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THE LIMIT CASE IN A NONLOCAL p-LAPLACIAN EQUATION WITH DYNAMICAL BOUNDARY CONDITIONS

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Abstract. In this paper we deal with the limit as $p \to \infty$ for the nonlocal analogous to the p-Laplacian evolution with dynamic boundary conditions. Our main result states that in both the elliptic and the parabolic case. We are interested in smooth and singular kernels and show the existence and uniqueness of a limit solution. We obtain that the limit solution of the elliptic problem turns out to be also a viscosity solution of a corresponding problem. We prove that the natural energy functionals associated with this problem converge in the sense of Mosco convergence to a limit functional and therefore we obtain convergence of the solutions to the evolution problems for the parabolic case. For the limit problem, we provide examples of explicit solutions for some particular data.

1. Introduction

Our main purpose in this paper is to study a nonlocal diffusion equation obtained as the limit as $p \to \infty$ to the p-Laplacian with dynamical boundary conditions, that is, we look for the limit as $p \to \infty$ of the solutions to the following problem (P1);

$$\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dy, & x \in \Omega_r,\ t>0, \\ \dfrac{\partial u}{\partial t}(x,t) = \displaystyle\int_{\Omega_r} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dy + f(x,t), & x \in \Gamma_r,\ t>0, \\ u(x,0) = u_0(x), & x \in \Gamma_r \end{cases}$$

where $\hat{\Omega}$ is a smooth bounded domain and inside this domain a narrow strip $\Gamma_r = \{x \in \hat{\Omega} : dist(x, \partial\hat{\Omega}) \leq r\}$, with $r \leq R$, $\Omega_r = \hat{\Omega} \setminus \Gamma_r$. We consider $1 < p < \infty$ and the limit $p \to \infty$. We assume that non-singular kernel $J : \mathbb{R}^n \to \mathbb{R}$ is a nonnegative, continuous, radially symmetric, decreasing, compactly supported (let $supp(J) = B_R(0)$) with $\int J(w)dw = 1$. We also analyze the case in which the kernel J can be singular.

First of all, we introduce the linear form of nonlocal equations,

$$\frac{\partial u}{\partial t}(x,t) = \int J(x-y)|u(y,t)-u(x,t)|(u(y,t)-u(x,t))dy.$$

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These types of equations have been widely used in the modeling of diffusion processes. More precisely, as stated in [23], if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int J(y-x)u(y,t)dy$ is the rate at which individuals are arriving at position x from all other places and $-\int J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. Then these nonlocal equations give that the change in time of the density of individuals at x at time t is just the balance between arriving to/leaving from x at time t (see [1, 20, 21, 23]). As it was pointed out that in [30] according to this probabilistic interpretation of the nonlocal terms, we can regard as a model for the following situation: particles leave or arrive from $x \in \Omega_r$ to $y \in \Omega_r \cup \Gamma_r$ in very fast time scales given rise to an "elliptic" equation inside Ω_r (notice that t is only a parameter in the first equation that appears in (P1)). On the other hand, individuals arrive to or leave from $x \in \Gamma_r$ from other sites $y \in \Omega_r$ in the slow time scale. This gives the second equation in (P1) in which a time derivative appears. In [32, 26]; the existence and uniqueness of mild and strong solutions of nonlocal nonlinear diffusion problems of p-Laplacian type with nonlinear boundary conditions posed in metric random walk spaces were studied. For recent works on nonlocal diffusion, see [6, 11, 12, 17, 19, 21, 23, 24, 33, 34].

Concerning limits as $p \to \infty$ one of the first papers that studies this kind of problems is [13], let u_p denote the solution to following problem

$$\begin{cases} -\Delta_p u_p = f \text{ in } \Omega \\ u_p = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $f \in C(\Omega)$ and $f > 0$. Then, from [13], we have that u_p converges uniformly as $p \to \infty$ to the distance to the boundary, that is, $\lim_{p\to\infty} u_p(x) = u_\infty(x) = d(x, \partial\Omega)$ for $x \in \overline{\Omega}$.

In [9] and [22], it was studied the limiting behavior as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) - \Delta_p v(x,t) = f(x,t), & \text{in } (0,T) \times \mathbb{R}^N, \\ v(x,0) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases} \tag{2}$$

In [9], assuming that u_0 is a Lipschitz function with compact support, satisfying $|\nabla u_0| \le 1$, it is proved that $v_p \to v_\infty$ and the limit function v_∞ satisfies

$$f(x,t) - \frac{\partial v_\infty}{\partial t}(x,t) \in \partial F_\infty(v_\infty(x,t)),$$

with

$$F_\infty(v) = \begin{cases} 0, & \text{if } |\nabla v| \le 1, \\ +\infty, & \text{in other case.} \end{cases}$$

The limit problem explains the movement of a sandpile ($v_\infty(t,x)$ describes the amount of the sand at the point x and time t), the main assumption being that

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the sandpile is stable when the slope is less than one or equal to one and unstable if not.

In [29], we looked for the nonlinear diffusion equation obtained as the limit as $p \to \infty$ to the p-Laplacian with dynamical boundary conditions, that is, we looked for the limit as $p \to \infty$ of the solutions to the following problem;

$$\begin{cases} 0 = \Delta_p u(x,t), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial t}(x,t) + |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta}(x,t) = f(x,t), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \partial\Omega. \end{cases} \tag{3}$$

We proved that the natural energy functionals associated with this problem converge in the sense of Mosco convergence to a limit functional and therefore we obtain convergence of the solutions to the evolution problems.

In [30], the authors deal both with smooth and singular kernels and show the existence and uniqueness of solutions and study their asymptotic behavior as t goes to infinity for (P1) with homogeneous dynamic boundary conditions.

In [2], the authors study on the nonlocal ∞-Laplacian type diffusion equation obtained as the limit as $p \to \infty$ to the nonlocal analogous to the p-Laplacian evolution,

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\mathbb{R}^N} J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy + f(x,t) \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^N,\ t > 0. \end{cases} \tag{4}$$

They prove existence and uniqueness of a limit solution that verifies an equation governed by the subdifferential of a convex energy functional associated to the indicator function of the set

$$K = \{u \in L^2(\mathbb{R}^N) : |u(x)-u(y)| \leq 1,\ \text{when } x-y \in \operatorname{supp}(J)\}.$$

In [8], the authors study on the nonlocal p-Laplacian Dirichlet problem

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\Omega} J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy + f(x,t), \\ u(x,t) = \varphi(x),\ x \in \Omega_J \setminus \overline{\Omega},\ t > 0, \\ u(x,0) = u_0(x),\ x \in \Omega. \end{cases} \tag{5}$$

where $\Omega_J = \Omega + supp(J)$ and φ is given function $\varphi : \Omega_J \setminus \overline{\Omega} \to \mathbb{R}$. In this paper, the authors prove the existence and uniqueness of strong solutions for the nonlocal p-Laplacian problem with Dirichlet boundary conditions for $p > 1$ and they showed that this model approaches the local p-Laplacian evolution equation with Dirichlet boundary condition. They study the Dirichlet problem for the nonlocal total variation flow, proving convergence to the local model when the problem is rescaled appropriately as well. Lastly they study the case $p = \infty$, obtaining a model for sandpiles with Dirichlet boundary conditions.

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We have been inspired by the works [2, 8], closely related to the present work, where the Neumann and Dirichlet boundary value problems and their limits as p goes to infinity are considered. The difference here is that we are now considering dynamic boundary conditions with the nonhomogeneous case.

The rest of the paper is organized as follows: in Section 2, we recall some useful results that will be used in the proofs of theorems, among them some technical tools from convex analysis; in Section 3, we consider the limit $p \to \infty$ in elliptic case with smooth and with singular kernels. And also, in the case of singular kernel, we obtain that the limit solution of the elliptic problem turns out to be also a viscosity solution of a corresponding problem; in Section 4, we prove that the natural energy functionals associated with (P1) converge in the sense of Mosco convergence to a limit functional and therefore we obtain convergence of the solutions to the evolution problems; in Section 5, for the limit problem we provide examples of explicit solutions for some particular data.

