Schur complement is homogeneous of degree 1, we arrive at

$$|\mu \mathcal{S}(A +_p B)| \geq 1,$$

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which is equivalent to

$$|\mathcal{S}(A+_p B)| \geq \left(|\mathcal{S}(A)|^{\frac{p}{n-k}} + |\mathcal{S}(B)|^{\frac{p}{n-k}}\right)^{\frac{n-k}{p}}.$$

That confirms (3.3). The equality case follows from that of (3.1) immediately. □

We denote the L_p combination of $A_1, \cdots, A_m \in \mathcal{H}^n$ as $\sharp_{i=1}^m A_i$, representing $A_1 +_p A_2 +_p \cdots +_p A_m$. By employing an induction argument, we can establish the following L_p Ky Fan determinant inequality for multiple positive definite matrices.

Theorem 3.3. *Suppose $p > 1$. Let $A_1, \cdots, A_m \in \mathcal{H}^n$ be positive definite and let $(A_i)_k$ denote the k-th leading principal submatrix of A_i, where $i = 1, 2, \cdots, m$. Then*

$$\left(\frac{|\sharp_{i=1}^m A_i|}{|(\sharp_{i=1}^m A_i)_k|}\right)^{\frac{p}{n-k}} \geq \sum_{i=1}^m \left(\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}}, \tag{3.4}$$

with equality if and only if $A_1, \cdots, A_m$ are multiples of each other.

Proof. We prove the inequality using mathematical induction.

Based on Theorem 3.2, inequality (3.4) is established for $m = 2$.

Now, assuming that inequality (3.4) holds for $m = r-1$ with $r \geq 3$, we proceed the case where $m = r$ as follows:

$$\begin{aligned}\left(\frac{|\sharp_{i=1}^r A_i|}{|(\sharp_{i=1}^r A_i)_k|}\right)^{\frac{p}{n-k}} &= \left(\frac{|\sharp_{i=1}^{r-1} A_i +_p A_r|}{|(\sharp_{i=1}^{r-1} A_i +_p A_r)_k|}\right)^{\frac{p}{n-k}} \\ &\geq \left(\frac{|\sharp_{i=1}^{r-1} A_i|}{|(\sharp_{i=1}^{r-1} A_i)_k|}\right)^{\frac{p}{n-k}} + \left(\frac{|A_r|}{|(A_r)_k|}\right)^{\frac{p}{n-k}} \\ &\geq \sum_{i=1}^r \left(\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}},\end{aligned}$$

where the last inequality follows from the induction hypothesis. The equality case is established based on Theorem 3.2 and the induction argument. □

Theorem 3.4. *For $i = 1, 2, \cdots, m$, let $A_i \in \mathcal{H}^n$ be symmetric, positive definite matrices of order n, with $(A_i)_k$ representing the k-th leading principal submatrices of A_i. Given $1 < p \leq n-k$ and non-negative real numbers α_i such that $A_i > \alpha_i I_n$. Then,*

$$\left(\frac{|\sharp_{i=1}^m A_i|}{|(\sharp_{i=1}^m A_i)_k|} - \left|\left(\sum_{i=1}^m \alpha_i^p\right)^{1/p} I_{n-k}\right|\right)^{\frac{p}{n-k}} \geq \sum_{i=1}^m \left(\frac{|A_i|}{|(A_i)_k|} - |\alpha_i I_{n-k}|\right)^{\frac{p}{n-k}}. \tag{3.5}$$

Equality holds in (3.5) if and only if either $p = n-k$ or $\alpha_i^{-1} A_i = \alpha_j^{-1} A_j$ for $1 \leq i, j \leq m$ with $i \neq j$.

