Revista de la Unión Matemática Argentina Accepted article · Early view version

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SUPERPOWER GRAPHS OF FINITE ABELIAN GROUPS

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Abstract. For a finite group G, the superpower graph $S(G)$ of G is an undirected simple graph with vertex set G and two distinct vertices are adjacent in $S(G)$ if and only if the order of one divides the order of the other in G. The aim of this paper is to provide the tight bounds for the vertex connectivity of $S(G)$ together with some structural properties like maximal domination set, Hamiltonicity and its variations of superpower graph of finite abelian groups. We conclude this paper by giving some open problems.

1. Introduction

The investigation of graphs associated with algebraic structures are very important, as graphs like these enrich both algebra and graph theory. Also, these graphs have practical applications and are related to automata theory which can be seen in [11, 12, 13, 14]. The study of graphical representations of semi-groups and groups has become an energizing research topic over a recent couple of decades, prompting many intriguing outcomes and questions. In this context, some of the most studied graphs are the Cayley graph [5, 17, 20], commuting graph [19] and power graph [1, 15] of finite groups. The power graph $P(G)$ of a group G is an undirected graph with vertex set G and any two distinct vertices are adjacent in $P(G)$ if and only if one can be written as the power of the other in G. The notion of the superpower graph $S(G)$ of finite group G is a very recent development in the domain of graphs from groups, and it was first introduced by Hamzeh and Ashrafi [8] (they refer it as the order supergraph $S(G)$ of G) in 2018. In fact, for a finite group G, the superpower graph $S(G)$ is the simple undirected graph with the vertex set G and two distinct vertices $a, b \in G$ are adjacent in $S(G)$ if and only if either $o(a) \mid o(b)$ or $o(b) \mid o(a)$ in G where $o(a)$ is the order of $a \in G$. The aim of their work was to exhibit a relationship between $P(G)$ and $S(G)$ together with some structural properties of the superpower graph. Further, they posed a question that what is the structure of G when $|\pi(G)| = \alpha(S(G))$ where $\pi(G)$ is

2020 *Mathematics Subject Classification.* 05C25, 05C40, 05C45, 05C78.

Key words and phrases. superpower graph, Power graph, Dominant set, Vertex connectivity, Hamiltonian cycle.

The research work of Ajay Kumar is supported by CSIR-UGC JRF, India through Ref No. 19/06/2016(i) EU-V/Roll No. 417267; Lavanya Selvaganesh is financially supported by SERB, India, through Grant No. MTR/2018/000254 under the scheme MATRICS; T. Tamizh Chelvam is supported by CSIR Emeritus Scientist Scheme of Council of Scientific and Industrial Research (No.21(1123)/20/EMR-II), Government of India.

Submitted: December 24, 2023
Accepted: November 22, 2024
Published (early view): December 3, 2024

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the set of all orders of elements in G and $\alpha(S(G))$ is the independence number of $S(G)$. Following this, Ma and Su [18], studied the independence number $\alpha(S(G))$ and answered this question. Hamzeh and Ashrafi [10] computed the automorphism group and the full automorphism group of the superpower graph of certain finite groups. They also proved that the automorphism group of this graph could be written as a combination of Cartesian and wreath products of some symmetric groups. In [7], the same authors computed the characteristic polynomial of these graphs for certain finite groups. Consequently, the spectrum and Laplacian spectrum of these graphs for dihedral, semi-dihedral, cyclic and dicyclic groups were computed. In [9], the authors investigate the Hamiltonicity, Eulerian and 2-connectedness of the superpower graph $S(G)$ of G. Recall that $PSL(2,p)$ and $PGL(2,p)$ are central quotient groups of special linear and general linear groups respectively. Asboei and Salehi [2] proved that the groups $PSL(2,p), PGL(2,p)$ and central simple groups uniquely determine their superpower graphs. That is, for any finite group G, if $S(G) \cong S(PSL(2,p))$ or $S(PGL(2,p))$, then $G \cong PSL(2,p)$ or $PGL(2,p)$ respectively. They also proved that if M is a central simple group and $S(M) \cong S(G)$, then $G \cong M$.

2. Preliminaries

In this section, we present some definitions and results from group theory as well as graph theory. We use standard definitions and results from [6] for group theory and [3] for graph theory which we restate here along with our notations whenever necessary. Throughout this paper, by a group G, we mean a finite group of order n with the identity e. The relation $\sim$ on G defined as: two elements $a, b \in G$ are related if and only if they are of same order forms an equivalence relation on G. For each positive divisor d of n, let $w_d(G) = \{x \in G : o(x) = d\}$ and for any subset $X \subseteq G$, $X^* = X \setminus w_1(G) = X \setminus \{e\}$. Let $\pi(G) = \{a_1, \dots, a_k\}$ denote the set of all orders of elements in G. As usual, $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$ denotes the cyclic group of order n. For a positive integer n, the *Euler's phi function* $\phi(n)$, denotes the number of positive integers at most n and are relatively prime to n. We always take the prime factorization of a positive integer $n \geq 2$ as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ and it is assumed that $m \geq 1$, $p_1 < \cdots < p_m$ are primes and $\alpha_i \in \mathbb{N}$, $\forall i$, $1 \leq i \leq m$. For a positive integer $n, \tau(n)$ denotes the number of divisors of n.