2. Preliminaries

In this section, we collect some preliminaries and notations that will be used in the paper. We refer the reader to [1, 10, 14, 27, 28].

First, we recall the definition of Mosco-convergence. If X is a metric space, and $\{A_n\}$ is a sequence of subsets of X, we define

$$\liminf_{n\to\infty} A_n := \Big\{ x \in X \ : \ \exists x_n \in A_n, \ x_n \to x \Big\},$$

and

$$\limsup_{n\to\infty} A_n := \Big\{ x \in X \ : \ \exists x_{n_k} \in A_{n_k}, \ x_{n_k} \to x \Big\}.$$

If X is a normed space, we denote by $s-\lim$ and $w-\lim$ the above limits associated, respectively, to the strong and to the weak topology of X.

Definition 2.1. Let H be a Hilbert space. Given $\Psi_n, \Psi : H \to (-\infty, +\infty]$ convex, lower-semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if

$$w - \limsup_{n\to\infty} \operatorname{Epi}(\Psi_n) \subset \operatorname{Epi}(\Psi) \subset s - \liminf_{n\to\infty} \operatorname{Epi}(\Psi_n), \tag{6}$$

where $\operatorname{Epi}(\Psi_n)$ and $\operatorname{Epi}(\Psi)$ denote the epigraphs of the functionals Ψ_n and Ψ, defined by

$$\operatorname{Epi}(\Psi_n) := \Big\{ (u, \lambda) \in L^2(\mathbb{R}^N) \times \ \mathbb{R} \ : \ \lambda \geq \Psi_n(u) \Big\},$$

and

$$\operatorname{Epi}(\Psi) := \Big\{ (u, \lambda) \in L^2(\mathbb{R}^N) \times \ \mathbb{R} \ : \ \lambda \geq \Psi(u) \Big\}.$$

Remark 2.2. We note that (6) is equivalent to the requirement that the following two conditions are simultaneously satisfied:

$$\forall\, u \in D(\Psi) \ \ \exists u_n \in D(\Psi_n) \ : \ u_n \to u \ \text{ and } \ \Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n); \tag{7}$$

$$\text{for every subsequence } \{n_k\}, \ \Psi(u) \leq \liminf_{k} \Psi_{n_k}(u_k) \text{ whenever } u_k \rightharpoonup u. \tag{8}$$

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Here $D(\Psi) := \{u \in H \ : \ \Psi(u) < \infty\}$ and $D(\Psi_n) := \{u \in H \ : \ \Psi_n(u) < \infty\}$ denote the domains of Ψ and Ψ_n, respectively.

If H is a real Hilbert space with inner product $(\cdot,\cdot)$ and $\Psi : H \to (-\infty,+\infty]$ is convex, then the subdifferential of Ψ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \geq (v, w-u) \quad \forall\, w \in H.$$

Recall that the epigraph of Ψ is defined by

$$\mathrm{Epi}(\Psi) = \Big\{(u,\lambda) \in H \times \mathbb{R} \ : \ \lambda \geq \Psi(u)\Big\}.$$

Given K a closed convex subset of H, we define the indicator function of K by

$$I_K(u) = \begin{cases} 0 & \text{if } \ u \in K, \\ +\infty & \text{if } \ u \notin K. \end{cases}$$

Then the subdifferential is characterized by

$$v \in \partial I_K(u) \iff u \in K \ \text{ and } \ (v, w-u) \leq 0 \quad \forall\, w \in K.$$

When the convex functional $\Psi : H \to (-\infty,+\infty]$ is proper, lower-semicontinuous, and such that $\min \Psi = 0$, it is well known (see [15]) that the abstract Cauchy problem

$$\begin{cases} u_t + \partial\Psi(u) \ni f, & \text{a.e } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

has a unique solution for any $f \in L^1(0,T;H)$ and $u_0 \in \overline{D(\partial\Psi)}$.

The Mosco convergence is a very useful tool to study the convergence of solutions of parabolic problems. The following theorem is a consequence of results in [10], [16].

Theorem 2.3. *Let $\Psi_n, \Psi : H \to (-\infty,+\infty]$ be convex and lower semicontinuous functionals. Then the following statements are equivalent:*

(i) *Ψ_n converges to Ψ in the sense of Mosco.*

(ii) $(I+\lambda\partial\Psi_n)^{-1}u \to (I+\lambda\partial\Psi)^{-1}u \quad \forall\,\lambda > 0,\ u \in H.$

Moreover, either one of the above conditions, (i) or (ii), imply that

(iii) *for every $u_0 \in \overline{D(\partial\Psi)}$ and $u_{0,n} \in \overline{D(\partial\Psi_n)}$ such that $u_{0,n} \to u_0$, and for every $f_n, f \in L^1(0,T;H)$ with $f_n \to f$, if $u_n(t)$, $u(t)$ are solutions of the abstract Cauchy problems*

$$\begin{cases} (u_n)_t + \partial\Psi_n(u_n) \ni f_n & \text{a.e. } \ t \in (0,T) \\ u_n(0) = u_{0,n}, \end{cases}$$

and

$$\begin{cases} u_t + \partial\Psi(u) \ni f & \text{a.e. } \ t \in (0,T) \\ u(0) = u_0, \end{cases}$$

respectively, then

$$u_n \to u \qquad \text{in } C([0,T]:H).$$

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3. Elliptic case

The first step we consider the $(P1)$ in elliptic case, that is, drop the time dependence and take the limit as $p \to \infty$ in the following problem (P1s);

$$\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy, & x \in \Omega_r, \\ f(x) = \displaystyle\int_{\Omega_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy, & x \in \Gamma_r. \end{cases}$$

Theorem 3.1. *Assume that $\int_{\Gamma_r} f(x)dx = 0$. For p fixed there exists a unique solution u_p (that depends on p) of $(P1s)$ such that*

$$\int_{\Gamma_r} u_p(x) = 0.$$

For the proof of this theorem, we use variational arguments. Let us define the functional $J_p(u)$ as the following;

$$J_p(u) = \frac{1}{2p} \iint_H J(x-y)|u(y)-u(x)|^p dydx - \int_{\Gamma_r} fu \tag{9}$$

with

$$H = (\Omega_r \cup \Gamma_r) \times (\Omega_r \cup \Gamma_r) \setminus \Gamma_r \times \Gamma_r.$$

We will prove some properties of $J_p(u)$ in the following lemmas.

Lemma 3.2. *There is a unique solution u_p, to the problem*

$$\min_{u \in B_p} J_p(u)$$

in the set

$$B_p = \Big\{ u \in L^p(\hat{\Omega}) : \int_{\Gamma_r} u(x) = 0 \Big\}.$$

Proof. To find the existence of such a minimum we need a Poincare type inequality: there is a constant c such that

$$\iint_H J(x-y)|u(y)-u(x)|^p dy\, dx \geq c \int_{\Gamma_r} |u|^p \tag{10}$$

for every $u \in B_p$. This inequality is proved in [30], Theorem 2.5. By using the above inequality, Hölder and Young inequality, we obtain the following;

$$J_p(u) = \frac{1}{2p} \iint_H J(x-y)|u(y)-u(x)|^p dydx - \int_{\Gamma_r} fu \geq \frac{c}{2p} \int_{\Gamma_r} |u|^p - c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)} - \varepsilon \|u\|^p_{L_p(\Gamma_r)}.$$