Proof. Since $A_i > \alpha_i I_n$, it follows from Lemma 2.2 that

$$\frac{|A_i|}{|(A_i)_k|} > \alpha_i^{n-k}, \qquad \text{for all } i = 1, 2, \cdots, m. \tag{3.6}$$

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Setting $q=\frac{n-k}{p}$, $l=2$, $a_{i1}^q=\frac{|A_i|}{|(A_i)_k|}$, and $a_{i2}=\alpha_i^p$ in Lemma 2.4, we obtain

$$\begin{aligned}\sum_{i=1}^m\left(\frac{|A_i|}{|(A_i)_k|}-|\alpha_i I_{n-k}|\right)^{\frac{p}{n-k}}&\le\left(\left(\sum_{i=1}^m\left(\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}}\right)^{\frac{n-k}{p}}-\left(\sum_{i=1}^m\alpha_i^p\right)^{\frac{n-k}{p}}\right)^{\frac{p}{n-k}}\\&\le\left(\frac{|\sharp_{i=1}^m A_i|}{|(\sharp_{i=1}^m A_i)_k|}-\left|\left(\sum_{i=1}^m\alpha_i^p\right)^{\frac{1}{p}}I_{n-k}\right|\right)^{\frac{p}{n-k}},\end{aligned}$$

where the last inequality follows from (3.4).

Furthermore, by the equality case of Lemma 2.4, we see that when $1<p<n-k$, equality holds in (3.5) if and only if $\frac{|A_i|}{|(A_i)_k|}$ is proportional to $|\alpha_i I_{n-k}|$ for all $i=1,2,\cdots,m$. This fact, combined with the equality case of Theorem 3.3, implies that $A_i=\mu\alpha_i I_n$ for some $\mu>0$ and all $i=1,2,\cdots,m$. Consequently, equality holds in (3.5) if and only if $\alpha_i^{-1}A_i=\alpha_j^{-1}A_j$ for all $i,j=1,2,\cdots,m$ such that $i\neq j$. This completes the proof. ☐

It is evident that when $m=2$, Theorem 3.4 reduces to Theorem 1.2 presented in the Introduction. Moreover, when $\alpha_i=0$ for all $i=1,2,\cdots,m$, it retrieves the multiple L_p Ky Fan inequality (Theorem 3.3) for $1<p\le n-k$.

Theorem 3.5. *For $i=1,2,\cdots,m$, let $A_i\in\mathcal{H}^n$ be symmetric, positive definite matrices of order n, with $(A_i)_k$ representing the k-th leading principal submatrices of A_i. Given $p>n-k$ and positive real numbers α_i such that $0<A_i<\alpha_i I_n$. Then,*

$$\left(\left|\left(\sum_{i=1}^m\alpha_i^p\right)^{1/p}I_{n-k}\right|-\frac{|\sharp_{i=1}^m A_i|}{|(\sharp_{i=1}^m A_i)_k|}\right)^{\frac{p}{n-k}}\le\sum_{i=1}^m\left(|\alpha_i I_{n-k}|-\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}}. \quad (3.7)$$

Equality holds in (3.7) if and only if $\alpha_i^{-1}A_i=\alpha_j^{-1}A_j$ for $1\le i,j\le m$ with $i\neq j$.

Proof. Since $0<A_i<\alpha_i I_n$, it follows from Lemma 2.2 that

$$\alpha_i^p=\left(\frac{|\alpha_i I_n|}{|(\alpha_i I_n)_k|}\right)^{\frac{p}{n-k}}>\left(\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}}>0,\qquad\text{for all } i=1,2,\cdots,m.$$

Setting $a_i=\alpha_i^p$, $b_i=\left(\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}}$, and $q=\frac{n-k}{p}$ in Lemma 2.6, we obtain

$$\begin{aligned}\sum_{i=1}^m\left(|\alpha_i I_{n-k}|-\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}}&\ge\left[\left(\sum_{i=1}^n\alpha_i^p\right)^{\frac{n-k}{p}}-\left(\sum_{i=1}^m\left(\frac{|A_i|}{|(A_i)_k|}\right)^{\frac{p}{n-k}}\right)^{\frac{n-k}{p}}\right]^{\frac{p}{n-k}}\\&\ge\left[\left|\left(\sum_{i=1}^m\alpha_i^p\right)^{\frac{1}{p}}I_{n-k}\right|-\frac{|\sharp_{i=1}^m A_i|}{|(\sharp_{i=1}^m A_i)_k|}\right]^{\frac{p}{n-k}},\end{aligned}$$

where the last inequality follows from Theorem 3.3.

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The equality case can be established using a similar argument to the one used for Theorem 3.4. □

It is easily seen that Theorem 1.3 from the Introduction is a special case ($m = 2$) of Theorem 3.5.

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