By a graph Γ, we mean a finite undirected simple graph with non-empty vertex set V and edge set E. A vertex of Γ is called a *dominant vertex* if it is adjacent to every other vertex of Γ. A graph Γ is said to be *dominatable* if it contains at least one dominant vertex and $dom(\Gamma)$ denotes the set of all dominant vertices in Γ. A *connected component* of Γ is a maximal connected subgraph of Γ. A subset $T \subset V$ of Γ is called a *separating set* of Γ if the number of connected components of the graph $\Gamma \setminus T$ is greater than the number of connected components of Γ. A separating set T is called minimal separating set if any proper subset $T^\dagger$ of T is not a separating set of Γ. A minimal separating set of minimum cardinality is called a minimum separating set and the cardinality of this minimum separating set is called the *vertex connectivity* of a connected graph Γ and is denoted by $\kappa(\Gamma)$.

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Accepted: November 22, 2024
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If a graph Γ has a path P (or a cycle C) which contains all the vertices of Γ, then P (or C) is called *Hamiltonian path (or Hamiltonian cycle)* of Γ. A graph Γ with a Hamiltonian cycle is called as a *Hamiltonian graph.* A graph Γ is called *1-Hamiltonian* if it is *Hamiltonian* and all of its 1-vertex-deleted subgraphs are *Hamiltonian.* A graph Γ is called *Hamiltonian connected* if any two vertices of Γ can be joined by a Hamiltonian path. A graph Γ on n vertices is called *pancyclic* if, for every $\ell(3 \leq \ell \leq n)$, there exists a cycle of length ℓ in Γ. If a graph Γ has the property that, for any two distinct vertices u and v of G, there exists a path of length ℓ for all possible ℓ for $d(u,v) \leq \ell \leq n$, then Γ is called *pan-connected.* A connected graph Γ is called *k-connected* if after removing any $k-1$ vertices from it, the remaining graph is connected. If a graph Γ has a closed walk W which traverse every edge of Γ exactly once, then W is called an *Euler circuit* and Γ is called an *Eulerian graph.* If Γ is a p-vertex labeled graph and $\Gamma_1,\dots,\Gamma_p$ are connected graphs then Γ-join of $\Gamma_1,\Gamma_2,\dots,\Gamma_p$ is denoted by $\Delta_\Gamma[\Gamma_1,\Gamma_2,\dots,\Gamma_p]$.

Let us recall certain results for our future use.

Theorem 2.1 ([6], Theorem 8.2)**.** *Let G_1 and G_2 be two finite cyclic groups. Then their external direct product $G_1 \times G_2$ is cyclic if and only if orders $o(G_1)$ and $o(G_2)$ are relatively prime.* □

Theorem 2.2 ([6], Theorem 11.1)**.** *Every finite abelian group is the direct product of cyclic groups of prime-power order. Moreover, the number of terms in the direct product and the orders of the cyclic groups are uniquely determined by the group.* □

The above theorem actually gives for a finite abelian group G, $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_m^{\alpha_m}}$, where the cyclic groups $\mathbb{Z}_{p_1^{\alpha_1}}$, $\mathbb{Z}_{p_2^{\alpha_2}}$ and $\mathbb{Z}_{p_m^{\alpha_m}}$ are uniquely determined by G. The following is a known characterization of finite abelian groups through elementary divisors of the same.

Theorem 2.3 ([6], Chapter 11)**.** *Let G be a finite abelian group of order $n = p_1^{\alpha_1}.p_2^{\alpha_2}.\cdots p_m^{\alpha_m}$, where each p_i is prime and $\alpha_i \in \mathbb{N}$. Then $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_1 \geq n_2 \geq \cdots \geq n_k, n_j | n_i$ for each $j \geq i, 1 \leq j \leq k$ and $o(G) = n_1 n_2 \cdots n_k$.* □

In rest of the paper, we use the above characterization for finite abelian groups.

Lemma 2.4 ([6], Chapter 11)**.** *Let G be a finite abelian group having an element x of maximum nontrivial order. Then $o(g)$ divides $o(x)$ $\forall\ g \in G$.* □

Theorem 2.5 ([8], Theorem 2.3)**.** *Let G be a finite group. The superpower graph $S(G)$ is complete if and only if G is a p-group.* □

In view of the above theorem, we consider only finite abelian non p-groups for the rest of the paper unless stated otherwise.

3. Dominant Set of $S(G)$

It is a well known fact that dominant vertices play an important role in characterization of graphs. In fact, if a graph contains a dominant vertex, then it is

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connected and diameter is at most two. Thus, it is interesting to find out the set of all dominant vertices in $S(G)$. In the following theorem, we find the number of dominant vertices in $S(G)$ for any finite abelian non p-group G.

Theorem 3.1. *Let G be a finite abelian non p-group of order n and $dom(S(G))$ be the set of all dominant vertices in the superpower graph $S(G)$ of G. Then $|dom(S(G))| = t\phi(n_1) + 1$, where n_1 is the largest order of an element in G and t is the number of cyclic subgroups of G of order n_1.*

Proof. Let G be a finite abelian group of order n with identity e and let n_1 be the largest order of an element in G. Let $n_1 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$, $\beta_i \in \mathbb{N}$ and $p_1 < \cdots < p_m$ are distinct primes. For a divisor d of n, let $w_d = \{x \in G : o(x) = d\}$. From Lemma 2.4, vertices in $w_{n_1}(G) \cup \{e\}$ are adjacent to every other vertex of $S(G)$ and hence $w_{n_1}(G) \cup \{e\} \subseteq dom(S(G))$. On the other hand, let $x \notin dom(S(G))$. This gives that $1 < o(x) < n_1 \leq o(G)$. By Lemma 2.4, $o(x) \mid n_1$. Let y be an element in G such that $o(y) = p_i^{\beta_i}$ for some $1 \leq i \leq m$ such that $p_i^{\beta_i}$ does not divides n_1. It is trivial that such an element y exists always and x is not adjacent to y. So x is not an element of $w_{n_1}(G) \cup \{e\}$. Hence $dom(S(G) \subseteq w_{n_1}(G) \cup \{e\}$ and so $w_{n_1}(G) \cup w_1(G) = dom(S(G))$. Also, $|w_{n_1}(G) \cup w_1(G)| = t\phi(n_1) + 1$, where t is the number of distinct cyclic subgroups of order n_1 in G. Therefore $|dom(S(G))| = t\phi(n_1) + 1$. □

As mentioned in the proof of Theorem 3.1, $S(G)$ always contains a dominant vertex other than identity and hence we have the following corollary.