Taking $\varepsilon \leq \frac{c}{2p}$ we get,

$$J_p(u) \geq -c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)}.$$

It means that we can obtain the existence of the infimum of $J_p(u)$ in B_p. Then $\exists \{u_n\} \in B_p$ such that

$$J_p(u_n) \to \inf_{B_p} J_p(u) > -\infty.$$

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Hence we have that $\{J_p(u_n)\}$ is bounded. Then

$$\frac{1}{2p}\iint_H J(x-y)|u_n(y)-u_n(x)|^p dydx - \int_{\Gamma_r} fu_n \leq C.$$

Taking into account (10) and using Hölder inequality we have the following,

$$\frac{c}{2p}\int_{\Gamma_r} |u_n|^p \leq C + \int_{\Gamma_r} fu_n dx \leq C + \|f\|_{L_{p'}(\Gamma_r)}\|u_n\|_{L_p(\Gamma_r)}.$$

Using Young inequality we get

$$\frac{c}{2p}\int_{\Gamma_r} |u_n|^p \leq C + c(\varepsilon)\frac{\|f\|^{p'}_{L_{p'}(\Gamma_r)}}{p'} + \varepsilon\frac{\|u_n\|^p_{L_p(\Gamma_r)}}{p}.$$

Then

$$\left(\frac{\frac{c}{2}-\varepsilon}{p}\right)^{\frac{1}{p}} \|u_n\|_{L_p(\Gamma_r)} \leq \left[C + \frac{c(\varepsilon)}{p'}\|f\|^{p'}_{L_{p'}(\Gamma_r)}\right]^{\frac{1}{p}}.$$

Hence we obtain that u_n is bounded in $L_p(\Gamma_r)$ independently of p. Hence for some subsequence $u_{n_j} \subset u_n$ and $u_p \in L_p(\Gamma_r)$ we have

$$u_{n_j} \rightharpoonup u_p \ \texttt{ in } \ L_p(\Gamma_r).$$

On the other hand, by weak convergence,

$$\frac{1}{2p}\iint_H J(x-y)|u_p(y)-u_p(x)|^p dydx \leq \frac{1}{2p}\lim_{n_j\to\infty}\inf \iint_H J(x-y)|u_{n_j}(y)-u_{n_j}(x)|^p dydx$$

and

$$-\int_{\Gamma_r} u_{n_j} f \to -\int_{\Gamma_r} u_p f.$$

Combining the previous two limits we obtain;

$$\frac{1}{2p}\iint_H J(x-y)|u_p(y)-u_p(x)|^p dydx - \int_{\Gamma_r} u_p f$$

$$\leq \lim_{n_j\to\infty}\inf \frac{1}{2p}\iint_H J(x-y)|u_{n_j}(y)-u_{n_j}(x)|^p dydx - \int_{\Gamma_r} u_{n_j} f.$$

Hence we can write

$$J_p(u_p) \leq \lim_{n_j\to\infty}\inf J_p(u_{n_j}) = \inf_{B_p} J_p(u)$$

which implies u_p is a minimizer of $J_p(u)$ in B_p. The uniqueness follows by the strict convexity of the functional J_p. Let us assume that we have two minimizers $u_1 \neq u_2 \in B_p$, $w := \frac{u_1+u_2}{2} \in B_p$ and

$$J_p(w) < \frac{1}{2}(J_p(u_1)+J_p(u_2)) = \inf_{B_p} J_p(u).$$

This implies that $J_p(w) < \inf_{B_p} J_p(u)$ which contradicts that u_p is the minimizer of $J_p(u)$ in B_p. □

Now we are ready to prove our existence and uniqueness result.

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Proof. (Proof of Theorem 3.1) As an immediate consequence of our previous results, unique minimizer $u_p \in B_p$ is the unique solution of $(P1s)$. □

Now we want to obtain an estimate on u_p.

Lemma 3.3. *Let u_p be the minimizer of $J_p(u)$ in B_p. Then there exists a constant C independent of p such that*

$$\|u_p\|_{L_p(\Gamma_r)} \leq C$$

for large p enough.

Proof. Let us take $u_0 \in L_\infty(\hat{\Omega})$ such that $|u_0(x) - u_0(y)| \leq 1$ for almost every $x, y \in H$,

$$J_p(u_0) = \frac{1}{2p} \iint_H J(x-y)|u_0(y) - u_0(x)|^p dydx - \int_{\Gamma_r} fu_0$$

$$\leq \frac{1}{2p} \iint_H J(x-y) dydx - \int_{\Gamma_r} fu_0 = C(u_0).$$

Since u_p is the minimizer of $J_p(u)$, we have

$$\frac{1}{2p} \iint_H J(x-y)|u_p(y) - u_p(x)|^p dydx \leq C(u_0) + \int_{\Gamma_r} fu_p.$$

Taking into account (10) and using Hölder inequality we have the following,

$$\frac{c}{2p} \int_{\Gamma_r} |u_p|^p \leq C(u_0) + \int_{\Gamma_r} fu_p dx \leq C(u_0) + \|f\|_{L_{p'}(\Gamma_r)} \|u_p\|_{L_p(\Gamma_r)}.$$

Using Young inequality we get

$$\left(\frac{\frac{c}{2} - \varepsilon}{p}\right)^{\frac{1}{p}} \|u_p\|_{L_p(\Gamma_r)} \leq \left[C(u_0) + \frac{c(\varepsilon)}{p'} \|f\|^{p'}_{L_{p'}(\Gamma_r)}\right]^{\frac{1}{p}}.$$

Hence we obtain that u_p is bounded in $L_p(\Gamma_r)$ independently of p;

$$\|u_p\|_{L_p(\Gamma_r)} \leq C$$

for large p enough.

□

We obtain the following trivial consequence of Lemma 3.3.

Remark 3.4. We have that for the unique minimizer u_p there exist some constant C independent of p such that

$$\frac{1}{2p} \iint_H J(x-y)|u_p(y) - u_p(x)|^p dydx \leq C.$$

As a consequence of Lemma 3.3 we can extract a subsequence $\{u_{p_j}\} \subset u_p$ and $u_{p_j} \rightharpoonup u_\infty$ weakly in $L_p(\Gamma_r)$.

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Lemma 3.5. *Let u_∞ be a limit of u_p (along a subsequence if necessary), then there exists a constant $C(q)$ such that*

$$\lim_{q\to\infty} C(q) = 1$$

and

$$\left(\iint_H J(x-y)|u_\infty(y)-u_\infty(x)|^q dxdy\right)^{1/q} \le C(q).$$

Proof. We have

$$\left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^q dxdy\right)^{1/q}$$
$$= \left(\iint_H (J(x-y))^{1-\frac{q}{p_j}} \left[(J(x-y))^{\frac{q}{p_j}} |u_{p_j}(y)-u_{p_j}(x)|^q\right]\right)^{\frac{1}{q}}.$$

Now, for $1 \le q < \infty$, we observe that for $p_j > q$, from Hölder's inequality with $\frac{p_j}{p_j-q}$ and $\frac{p_j}{q}$, we obtain

$$\le \left(\iint_H J(x-y)dydx\right)^{\frac{p_j-q}{p_j q}} \left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^{p_j}\right)^{\frac{1}{p_j}}$$
$$\le \left(\iint_H J(x-y)dydx\right)^{\frac{p_j-q}{p_j q}} (2p_jC)^{\frac{1}{p_j}}$$

let $p_j \to \infty$ to obtain,

$$\left(\iint_H J(x-y)|u_\infty(y)-u_\infty(x)|^q dxdy\right)^{1/q} \le \left(\iint_H J(x-y)dydx\right)^{\frac{1}{q}}$$

by taking as $C(q) := \left(\iint_H J(x-y)dydx\right)^{\frac{1}{q}}$ and $q\to\infty$ we obtain desired result. □

And also we have

$$\left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^q dxdy\right)^{1/q} \le C(p,q),$$

$$\left(\iint_H J(x-y)|u_\infty(y)-u_\infty(x)|^q dxdy\right)^{1/q}$$
$$\le \lim_{p_j\to\infty} \inf \left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^q dxdy\right)^{1/q} \le \lim_{p_j\to\infty} \inf C(p,q) \le 1.$$

Now letting $q\to\infty$, we get

$$|u_\infty(y)-u_\infty(x)| \le 1 \quad \text{for a.e.} \quad x,y\in H, \quad x-y\in \operatorname{supp}(J).$$

Hence we conclude that

$$u_{p_j} \rightharpoonup u_\infty \text{ weakly in } L_q(\Gamma_r) \quad \text{and} \quad u_\infty \in B_\infty$$

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$$B_\infty := \Big\{ u : |u(x) - u(y)| \le 1, \text{ for a.e. } x, y \in H, x - y \in \operatorname{supp}(J) \Big\}.$$

Lemma 3.6. *Let u_∞ be a limit of u_p (along a subsequence if necessary), then u_∞ is a solution to*

$$\sup_{u \in B_\infty} \int_{\Gamma_r} fu \tag{11}$$

with

$$B_\infty = \Big\{ u : |u(x) - u(y)| \le 1, \textit{ for a.e. } x, y \in H, x - y \in \textit{supp}(J) \Big\}.$$

Proof. Since u_p is the minimizer of J_p and $B_\infty \subset B_p$ then we have the following for all $v \in B_\infty$;

$$\begin{aligned} &\frac{1}{2p} \iint_H J(x-y)|u_p(y) - u_p(x)|^p dxdy - \int_{\Gamma_r} fu_p \\ &\le \frac{1}{2p} \iint_H J(x-y)|v(y) - v(x)|^p dxdy - \int_{\Gamma_r} fv \le \frac{1}{2p} \iint_H J(x-y) dxdy - \int_{\Gamma_r} fv \end{aligned}$$

then

$$-\int_{\Gamma_r} fu_p \le \frac{1}{2p} \iint_H J(x-y) dxdy - \int_{\Gamma_r} fv.$$

The idea now is to pass to the limit as p goes to infinity in the previous inequality. On the left hand side, we can neglect the first integral while

$$\int_{\Gamma_r} fu_p(x)dx \to \int_{\Gamma_r} fu_\infty(x)dx, \quad as \quad p \to \infty.$$

For the right hand side, we know that the first integral goes to zero and then (11) holds. □

3.1. **Singular case.** We also include here the case in which the kernel J can be singular. For $0 < s < 1$,

$$(P2) \begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} \frac{1}{|x-y|^{n+ps}} |u(y) - u(x)|^{p-2}(u(y) - u(x))dy, & x \in \Omega_r, \\ f(x) = \displaystyle\int_{\Omega_r} \frac{1}{|x-y|^{n+ps}} |u(y) - u(x)|^{p-2}(u(y) - u(x))dy, & x \in \Gamma_r. \end{cases}$$

To deal with this problem, we consider the space

$$\mathbb{X}_{s,p} = \{ u \in L_p(\Omega_r \cup \Gamma_r) : \|u\|_{s,p} < +\infty \}$$

where

$$\|u\|_{s,p} := (\|u\|^p_{L_p(\Gamma_r)} + \iint_H \frac{1}{|x-y|^{n+ps}} |u(y) - u(x)|^p dydx)^{\frac{1}{p}}.$$

In the next lemma as in Lemma 3.2, we will find a unique minimizer of $J_p(u)$ with singular kernel.

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Lemma 3.7. *There is a unique solution u_p, to the problem*

$$\min_{u\in B_p^s} J_p^s(u)$$

in the set

$$B_p^s := \Big\{u \in \mathbb{X}_{s,p} : \int_{\Gamma_r} u(x) = 0\Big\},$$

where

$$J_p^s(u) := \frac{1}{2p}\iint_H \frac{1}{|x-y|^{n+ps}}|u(y)-u(x)|^p dydx - \int_{\Gamma_r} fudx.$$

Proof. For the proof of this lemma if we replace $J(x-y)$ by the singular kernel and use similar techniques as in the proof of Lemma 3.2 then we can obtain the desired result easily. □

Corollary 3.8. *Let u_p be the minimizer of $J_p^s(u)$ in B_p^s, as a consequence of Lemma 3.7 there exists a constant $K > 0$ independent of p such that $\|u_p\|_{s,p} \le K$, then by compact embedding (see [25]) we can extract a subsequence $\{u_{p_j}\} \subset u_p$ such that $u_{p_j} \to u_\infty$ in $L_q(\Gamma_r)$ for $p > q$.*

Lemma 3.9. *Let u_p be the minimizer of $J_p^s(u)$ in B_p^s. There exists a constant $C(p)$ such that*

$$\Big(\iint_H \frac{1}{|x-y|^n}\frac{|u_p(y)-u_p(x)|^p}{|x-y|^{sp}}dxdy\Big)^{1/p} \le C(p)$$

and

$$\lim_{p\to\infty} C(p) = C_\infty$$

where $C_\infty = \max\{1, (diam(\Omega))^\delta\}$ for small $\delta > 0$.

Proof. Let us take fixed $v(x) \in C^\infty(\Omega)$ such that $\int_{\Gamma_r} v = 0$ and $\frac{v(x)-v(y)}{|x-y|^{s+\delta}} \le 1$ for sufficiently small δ and for a.e. $x, y \in H$. Since u_p is the minimizer of J_p^s in B_p^s and $v \in B_p^s$ then we have:

$$\begin{aligned}
&\frac{1}{2p}\iint_H \frac{1}{|x-y|^{n+sp}}|u_p(y)-u_p(x)|^p dxdy\\
&\qquad\le \int_{\Gamma_r} fu_p + \frac{1}{2p}\iint_H \frac{1}{|x-y|^{n+sp}}|v(y)-v(x)|^p dxdy - \int_{\Gamma_r} fv\\
&\qquad\quad= \int_{\Gamma_r} fu_p - \int_{\Gamma_r} fv + \frac{1}{2p}\iint_H \frac{1}{|x-y|^{n-\delta p}}\Big[\frac{|v(y)-v(x)|}{|x-y|^{s+\delta}}\Big]^p dxdy.
\end{aligned}$$

For sufficiently large p, we have,

$$\le \int_{\Gamma_r} fu_p - \int_{\Gamma_r} fv + \frac{1}{2p}|H|(\mathrm{diam}(\Omega))^{\delta p-n}$$

$$\le c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)} + \varepsilon\|u_p\|^p_{L_p(\Gamma_r)} - \int_{\Gamma_r} fv + \frac{1}{2p}|H|(\mathrm{diam}(\Omega))^{\delta p-n}$$

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$$\leq c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)}+\frac{\varepsilon}{c}\iint_H \frac{1}{|x-y|^{n+sp}}|u_p(y)-u_p(x)|^p dxdy+\int_{\Gamma_r}|fv|+\frac{1}{2p}|H|(\mathrm{diam}(\Omega))^{\delta p-n}.$$

This gives us;

$$(\frac{1}{2p}-\frac{\varepsilon}{c})^{\frac{1}{p}}\left(\iint_H \frac{1}{|x-y|^n}\frac{|u_p(y)-u_p(x)|^p}{|x-y|^{sp}}dxdy\right)^{1/p} \leq \left(c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)}+\int_{\Gamma_r}|fv|+\frac{1}{2p}|H|(\mathrm{diam}(\Omega))^{\delta p-n}\right)^{\frac{1}{p}}.$$

Here

$$\left(c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)}+\int_{\Gamma_r}|fv|+\frac{1}{2p}|H|(\mathrm{diam}(\Omega))^{\delta p-n}\right)^{\frac{1}{p}} \to C_\infty \text{ and } (\frac{1}{2p}-\frac{\varepsilon}{c})^{\frac{1}{p}} \to 1 \text{ as } p\to\infty.$$