Corollary 3.2. *For any finite abelian group G, the superpower graph $S(G^*)$ is dominatable.* □

In view of the above theorem, we take $H_{\overline{G}}$ as the induced subgraph of $S(G)$ induced by the subset $\overline{G} = G \setminus (w_{n_1} \cup \{e\})$ of G.

4. Hamiltonicity of $S(G)$

In [4], it was proved that the power graph $P(G)$ of any cyclic group G of order at least three is Hamiltonian[see Theorem 4.13]. In [8], it was proved that $S(G) = P(G)$ if and only if G is a finite cyclic group. Thus $S(G)$ is Hamiltonian for any cyclic group of order at least three. Now a natural question arises that can we extend this result for finite abelian groups? Unfortunately, the same question has got a negative answer in the case of $P(G)$. For example, $P(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is not Hamiltonian. However, we can extend this result to $S(G)$ and the same is proved in the following theorem. For any divisor d of $o(G)$, the induced subgraph H_d induced by the vertex subset $w_d(G) = \{x \in G : o(x) = d\}$ is a clique in $S(G)$.

Theorem 4.1. *For any finite abelian group G of order greater than two, the superpower graph $S(G)$ is Hamiltonian.*

Proof. Let G be a finite abelian group of order n and n_1 be the largest order of an element in G. Let $\{d_1, d_2, \ldots, d_\ell, n_1\}$ be the divisors of n such that $1 < d_1 < \cdots < d_\ell < n_1 \leq n$. By Lemma 2.4, $d_i | n_1 \ \forall\ i (1 \leq i \leq \ell)$. Note that $\ell < \tau(n_1)$.

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Accepted: November 22, 2024
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For any divisor d of n, the induced subgraph H_d induced by the vertex subset $w_d(G) = \{x \in G : o(x) = d\}$ is a clique in $S(G)$. Now let us trace a Hamiltonian cycle in $S(G)$ as follows:

Start from the vertex $v_1 \in dom(S(G))$. From v_1, go to any vertex of the clique H_{d_1} and traverse all vertices of H_{d_1}. Now we have a Hamilton path containing all the vertices in $H_{d_1} \cup \{v_1\}$. Note that the terminal vertex of this Hamiltonian path is adjacent to a vertex $v_2 \in dom(S(G))$ and $v_2 \neq v_1$. From v_2, go to any vertex of the clique H_{d_2} and traverse all vertices of H_{d_2}. Now the terminating vertex of the resulting Hamiltonian path is adjacent to a vertex vertex $v_3 \in dom(S(G))$ and $v_3 \notin \{v_1, v_2\}$. Repeat this until all the cliques $H_{d_i}(G), 1 \leq i \leq \ell$ are covered. Since $\ell < \tau(n_1) \leq \phi(n_1) < |dom(S(G))|$, there are sufficient number of vertices in $dom(S(G))$ to connect all disjoint cliques. Finally, complete the cycle by joining all the uncovered vertices of $dom(S(G))$ by path to v_1. The entire process of identifying a Hamiltonian cycle is given in Figure 1. □

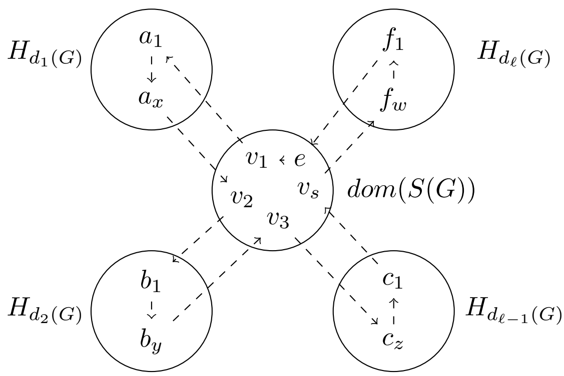

FIGURE 1. Hamiltonian cycle in $S(G)$

Remark 4.2. In [9], it was conjectured that, for any finite simple non-abelian group G, $S(G)$ is non Hamiltonian. In view of this, we observe that $S(G)$ may or may not be Hamiltonian for any non-abelian group G. For instance, we have shown that the superpower graph $S(D_{2n})$ of the dihedral group D_{2n} is Hamiltonian if and only if n is an even integer [16], whereas $S(T_{4n})$ of dicyclic group is Hamiltonian for any integer n (one can prove this in similar lines as we proved for $S(D_{2n})$).

Recall that a graph Γ is called *1-Hamiltonian* if it is *Hamiltonian* and all of its 1-vertex-deleted subgraphs are *Hamiltonian*. Now, we prove that the superpower graph $S(G)$ is 1-Hamiltonian.

Corollary 4.3. *For any finite abelian group G with $o(G) \geq 4$, the superpower graph $S(G)$ is 1-Hamiltonian.*

Proof. Let n_1 be the largest order of an element in G. For $g \in G$, if $o(g) = d < n_1 (d \neq 1, 2)$, then g is a vertex in the clique induced by w_d for the divisor d of

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$o(G)$. Further, $H_d \setminus \{g\}$ remains as a clique and so it has a spanning path whose initial and terminal vertices can be joined by two different vertices of $dom(S(G))$. Now, the proof can be completed as proved in Theorem 4.1.

If $o(g) = 2$, then H_2 is a single vertex and removing this vertex does not disconnect the graph as the vertices of $S(G))$ which are joined through this vertex can also be joined through vertices of $dom(S(G))$.

If $o(g) = n_1$ or 1, then $g \in dom(S(G))$. As seen in the proof of Theorem 4.1, there are sufficient number of vertices in $dom(S(G))\setminus\{g\}$ to connect all the disjoint cliques corresponding to all proper divisors of $o(G)$. Hence the required Hamiltonian cycle can be obtained as in Theorem 4.1. Thus $S(G)$ is 1-Hamiltonian. □

Corollary 4.4. *For any finite abelian group G of order at least three, the superpower graph $S(G)$ is pancyclic.*

Proof. Let $o(G) = n \geq 3$ and n_1 be the largest order of an element in G. Let $\{d_1, d_2, \ldots, d_\ell\}$ be the set of all nontrivial divisors of n_1 with $d_1 < d_2 < \cdots < d_\ell < n_1 \leq n$. By Theorem 3.1, the subgraph induced by $dom(S(G))$ is a clique of size $t\phi(n_1) + 1$ and so we have cycles of lengths from 3 to $t\phi(n_1) + 1$ in $S(G)$. Also, from Theorem 4.1 we know that $S(G)$ contains a cycle of length n.

For any $g_1 \in V(S(G))$, by Corollary 4.3, $S(G) \setminus \{g_1\}$ is Hamiltonian and thus $S(G)$ contains a cycle of length $n - 1$. Note that, in the proof of Corollary 4.3, we see that as long as we keep choosing a vertex $g \in w_{d_1} \subset V(G) \setminus dom(S(G))$, obtaining a cycle containing remaining vertices is immediate. Choose $g_2 \in w_{d_1}(G)$ (if exists), otherwise choose $\{g_2\} \in w_{d_2}(G)$ for some non-trivial divisor $d_2 \neq d_1$ of n and we immediately get that $S(G) \setminus \{g_1, g_2\}$ is Hamiltonian. So $S(G)$ contains a cycle of length $n - 2$. Recursively deleting the vertices of w_{d_i} for each i, $1 \leq i \leq l$, we can get cycles of length $n-2$ to $t\phi(n_1)+2$. Thus $S(G)$ contains cycles of length ℓ for $3 \leq \ell \leq n$ and hence $S(G)$ is pancyclic. □

It is not always true that there exists a Hamiltonian path between any pair of distinct vertices in a graph even if it a Hamiltonian. However, this happens in the case of the superpower graph $S(G)$ of any finite abelian group G and hence we have the following result.

Corollary 4.5. *For any finite abelian group G, the superpower graph $S(G)$ is Hamiltonian connected.*

Proof. Let $u, v \in V(S(G))$ be two distinct vertices in $S(G)$. Without loss of generality, one can take $u = v_1 \in w_{d_1}(G)$ and $v = v_2 \in w_{d_2}(G)$ where d_1 and d_2 are two non-trivial distinct divisors of $o(G)$. Start from the vertex v_1 and traverse along the spanning path in $H_{d_1}(G)$ and join it with a vertex v_3 of $dom(S(G))$. From v_3 go to any vertex of $H_{d_3}(G)$ and repeat the process until all vertices of the cliques $H_{d_i}(G) \cup dom(S(G)), 3 \leq i \leq \ell$ belongs to the path such that $v_\ell \in dom(S(G))$ is the last vertex of this path. Now, join v_ℓ to a vertex $x \neq v_2$ of $H_{d_2(G)}$. Upon completing the path from x to v_2 in $H_{d_2}(G)$, we obtain the required Hamiltonian path between u and v in $S(G)$. □

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Corollary 4.6. *For any finite abelian group G, the superpower graph $S(G)$ is pan-connected.*

Proof. Let $u, v \in V(S(G))$ be two distinct vertices in $S(G)$. For every k, $d(u,v) \leq k \leq n$, a path of length k, can be obtained by inserting the required number of vertices from $H_{d_i} \cup dom(S(G)), 1 \leq i \leq \ell$ in such a way that any two vertices of cliques H_{d_i}, H_{d_j} can be joined through a vertex of $dom(S(G))$. This implies that $S(G)$ is pan-connected. □

Now a natural question arises that what will be the effect on Hamiltonicity of the graph $S(G)$, if we remove all dominant vertices from it? In the following theorem, we are giving the answer for this question.

Theorem 4.7. *Let G be a finite abelian non p-group and n_1 be the largest order of an element in G. Let $w_{n_1}(G) = \{x \in G : o(x) = n_1\}$. Then the induced subgraph $H_{\overline{G}}$ of the superpower graph $S(G)$ induced by $\overline{G} = G \backslash (\{e\} \cup w_{n_1}(G))$ is Hamiltonian if and only if n_1 is not a product of two distinct primes.*

Proof. Assume that $H_{\overline{G}}$ is Hamiltonian. If $n_1 = pq$ for two distinct primes p and q, then $H_{\overline{G}} = H_p \cup H_q$ is disconnected as there is no path connecting the vertices of H_p and H_q, a contradiction.