Hence we obtain that

$$\left(\iint_H \frac{1}{|x-y|^n}\frac{|u_p(y)-u_p(x)|^p}{|x-y|^{sp}}dxdy\right)^{1/p} \leq C(p) \text{ and } \lim_{p\to\infty} C(p)=C_\infty$$

where

$$C(p)=\frac{\left(c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)}+\int_{\Gamma_r}|fv|+\frac{1}{2p}|H|(\mathrm{diam}(\Omega))^{\delta p-n}\right)^{\frac{1}{p}}}{(\frac{1}{2p}-\frac{\varepsilon}{c})^{\frac{1}{p}}}.$$

□

Lemma 3.10. *Let u_p be the minimizer of $J^s_p(u)$ in B^s_p. There exists a constant $C(p,q)$ such that*

$$\lim_{p,q\to\infty} C(p,q)=C_\infty$$

and

$$\left(\iint_H \frac{1}{(diam(\Omega))^n}\Big(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\Big)^q dxdy\right)^{\frac{1}{q}} \leq C(p,q). \tag{12}$$

Proof. We have

$$\left(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n}\Big(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\Big)^q dxdy\right)^{\frac{1}{q}} = \left(\iint_H (\mathrm{diam}(\Omega))^{\frac{nq}{p}-n}\frac{1}{(\mathrm{diam}(\Omega))^{\frac{nq}{p}}}\Big(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\Big)^q dxdy\right)^{\frac{1}{q}}$$

Now, for $1\leq q<\infty$, we observe that for $p>q$, from Hölder's inequality, we obtain

$$\leq \left(\iint_H \Big((\mathrm{diam}(\Omega))^{\frac{nq}{p}-n}\Big)^{\frac{p}{p-q}}dxdy\right)^{\frac{p-q}{pq}}\left(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n}\Big(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\Big)^p dxdy\right)^{\frac{1}{p}}$$

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$$\leq \Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} dxdy\Big)^{\frac{p-q}{pq}} \Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} \Big(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\Big)^p dxdy\Big)^{\frac{1}{p}}$$

$$\leq \Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} dxdy\Big)^{\frac{p-q}{pq}} \Big(\iint_H \frac{1}{|x-y|^n} \frac{|u_p(y)-u_p(x)|^p}{|x-y|^{sp}} dxdy\Big)^{1/p}$$

$$\leq \Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} dxdy\Big)^{\frac{p-q}{pq}} C(p).$$

Hence we obtain that

$$\Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} \Big(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\Big)^q dxdy\Big)^{\frac{1}{q}} \leq C(p,q)$$

where

$$C(p,q) = \Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} dxdy\Big)^{\frac{p-q}{pq}} C(p) \text{ and } \lim_{p,q\to\infty} C(p,q) = C_\infty$$

□

Therefore, as a consequence of Lemma (3.10),

$$\begin{aligned}\Big(\iint_H &\frac{1}{(\mathrm{diam}(\Omega))^n} \Big(\frac{|u_\infty(y)-u_\infty(x)|}{|x-y|^s}\Big)^q dxdy\Big)^{1/q} \\ &\leq \lim_{p_j\to\infty} \inf \Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} \Big(\frac{|u_{p_j}(y)-u_{p_j}(x)|}{|x-y|^s}\Big)^q dxdy\Big)^{1/q} \\ &\leq \lim_{p_j\to\infty} \inf C(p_j,q) \leq \Big(\iint_H \frac{1}{(\mathrm{diam}(\Omega))^n} dxdy\Big)^{\frac{1}{q}} C_\infty\end{aligned}$$

Now letting $q\to\infty$, we get

$$|u_\infty(y)-u_\infty(x)| \leq \max\{1,(diam(\Omega))^\delta\}|x-y|^s \quad \text{a.e. } x,y\in H \text{ such that } x\neq y.$$

Then we have,

$$u_{p_j}\to u_\infty \quad \text{in} \quad L_q(\Gamma_r) \quad \text{and} \quad u_\infty \in B^s_\infty,$$

$$B^s_\infty := \Big\{u : \frac{|u(y)-u(x)|}{|x-y|^s} \leq \max\{1,(diam(\Omega))^\delta\} \text{ a.e. } x,y\in H \text{ such that } x\neq y \Big\}.$$

Now our aim is to obtain an equation satisfied by the limit u_∞ in the usual viscosity sense.

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Theorem 3.11. *Assume that $f \in C(\hat{\Omega})$. Then u_∞ is the solution of the followings:*

$$\begin{cases} \sup\limits_{x\neq y} \dfrac{u_\infty(y)-u_\infty(x)}{|y-x|^s} + \inf\limits_{x\neq y} \dfrac{u_\infty(y)-u_\infty(x)}{|y-x|^s} = 0, \ x \in \Omega_r \\[2ex] \sup\limits_{x\neq y} \dfrac{u_\infty(y)-u_\infty(x)}{|x-y|^s} - \max\left\{ \sup\limits_{x\neq y} \dfrac{-u_\infty(y)+u_\infty(x)}{|x-y|^s},\ 1\right\} = 0, \ f(x)>0, \ x\in\Gamma_r \\[2ex] \max\left\{ \sup\limits_{x\neq y} \dfrac{u_\infty(y)-u_\infty(x)}{|x-y|^s},\ 1\right\} - \sup\limits_{x\neq y} \dfrac{-u_\infty(y)+u_\infty(x)}{|x-y|^s} = 0, \ f(x)<0, \ x\in\Gamma_r. \end{cases} \tag{13}$$

in the usual viscosity sense.

Proof. Let us start by showing that u_∞ is a viscosity subsolution of (13). Let $x_0 \in \Omega_r$ and let φ be a $C^2(\hat{\Omega})$ function such that $\varphi - u_\infty$ has a minimum at x_0 and $u(x_0) = \varphi(x_0)$. We want to prove that

$$\sup_{x_0\neq y} \frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^s} + \inf_{x_0\neq y} \frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^s} \geq 0$$

Thanks to the uniform convergence of u_p to u_∞ (see Proposition 6.1, [18]), there exists a sequence of point $x_p \to x_0$ such that $\varphi - u_p$ has a minimum at $x_p \in \Omega_r$ then for all $y \in \Omega_r \cup \Gamma_r$ we have the followings,

$$\varphi(y) - u_p(y) \geq \varphi(x_p) - u_p(x_p) \Rightarrow \varphi(y) - \varphi(x_p) \geq u_p(y) - u_p(x_p)$$

then

$$|\varphi(y)-\varphi(x_p)|^{p-2}(\varphi(y)-\varphi(x_p)) \geq |u_p(y)-u_p(x_p)|^{p-2}(u_p(y)-u_p(x_p))$$

and

$$\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-2}(\varphi(y)-\varphi(x_p)) \geq \frac{1}{|x_p-y|^{n+ps}}|u_p(y)-u_p(x_p)|^{p-2}(u_p(y)-u_p(x_p)).$$

Integrating on $\Omega_r \cup \Gamma_r$,

$$\int_{\Omega_r\cup\Gamma_r} \frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-2}(\varphi(y)-\varphi(x_p))dy$$
$$\geq \underbrace{\int_{\Omega_r\cup\Gamma_r} \frac{1}{|x_p-y|^{n+ps}}|u_p(y)-u_p(x_p)|^{p-2}(u_p(y)-u_p(x_p))dy}_{=0}.$$

Then, we have

$$\int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)<\varphi(x_p)\}} \frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-1}dy$$
$$\leq \int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)>\varphi(x_p)\}} \frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-1}dy,$$

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therefore we get

$$\left(\int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)<\phi(x_p)\}}\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\phi(x_p)|^{p-1}dy\right)^{1/(p-1)}$$
$$\leq\left(\int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)>\varphi(x_p)\}}\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-1}dy\right)^{1/(p-1)}.$$

and taking $p\to\infty$ we conclude that

$$\sup_{\varphi(y)>\varphi(x_0)}\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^s}\geq\sup_{\varphi(y)<\varphi(x_0)}\frac{-\varphi(y)+\varphi(x_0)}{|y-x_0|^s}.$$

Hence,

$$\sup_{x_0\neq y}\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^s}+\inf_{x_0\neq y}\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^s}\geq 0.$$

Now, for the second equation, let us consider $x_0\in\Gamma_r$ and let φ be a $C^2(\hat{\Omega})$ function such that $\varphi-u_\infty$ has a minimum at x_0 and $u_\infty(x_0)=\varphi(x_0)$.

Thanks to the uniform convergence of u_p to u_∞, there exists a sequence of point $x_p\to x_0$ such that $\varphi-u_p$ has a minimum at $x_p\in\Gamma_r$ then for all $y\in\Omega_r$ we have the following:

$$\int_{\Omega_r}\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-2}(\varphi(y)-\varphi(x_p))dy$$
$$\geq\int_{\Omega_r}\frac{1}{|x_p-y|^{n+ps}}|u_p(y)-u_p(x_p)|^{p-2}(u_p(y)-u_p(x_p))dy=f(x_p)$$

Then, we have

$$\int_{\Omega_r\cap\{\varphi(y)>\varphi(x_p)\}}\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-1}dy$$
$$\geq\int_{\Omega_r\cap\{\varphi(y)<\varphi(x_p)\}}\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-1}dy+f(x_p),$$

therefore we get

$$\left(\int_{\Omega_r\cap\{\varphi(y)>\varphi(x_p)\}}\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-1}dy\right)^{1/(p-1)}$$
$$\geq\left(\int_{\Omega_r\cap\{\varphi(y)<\varphi(x_p)\}}\frac{1}{|x_p-y|^{n+ps}}|\varphi(y)-\varphi(x_p)|^{p-1}dy+f(x_p)\right)^{1/(p-1)}.$$

Recall that given two sequences of nonnegative real numbers $a_p,\ b_p\subset\mathbb{R}$ where $a_p^{\frac{1}{p}}\to a$ and $b_p^{\frac{1}{p}}\to b$ we have $(a_p+b_p)^{\frac{1}{p-1}}\to\max\{a,b\}$.

In the case $f(x_p)>0$, if we take $p\to\infty$ and use the above property we conclude that

$$\sup_{x_0\neq y}\frac{\varphi(y)-\varphi(x_0)}{|x_0-y|^s}\geq\max\left\{\sup_{x_0\neq y}\frac{-\varphi(y)+\varphi(x_0)}{|x_0-y|^s},\ 1\right\}.$$

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Observe also that, in the case $f(x_p) < 0$,

$$\left(\int_{\Omega_r \cap \{\varphi(y) > \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1} dy - f(x_p)\right)^{1/(p-1)}$$
$$\geq \left(\int_{\Omega_r \cap \{\varphi(y) < \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1} dy\right)^{1/(p-1)}$$

and taking $p \to \infty$, we obtain

$$\max\left\{ \sup_{x_0 \neq y} \frac{\varphi(y) - \varphi(x_0)}{|x_0 - y|^s},\ 1\right\} \geq \sup_{x_0 \neq y} \frac{-\varphi(y) + \varphi(x_0)}{|x_0 - y|^s}.$$

We conclude that u_∞ is a viscosity subsolution to

$$\sup_{x \neq y} \frac{u_\infty(y) - u_\infty(x)}{|x - y|^s} - \max\left\{ \sup_{x \neq y} \frac{-u_\infty(y) + u_\infty(x)}{|x - y|^s},\ 1\right\} = 0,\ \ f(x) > 0$$

$$\max\left\{ \sup_{x \neq y} \frac{u_\infty(y) - u_\infty(x)}{|x - y|^s},\ 1\right\} - \sup_{x \neq y} \frac{-u_\infty(y) + u_\infty(x)}{|x - y|^s} = 0,\ \ f(x) < 0$$

for $x \in \Gamma_r$.

In order to prove that u is also a supersolution, let $x_0 \in \Omega_r$ and let ψ be a $C^2(\hat{\Omega})$ such that $\psi - u_\infty$ has a maximum at x_0 and $u_\infty(x_0) = \psi(x_0)$.

As in the previous case, we can infer from the uniform convergence of $u_p \to u_\infty$ that there exists a sequence of point $x_p \to x_0$ such that $\psi - u_p$ has a maximum at x_p, with $u_p(x_p) = \psi(x_p)$. By using similar arguments, we have

$$\sup_{\psi(y) > \psi(x_0)} \frac{\psi(y) - \psi(x_0)}{|y - x_0|^s} + \inf_{\psi(y) < \psi(x_0)} \frac{\psi(y) - \psi(x_0)}{|y - x_0|^s} \leq 0.$$

In the case, $x_0 \in \Gamma_r$ and let ψ be a $C^2(\hat{\Omega})$ function such that $\psi - u_\infty$ has a maximum at x_0 and $u_\infty(x_0) = \psi(x_0)$. Thanks to the uniform convergence of u_p to u_∞, there exists a sequence of point $x_p \to x_0$ such that $\psi - u_p$ has a maximum at $x_p \in \Gamma_r$ then for all $y \in \Omega_r$ we have the followings,

$$\sup_{x_0 \neq y} \frac{\psi(y) - \psi(x_0)}{|x_0 - y|^s} \leq \max\left\{ \sup_{x_0 \neq y} \frac{-\psi(y) + \psi(x_0)}{|x_0 - y|^s},\ 1\right\},\ \ f(x_0) > 0$$

$$\max\left\{ \sup_{x_0 \neq y} \frac{\psi(y) - \psi(x_0)}{|x_0 - y|^s},\ 1\right\} \leq \sup_{x_0 \neq y} \frac{-\psi(y) + \psi(x_0)}{|x_0 - y|^s},\ \ f(x_0) < 0.$$

We conclude that u_∞ is a viscosity solution to (13). ∎

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4. Parabolic Case

Recall from the Introduction that the nonlocal p-Laplacian evolution problem (P1):

$$\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dy, & x \in \Omega_r,\ t>0, \\ \dfrac{\partial u}{\partial t}(x,t) = \displaystyle\int_{\Omega_r} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dy + f(x,t), & x \in \Gamma_r,\ t>0, \\ u(x,0) = u_0(x), & x \in \Gamma_r. \end{cases}$$

Our aim in this section concerns the limit as $p \to \infty$ in (P1). Firstly, let us show that the existence and uniqueness of solutions to the problem (P1) by using abstract semigroup theory. Let us define functional $E_p(u)$ associated with (P1):

$$E_p(u) := \begin{cases} \dfrac{1}{2p} \displaystyle\iint_H J(x-y)|u(y,t)-u(x,t)|^p dydx & \text{if } \ u \in A_p, \\ +\infty, \ \text{if } \ u \notin A_p \end{cases} \tag{14}$$

where

$$A_p := \{u \in L_p(H) : \iint_H J(x-y)|u(y,t)-u(x,t)|^p dydx < +\infty\}.$$

Then the nonlocal problem can be written as the abstract Cauchy problem associated to the subdifferential of E_p, that is:

$$(P1-s) \begin{cases} f(\cdot,t) - \dfrac{\partial u}{\partial t}(\cdot,t) = \partial E_p(u(\cdot,t)), \ \text{a.e. } t \in (0,T), \\ u(x,0) = u_0(x), \ x \in \Gamma_r. \end{cases}$$

With a formal computation, taking limits, we arrive at the functional:

$$E_\infty(u) = \begin{cases} 0 \ \text{ if } \ u \in A_\infty, \\ +\infty, \ \text{if } \ u \notin A_\infty \end{cases}$$

where

$$A_\infty := \{u : \ \ |u(x)-u(y)| \le 1, \ \ \text{for a.e. } x,y \in H, \ \ |x-y| \in supp(J)\}.$$

Then the nonlocal limit problem can be written as

$$(P_\infty) \begin{cases} f(\cdot,t) - \dfrac{\partial u}{\partial t}(\cdot,t) \in \partial E_\infty(u(\cdot,t)), \ \text{a.e. } t \in (0,T), \\ u(x,0) = u_0(x), \ \ x \in \Gamma_r. \end{cases}$$

We have the following existence result.