Conversely, assume that n_1 is not a product of two distinct primes. Since G is not a p-group, we have $n_1 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ and $\beta_i \in \mathbb{N}, m \geq 2$ and $p_1 < \cdots < p_m$ are distinct primes. By the assumption on n_1, it can be seen that when $m = 2$ either $\beta_1 > 1$ or $\beta_2 > 1$. Now the largest order of an element in the set $\overline{G} = G \setminus (\{e\} \cup w_{n_1}(G))$ will be $\frac{n_1}{p_1} (= \overline{n}_1, \text{say})$.

Let $w_{\overline{n}_1}(\overline{G}) = \{b_1, \cdots, b_s\}$ be the set of all elements of order $\overline{n}_1$. Let $\{\overline{d_1}, \ldots, \overline{d_\ell}\}$ be the set of all non trivial divisors of $\overline{n_1}$ with $1 < \overline{d}_1 < \overline{d}_2 < \cdots < \overline{d_\ell} < \overline{n}_1$. Let $w_{\overline{d}_i}(\overline{G})$ be the set of all elements of order $\overline{d}_i$ in $\overline{G}$ and let $H_{\overline{d}_i}$ be the subgraph induced by $w_{\overline{d}_i}(\overline{G})$. For each $i, 1 \leq i \leq \ell$, let $P_{H_{\overline{d}_i}} =< v_i, u_i, \cdots, x_i >$ be the Hamiltonian path in $H_{\overline{d}_i}$. Then the induced subgraph H of $H_{\overline{G}}$ on the vertices of $\bigcup_{1 \leq i \leq \ell} w_{\overline{d_i}} \cup w_{\overline{n_1}}$ is Hamiltonian, since

$$C =< b_1, P_{H_{\overline{d}_1}}, b_2, P_{H_{\overline{d}_2}}, b_3, \cdots, b_\ell, P_{H_{\overline{d_\ell}}}, b_{\ell+1}, \cdots, b_s >$$

is a Hamiltonian cycle in H.

It remains for us to include remaining vertices in $H_{\overline{G}} \setminus H$ into C appropriately to get Hamiltonian cycle in $H_{\overline{G}}$. Based on the condition on $\overline{n}_1$, we observe that the only possible subsets of different orders in $\overline{G} \setminus \{\bigcup_{1 \leq i \leq \ell} w_{\overline{d}_i}(\overline{G}) \cup w_{\overline{n}_1}(\overline{G})\}$ are of the form $w_{p_1{}^{\beta_1}}(\overline{G})$ and $w_{p_1^{\beta_1} r}(\overline{G})$, where $r = \overline{d_i}$ for some $i, 1 < i \leq \ell$. If cliques $H_{p_1^{\beta_1}}$, $H_{p_1{}^{\beta_1}\overline{d_j}}$ exist in $H_{\overline{G}} \setminus H$, then the spanning paths $P(v_1^{'}, u_1^{'})$ of $H_{p_1^{\alpha_1}}$ and $P(v_j^{'}, u_j^{'})$ for $1 < j \leq \ell$ of $H_{p_1{}^{\alpha_1} d_j}$ are inserted into the spanning path of $H_{\overline{d_1}}$ and $H_{\overline{d_j}}$ respectively, as shown in Figure 2. That is, the required Hamiltonian cycle $C_{H_{\overline{G}}}$ in $H_{\overline{G}}$ is given by $< b_1, P_1, b_2, P_2, \cdots, b_\ell, P_l, b_{\ell+1}, \cdots, b_s >$, where $P_j =< v_j, v_j^{'}, P(v_j^{'}, u_j^{'}), u_j, P(u_j, x_j) >$(if it exists). □

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Revista de la Unión Matemática Argentina Accepted article · Early view version
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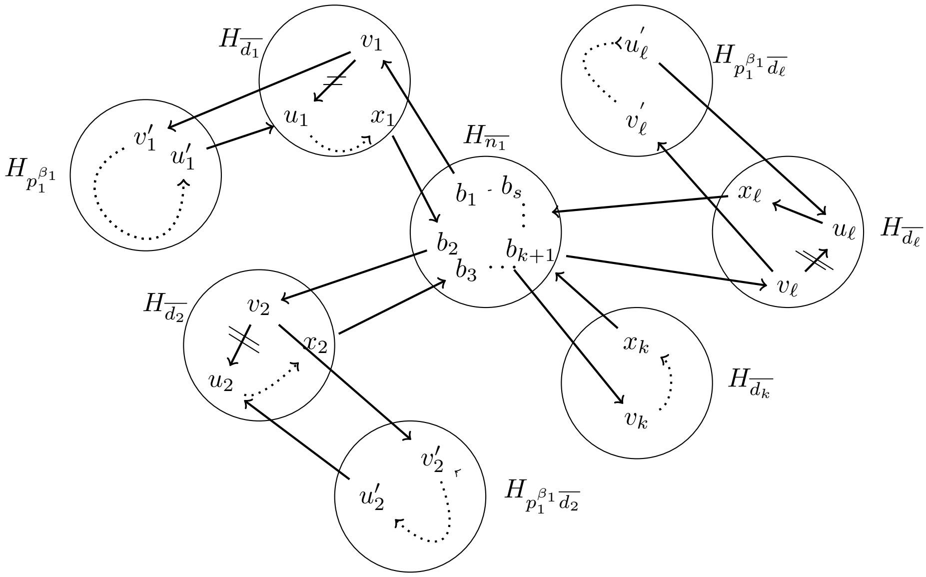

FIGURE 2. Hamiltonian cycle in $H_{\overline{G}}$

Recall that that $S(\mathbb{Z}_n) = P(\mathbb{Z}_n)$, for every positive integer n. Since the dominant vertices of both $S(\mathbb{Z}_n)$ and $P(\mathbb{Z}_n)$ are the only generators of $\mathbb{Z}_n$, we have $dom(S(\mathbb{Z}_n)) = dom(P(\mathbb{Z}_n))$. Let $P(\overline{\mathbb{Z}_n})$ be the induced power graph of $P(\mathbb{Z}_n)$ by removing all the dominant vertices from it. This observation gives us the following interesting property of $P(\overline{\mathbb{Z}_n})$ from $H_{\overline{\mathbb{Z}_n}}$.

Corollary 4.8. *For any positive integer n, $H_{\overline{\mathbb{Z}_n}} = P(\overline{\mathbb{Z}_n})$ is Hamiltonian if and only if n is not a product of two distinct primes.*

In [9], it was proved that $S(G)$ is Eulerian if and only if G is a group of odd order. What will be the effect on order of G if we remove all dominant vertices from $S(G)$? Following theorem gives answer for this question.

Theorem 4.9. *Let G be a finite abelian non p-group. For any proper divisor d of n, let $w_d = \{x \in G : o(x) = d\}$. Let n_1 be the largest order of an element in the group G and $\overline{G} = G \setminus (\{e\} \cup w_{n_1}(G))$. Then the subgraph $H_{\overline{G}}$ of the superpower graph $S(G)$ induced by $\overline{G}$ is Eulerian if and only if n is an even integer.*

Proof. Suppose $H_{\overline{G}}$ is Eulerian and $o(G) = n$. Let $\pi(\overline{G}) = \{a_1, a_2, \cdots, a_r\}$ be the set of all orders of elements in $\overline{G}$. For any $x \in \overline{G}$ with $o(x) = a_i$,

$$deg_{H_{\overline{G}}}(x) = w_{a_i}(\overline{G}) + \sum_{a_i|a_j \text{ or } a_j|a_i,\ a_i \neq a_j} w_{a_j}(\overline{G}) - 1. \tag{1}$$

For each $i, 1 \leq i \leq k$, number of elements of order $a_i = t_i\phi(a_i)$, where t_i is the number of cyclic subgroups of order a_i in $\overline{G}$. Also, it is a well known fact that $\phi(k)$ is odd if and only if $k \in \{1, 2\}$. Clearly, $deg_{H_{\overline{G}}}(x)$ is even if and only if

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Accepted: November 22, 2024
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expression in (1) is even which is possible if and only if G must have odd number of involution elements. Thus, n is even. Conversely, assume that n is even. By Cauchy theorem, G and hence $\overline{G}$ must have an odd number of involutions and these are odd numbers. By expression in (1), degree of any element in $H_{\overline{G}}$ is even. Thus $H_{\overline{G}}$ is Eulerian. □

5. Vertex Connectivity of $S(G)$

It is well known that any graph containing Hamiltonian cycle is 2-connected and hence $S(G)$ is 2-connected for any finite abelian group. Hence we have the following observation from Corollary refcor4.8.

Corollary 5.1. *Let G be a finite abelian non p-group such that maximum order of an element is not a product of two primes. Then $H_{\overline{G}}$ is 2-connected.*

In the following theorem, we provide a lower bound for the vertex connectivity $\kappa(S(G))$ of $S(G)$ for any finite abelian group G which extend Theorem 2.7[9] and [Theorem 2.11[8].

Theorem 5.2. *Let G be a finite abelian group of order n, let n_1 be the largest order of elements in G and t be the number of cyclic subgroups of order n_1 in G. Then $\kappa(S(G)) \geq t\phi(n_1)+1$. Further, $\kappa(S(G)) = t\phi(n_1)+1$ if and only if $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_p^{s_1}$ or $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_q^{s_2}$ where p and q are two distinct primes and $r_1 \geq 1$ and $s_1, s_2 \geq 0$ are integers.*

Proof. Let G be a finite abelian group n, let n_1 be the largest order of elements in G and t be the number of cyclic subgroups of order n_1 in G. Note that, one needs to remove at least all of the vertices of $dom(S(G)) = t\phi(n_1)+1$ to disconnect $S(G)$. Thus by Theorem 3.1, we have $\kappa(S(G)) \geq t\phi(n_1) + 1$.

Next we prove the second part of the statement. Assume that $G = \mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_p^{s_1}$ or $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_q^{s_2}$ where p and q are two distinct primes and $r_1 \geq 1$ and $s_1, s_2 \geq 0$ are integers. Note that the largest order of elements in G is $n_1 = pq$. Let $\overline{G} = G \setminus (\{e\} \cup w_{n_1}(G))$. Since there is no path between elements of orders p and q, the subgraph $H_{\overline{G}}$ induced by $\overline{G} = G \setminus (\{e\} \cup w_{n_1}(G))$ is disconnected. Thus $\kappa(S(G)) = t\phi(n_1) + 1$.