Theorem 4.1. *Suppose $p > 1$ and let $u_0 \in L_p(\Gamma_r)$ then, for any $T > 0$ and $f(t,x) \in C([0,T] \times \Gamma_r)$, there exists a unique solution $u_p(x,t) \in C([0,\infty); L_p(\Gamma_r))$ to (P1).*

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For the proof of this theorem as we have mentioned we will use from the point of view of nonlinear semigroup theory. To go on with this task, we first need the following lemma in which we prove that operator $\partial E_p(u)$ satisfies adequate conditions to apply the nonlinear semigroup theory to solve (P1).

Lemma 4.2. *The operator $\partial E_p(u) = B_p(u)$ is an m-accretive in $L_p(\Gamma_r)$.*

Proof. For the proof of m-accretive we should have the following two conditions:

(i) Given $u_1, u_2 \in Dom(B_p)$ and $q \in P_0$,
$$\int_{\Gamma_r} (B_p(u_1)(x) - B_p(u_2)(x))q(u_1(x) - u_2(x))dx \geq 0$$
where $P_0 = \{q \in C^\infty(\mathbb{R}) : \ 0 \leq q' \leq 1, \ \ \text{supp}(q') \ \text{ is compact and } \ 0 \notin supp(q)\}$

(ii) B_p should satisfies the range condition,
$$L_p(\Gamma_r) \subset R(B_p + I).$$

By [30], Theorem 3.4; (i), (ii) are proved. □

Proof of Theorem 4.1. By [30], Corollary A.3, Theorem 4.1 is proved.

Theorem 4.3. *Let $T > 0$, $f(t,x) \in C([0,T] \times \Gamma_r)$, $u_p(x,t)$ be the solution to (P1) with a fixed initial condition*
$$u_0 \in \overline{A_\infty}^{L^2(\Gamma_r)}.$$
Then if u_∞ is the unique solution of P_∞,
$$u_p \to u_\infty \tag{15}$$
as $p \to \infty$ in $C([0,T] : L^2(\Gamma_r))$, that is,
$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(\cdot,t) - u_\infty(\cdot,t)\|_{L^2(\Gamma_r)} = 0.$$

Proof. Let us show that the functional
$$E_p(u) = \frac{1}{2p} \iint_H J(x-y)|u(y,t) - u(x,t)|^p dydx$$
converge to
$$E_\infty(u) = \begin{cases} 0, & \text{if } \ u \in A_\infty, \\ +\infty, & \text{if } \ u \notin A_\infty. \end{cases}$$
as $p \to \infty$ in the sense of Mosco. For the Mosco convergence of E_p to E_∞, first let us prove that
$$\text{Epi}(E_\infty) \subset s - \lim_{p\to\infty} \inf \text{Epi}(E_p).$$
Let us take $(u,k) \in \text{Epi}(E_\infty)$ then our claim is there exists (u_p, k_p) such that $u_p \to u$ in $L_2(H)$ and $k_p \to k$. Now take
$$u_p = u, \ \ \text{and} \ \ k_p = c_p + k, \ \text{where } c_p = E_p(u).$$

$$\text{If } u \in A_\infty \text{ then } E_\infty(u) = 0 \text{ and } k \geq 0.$$

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Also

$$c_p \to 0 \text{ then } k_p = c_p + k \to k \text{ and } k_p \geq E_p(u).$$

This gives us $(u_p, k_p) \in \mathrm{Epi}(E_p)$ and since $(u_p, k_p) \to (u, k)$ we obtain that $(u, k) \in s - \lim_{p\to\infty} \inf \mathrm{Epi}(E_p)$.

On the other hand if $u \notin A_\infty$ then $E_\infty(u) = \infty$ and $(u, +\infty) \in \mathrm{Epi}(E_\infty)$. By taking $u_p = u$, $k_p = +\infty$, we get that $k_p = +\infty \to +\infty$, $u_p \to u$ in $L_2(H)$ and $(u_p, +\infty) \in \mathrm{Epi}(E_p)$ so we obtain that $(u, k) \in s - \lim_{p\to\infty} \inf \mathrm{Epi}(E_p)$. Secondly, let us prove that

$$w - \lim_{p\to\infty} \sup \mathrm{Epi}(E_p) \subset \mathrm{Epi}(E_\infty).$$

Given $(u, k) \in w - \lim_{p\to\infty} \sup \mathrm{Epi}(E_p)$ there exists a sequence $(u_p, k_p) \in \mathrm{Epi}(E_p)$.

$$u_p \rightharpoonup u \text{ in } L_2(H), \ k_p \to k \text{ in } \mathbb{R}.$$

We can assume that there exists a constant $\tilde{c}$ such that $k_p \leq \tilde{c}$. Otherwise $k_p \to \infty$ and since $k = +\infty \geq E_\infty(u)$ then we get $(u, +\infty) \in \mathrm{Epi}(E_\infty)$.
On the other hand, we have that

$$\frac{1}{2p} \iint_H J(x-y)|u_p(y,t) - u_p(x,t)|^p dydx \leq \tilde{c}$$

$$\left(\iint_H J(x-y)|u_p(y,t) - u_p(x,t)|^p dydx \right)^{1/p} \leq (2p\tilde{c})^{1/p}.$$

We have also

$$\left(\iint_H J(x-y)|u_p(y) - u_p(x)|^q dxdy \right)^{1/q}$$

$$= \left(\iint_H (J(x-y))^{1-\frac{q}{p}} \left[(J(x-y))^{\frac{q}{p}} |u_p(y) - u_p(x)|^q \right] \right)^{\frac{1}{q}}.$$

Now, for $1 \leq q < \infty$, we observe that for $p > q$, from Hölder's inequality with $\frac{p}{p-q}$ and $\frac{p}{q}$, we obtain

$$\leq \left(\iint_H J(x-y)dydx \right)^{\frac{p-q}{pq}} \left(\iint_H J(x-y)|u_p(y) - u_p(x)|^p \right)^{\frac{1}{p}}$$

$$\leq \left(\iint_H J(x-y)dydx \right)^{\frac{p-q}{pq}} (2p\tilde{c})^{1/p}$$

let $p \to \infty$ to obtain,

$$\left(\iint_H J(x-y)|u(y) - u(x)|^q dxdy \right)^{1/q} \leq \left(\iint_H J(x-y)dydx \right)^{\frac{1}{q}}.$$

Now letting $q \to \infty$, we get

$$|u(y) - u(x)| \leq 1 \quad \text{for a.e.} \quad x, y \in H, \quad x - y \in \mathrm{supp}(J).$$

Hence we obtain that $u \in A_\infty$ it means that $E_\infty(u) = 0$ and $k \geq 0$, we get that $(u, k) \in \mathrm{Epi}(E_\infty)$. $\square$

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5. Explicit Solutions

In this section, we give some explicit examples of limit solutions for the problems. Firstly we will give an explicit example in elliptic case,

$$(P1s)\begin{cases} 0=\displaystyle\int_{\Omega_r\cup\Gamma_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy, & x\in\Omega_r,\\ f(x)=\displaystyle\int_{\Omega_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy, & x\in\Gamma_r.\end{cases}$$

In order to verify that a function $u_\infty(x)$ is a limit solution to (P1s) we need to check that $u_\infty(x)$ is the solution of

$$\sup_{u\in B_\infty}\int_{\Gamma_r} fu$$

where

$$B_\infty=\Big\{u:|u(x)-u(y)|\le 1,\ \text{for a.e. } x,y\in H,\ x-y\in \operatorname{supp}(J)=B_R(0)\Big\}.$$