Conversely, assume that $\kappa(S(G)) = t\phi(n_1)+1$. Let G be a group such that n_1 is the largest order of elements in G. If $n_1 \neq pq$, by Theorem 4.7, $H_{\overline{G}}$ is Hamiltonian which implies that $\kappa(S(G)) > t\phi(n_1) + 1$.. Note that, only possible abelian groups having two prime divisors p and q for $o(G)$ and with largest order as $n_1 = pq$ are $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_p^{s_1}$ or $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_q^{s_2}$ where p and q are two distinct primes and $r_1 \geq 1$ and $s_1, s_2 \geq 0$ are integers. □

Let G be a finite abelian group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 2$. Assume that $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, where n_i for $1 \leq i \leq k$ are elementary divisors of G. As stated in Theorem 2.3, $n_j | n_i$ for each $j \geq i, n_i \in \mathbb{N}, 1 \leq i, j \leq k$ and $n_1 n_2 \cdots n_k = n$. Let $n_1 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}, 1 \leq s \leq k, \beta_i \geq 0, 1 \leq i \leq s$ be the prime decomposition of the largest order n_1. Let $a_0 = p_s^{\beta_s}, a_1 = \frac{n_1}{a_0}$ and $\pi(G) = \{a_0, a_1, a_2, \cdots, a_r\}$ be the set of all orders of elements in G.

Submitted: December 24, 2023
Accepted: November 22, 2024
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In the following theorem, we find an upper bound for the vertex connectivity of $S(G)$ using the notations defined above.

Theorem 5.3. *Let G be a finite abelian group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 2$. Let $n_1 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}$ be the largest order of elements in G. Let $a_0 = p_s^{\beta_s}, a_1 = \frac{n_1}{a_0}$ and let $\pi(G) = \{a_0, a_1, \cdots, a_r\}$ be the set of all orders of elements in G. Then there exists a minimal separating set T of $S(G)$ with*

$$|T| = \sum_{(a_i|a_1 \ or \ a_1|a_i, \ a_i \neq a_1)} t_i \phi(a_i)$$

where t_i is the number of cyclic subgroups of order a_i in G.

Proof. Consider the set

$$T = \{w_{a_i}(G) : \ a_i|a_1 \ or \ a_1|a_i, \text{ for } 2 \leq i \leq r\}.$$

Since there is no path between vertices of the cliques $w_{a_0}(G)$ and $w_{a_1}(G)$, T is a separating set of $S(G)$. Let A and B are two connected components of $S(G) \setminus T$ such that $w_{a_0}(G) \subseteq V(A)$ and $w_{a_1}(G) \subseteq V(B)$. Now, we prove that T is a minimal separating set by showing that, for any proper subset $T^\dagger$ of T, there is a path connecting $u \in w_{a_0}(G)$ and $v \in w_{a_1}(G)$ in $S(G) \setminus T^\dagger$. Without loss of generality, one can take $T^\dagger = T \setminus \{x\}$ for $x \in w_{a_r}(G)$. Since either $a_r|a_1$ or $a_1|a_r$, there exists a path $P_1(u, x)$ connecting u and x in $A \cup w_r(G)$. Similarly, for $y \in B$ with $o(y) = a_r p_s^{\beta_s}$, there exists a path $P_2(y, v)$ connecting y to v in B. Now, $P =:< P_1(u, x), x, y, P_2(y, v) >$ is a path connecting u and v in $S(G) \setminus T^\dagger$. Thus, T is a minimal separating set of $S(G)$. If t_i is the number of cyclic subgroups of order a_i in G for $2 \leq i \leq r$, then

$$|T| = \sum_{(a_i|a_1 \ or \ a_1|a_i, \ a_i \neq a_1)} t_i \phi(a_i).$$

□

Remark 5.4. The bound obtained in Theorem 5.3 is tight. For three distinct primes p, q, r with $p < q < r$, we shall show that the bound is tight for groups $G \in \{\mathbb{Z}_{pq}, \mathbb{Z}_{pqr}, \mathbb{Z}_{pqr} \times \mathbb{Z}_r \times \mathbb{Z}_r \times \cdots \times \mathbb{Z}_r\}$.

When $G \cong \mathbb{Z}_{pq}$, by Theorem 5.2, we have that $\kappa(S(G)) = \phi(1) + \phi(pq) = |T|$.

Let $G \cong \mathbb{Z}_{pqr} \times \mathbb{Z}_r \times \cdots \times \mathbb{Z}_r$. By Theorem 5.2, $H_{\overline{G}}$ is connected. If $\overline{T}$ is a minimum separating set of $H_{\overline{G}}$, then $T = \overline{T} \cup dom(S(G))$ is a minimum separating set of $S(G)$. Hence to get the vertex connectivity of $S(G)$, it is enough to find the vertex connectivity of $H_{\overline{G}}$. Notice that the equivalence classes of $\overline{G}$ with respect to $\sim$ are precisely $w_p(G)$, $w_q(G)$, $w_r(G)$, $w_{pq}(G)$, $w_{pr}(G)$, $w_{qr}(G)$ and each of these equivalence classes is a clique in $H_{\overline{G}}$. Thus,

$$H_{\overline{G}} = \Delta_{\overline{G}}[K_{w_p(G)}, K_{w_q(G)}, K_{w_r(G)}, K_{w_{pq}(G)}, K_{w_{pr}(G)}, K_{w_{qr}(G)}],$$

as given in Figure 3.

It is clear that deletion of any one of the cliques in $H_{\overline{G}}$ does not disconnect $H_{\overline{G}}$. However, deletion of any two cliques in $H_{\overline{G}}$ that are not adjacent can

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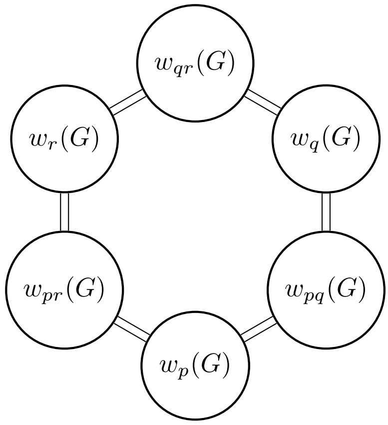

FIGURE 3. Connectivity in $H_{\overline{G}}$.

disconnect the same. This imply that a minimal separating set of $H_{\overline{G}}$ is precisely the union of any two non adjacent cliques. Also, we have the inequality $|w_p(G)| < |w_q(G)| < |w_r(G)|$ and $|w_{pq}(G)| < |w_{pr}(G)| < |w_{qr}(G)|$. Heuristically, we first add the smallest clique namely $w_p(G)$ into the minimum separating set $\overline{T}$, we find that next best possible non adjacent clique having minimum cardinality is $w_q(G)$. Thus, $\overline{T} = w_p(G) \cup w_q(G)$ is a minimum separating set of $H_{\overline{G}}$ implying that $T = w_p(G) \cup w_q(G) \cup w_{pqr}(G) \cup w_1(G)$ is a minimum separating set of $S(G)$ having the cardinality $\phi(pqr) + \phi(p) + \phi(q) + \phi(1)$.

A similar proof holds when $G \cong \mathbb{Z}_{pqr}$.

Corollary 5.5. *Let G be an finite abelian non p-group of order n. If largest order n_1 of elements in G is not a product of two distinct primes, then*

$$t\phi(n_1) + 3 \leq \kappa(S(G)) \leq |T|,$$

where t denotes the number of cyclic subgroups of order n_1 in G and T is the minimal separating set of $S(G)$ as obtained in Theorem 5.3.

Proof. Note that the required lower bound follows from Corollary 5.1 and Theorem 5.2 while the upper bound follows from Theorem 5.3. □

We conclude this article by giving some open problems.

6. Open Problems

Problem 6.1. Characterize all non-abelian groups G for which $S(G)$ is Hamiltonian.

Problem 6.2. Obtain a tight bound for the edge connectivity of $S(G)$ where G is a finite abelian group.

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Accepted: November 22, 2024
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References

[1] Jemal Abawajy, Andrei Kelarev, and Morshed Chowdhury. Power graphs: A survey. *Electronic Journal of Graph Theory and Applications (EJGTA)*, 1(2):125–147, 2013.

[2] Alireza Khalili Asboei and Seyed Sadegh Salehi. Some results on the main supergraph of finite groups. *Algebra and Discrete Mathematics*, 30(2):172–178, 2020.

[3] J.A. Bondy and U.S.R Murty. *Graph Theory*. Springer Publishing Company, Incorporated, 1st edition, 2008.

[4] Ivy Chakrabarty, Shamik Ghosh, and MK Sen. Undirected power graphs of semigroups. In *Semigroup Forum*, volume 78, pages 410–426. Springer, 2009.

[5] Stephen J Curran and Joseph A Gallian. Hamiltonian cycles and paths in cayley graphs and digraphs a survey. *Discrete Mathe.*, 156(1-3):1–18, 1996.

[6] Joseph A. Gallian. *Contemporary Abstract Algebra*. Narosa publishing house, fourth edition, 1999.

[7] Asma Hamzeh and Ali Reza Ashrafi. Spectrum and $L-$spectrum of the power graph and its main supergraph for certain finite groups. *Filomat*, 31(16):5323–5334, 2017.

[8] Asma Hamzeh and Ali Reza Ashrafi. The order supergraph of the power graph of a finite group. *Turkish J. Math.*, 42(4):1978–1989, 2018.

[9] Asma Hamzeh and Ali Reza Ashrafi. Some remarks on the order supergraph of the power graph of a finite group. *Int. Electron. J. Algebra*, 26(26):1–12, 2019.

[10] Asma Hamzeh and Ali Reza Ashrafi. Automorphism groups of supergraphs of the power graph of a finite group. *Eur J Combin*, 60:82–88, 2017.

[11] Andrei Kelarev. *Graph algebras and automata*. CRC Press, 2003.

[12] Andrei Kelarev. Labelled cayley graphs and minimal automata. *Australas. J. Combin.*, 30:95–101, 2004.

[13] Andrei Kelarev, Joe Ryan, and John Yearwood. Cayley graphs as classifiers for data mining: the influence of asymmetries. *Discrete Math.*, 309(17):5360–5369, 2009.

[14] Andrei V Kelarev. *Ring constructions and applications*, volume 9. World Scientific, 2002.

[15] Ajay Kumar, Lavanya Selvaganesh, Peter J Cameron, and T Tamizh Chelvam. Recent developments on the power graph of finite groups–a survey. *AKCE Int. J. Graphs Comb.*, 18(2):65–94, 2021.

[16] Ajay Kumar, Lavanya Selvaganesh, and T Tamizh Chelvam. Structural Properties of Super PowerGraph of Dihedral Group D_{2n}. *preprint: submitted for Publication*, pages 1–11, 2021.

[17] Cai Heng Li. On isomorphisms of finite cayley graphs a survey. *Discrete Math.*, 256(1-2):301–334, 2002.

[18] Xuanlong Ma and Huadong Su. On the order supergraph of the power graph of a finite group. *Ric. Mat.*, pages 1–10, 2020.

[19] Mahsa Mirzargar. A survey on the automorphism groups of the commuting graphs and power graphs. *Facta Universitatis, Series: Mathematics and Informatics*, pages 729–743, 2019.

[20] David Witte and Joseph A Gallian. A survey: Hamiltonian cycles in cayley graphs. *Discrete Math.*, 51(3):293–304, 1984.

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Submitted: December 24, 2023
Accepted: November 22, 2024
Published (early view): December 3, 2024