Example 5.1. Let us assume that one dimension space, $\Omega_r=(0,1)$ and $\Gamma_r=(-\delta,0)\cup(1,1+\delta)$,

$$f(x)=\begin{cases}1, & x\in(1,1+\delta),\\ -1, & x\in(-\delta,0).\end{cases}$$

$$u_\infty(x)=\begin{cases} c, & x\in(1,1+\delta),\\ c-1, & x\in(1-\delta,1),\\ c-2, & x\in(1-2\delta,1-\delta),\\ \vdots & \vdots\\ c-(k-1), & x\in(1-(k-1)\delta,1-(k-2)\delta),\\ c-k, & x\in(-\delta,0).\end{cases}$$

Indeed, if we take $k=\frac{1}{\delta}+1$ then we obtain interval $(-\delta,0)$. On the other hand $\int_{\Gamma_r}u_\infty(x)dx=0$, then we have

$$\int_{\Gamma_r}u_\infty(x)dx=c\delta+(c-k)\delta=c\delta+(c-(\frac{1}{\delta}+1))\delta=0$$

hence

$$c=\frac{1}{2}+\frac{1}{2\delta}.$$

From our choice of u_∞ clearly $|u_\infty(x)-u_\infty(y)|\le 1$ for $|x-y|\le\delta$ and also u_∞ is the solution of $\sup_{u\in B_\infty}\int_{\Gamma_r}fu$.

Now we will give some explicit examples for parabolic case,

$$(P_\infty)\begin{cases} f(\cdot,t)-\dfrac{\partial u}{\partial t}(\cdot,t)\in\partial E_\infty(u(\cdot,t)), \text{ a.e. } t\in(0,T),\\ u(x,0)=u_0(x).\end{cases}$$

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where

$$E_\infty(u) = \begin{cases} 0, & \text{if } \ u \in A_\infty, \\ +\infty, & \text{if } \ u \notin A_\infty. \end{cases}$$

In order to verify that a function $u(x,t)$ is a solution to (P_∞) we need to check that

$$\int_{\Gamma_r} (f - \frac{\partial u}{\partial t})(v-u) \le 0, \quad \forall v \in D(E_\infty). \tag{16}$$

Example 5.2. Let us assume that one dimension space, $\Omega_r = (0,1)$ and $\Gamma_r = (-\delta,0) \cup (1, 1+\delta)$, $t>0$

$$f(x,t) := \begin{cases} 1, & x \in (1,1+\delta), \ t \ge 0 \\ \\ 0, & x \in (-\delta, 0), \ t \ge 0 \end{cases}$$

and as initial data

$$u_0(x) = 0.$$

Now, let us find the solution by looking at its evolution between some critical times. Let us take $u_\infty(x,t)$ as the following,

$$u_\infty(x,t) = \begin{cases} t, & x \in [1,1+\delta), \ t \in [0,t_1) \\ \\ 0, & \text{otherwise.} \end{cases}$$

Clearly from our choice of $u_{\infty(x,t)}$, if $t_1 \le 1$ and $|x-y| \le \delta$ then $|u_\infty(x,t) - u_\infty(y,t)| \le 1$, we get that $u_\infty(x,t) \in A_\infty$ and (16) holds. Hence, for small times $t_1 \le 1$, the solution to (P_∞) is given by $u_\infty(x,t)$.
For times greater than t_1, let us take $u_\infty(x,t)$ as the following,

$$u_\infty(x,t) = \begin{cases} t, & x \in [1,1+\delta), \ t \in [t_1,t_2) \\ t-1, & x \in [1-\delta,1), \ t \in [t_1,t_2) \\ 0, & \text{otherwise.} \end{cases}$$

If $t_2 \le 2$, $\delta \le 1$ then $|u_\infty(x,t) - u_\infty(y,t)| \le 1$ as $|x-y| \le \delta$, we get that $u_\infty(x,t) \in A_\infty$ and (16) holds.
Now, it is easy to generalize and verify the following general formula that describes the solution for time $t_j \le j$, for any given integer j. We have

$$u_\infty(x,t) = \begin{cases} t, & x \in x \in [1,1+\delta), \ t \in [t_{j-1},t_j) \\ t-1, & x \in [1-\delta,1), \ t \in [t_{j-1},t_j) \\ t-2, & x \in [1-2\delta,1-\delta), \ t \in [t_{j-1},t_j) \\ \ \vdots & \quad \vdots \\ t-(j-1), & x \in [1-(j-1)\delta, 1-(j-2)\delta), \ t \in [t_{j-1},t_j) \\ 0, & \text{otherwise.} \end{cases}$$

where $\delta \le \frac{1}{j-1}$, for $j \ge 2$.

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Example 5.3. Let us assume that one dimension space, $\Omega_r = (0,1)$ and $\Gamma_r = (-\delta,0)\cup(1,1+\delta)$, $t>0$

$$f(x,t) := \begin{cases} 1, & x\in(1,1+\delta),\ t\geq 0 \\ -1, & x\in(-\delta,0),\ t\geq 0. \end{cases}$$

and as initial data

$$u_0(x) = 0.$$

As in the previous example, the evolution follows the same scheme differently from $(-\delta,0)$. Let us take $u_\infty(x,t)$ as the following,

$$u_\infty(x,t) = \begin{cases} t, & x\in[1,1+\delta),\ t\in[0,t_1) \\ -t, & x\in(-\delta,0),\ t\in[0,t_1) \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, if $t_1 \leq 1$ and $|x-y|\leq\delta$ then $|u_\infty(x,t)-u_\infty(y,t)|\leq 1$, we get that $u_\infty(x,t)\in A_\infty$ and (16) holds. Hence, for small times $t_1\leq 1$, the solution to (P_∞) is given by $u_\infty(x,t)$.
For times greater than t_1, let us take $u_\infty(x,t)$ as the following,

$$u_\infty(x,t) = \begin{cases} t, & x\in[1,1+\delta),\ t\in[t_1,t_2) \\ t-1, & x\in[1-\delta,1),\ t\in[t_1,t_2) \\ -(t-1), & x\in(-\delta,0),\ t\in[t_1,t_2) \\ 0, & \text{otherwise}. \end{cases}$$

If $t_2\leq 2$, $0<\delta<1$ then $|u_\infty(x,t)-u_\infty(y,t)|\leq 1$ as $|x-y|\leq\delta$, we get that $u_\infty(x,t)\in A_\infty$ and (16) holds.
Now, let us generalize and verify the following general formula that describes the solution for time $t_j\leq j$, for any given integer j. We have

$$u_\infty(x,t) = \begin{cases} t, & x\in[1,1+\delta),\ t\in[t_{j-1},t_j) \\ t-1, & x\in[1-\delta,1),\ t\in[t_{j-1},t_j) \\ t-2, & x\in[1-2\delta,1-\delta),\ t\in[t_{j-1},t_j) \\ \quad\vdots & \qquad\vdots \\ t-(j-1), & x\in[1-(j-1)\delta,1-(j-2)\delta),\ t\in[t_{j-1},t_j) \\ 0, & x\in(0,1-(j-1)\delta),\ t\in[t_{j-1},t_j) \\ -(t-(j-1)), & x\in(-\delta,0),\ t\in[t_{j-1},t_j) \\ 0, & \text{otherwise}. \end{cases}$$

where $\delta<\frac{1}{j-1}$.

Example 5.4. For two or more dimensions we can obtain similar examples. Given a bounded domain $\Omega_r := B_1(0)\subset\mathbb{R}^N$ and $\Gamma_r := B_{1+\delta}(0)\setminus B_1(0)$ in the following

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for all $t \geq 0$:

$B_1(0) = \{x \in \mathbb{R}^N : \|x\| < 1\}$, $B_{1+\delta}(0) = \{x \in \mathbb{R}^N : \|x\| < 1+\delta\}$, $f(x,t) = 1$ for all $x \in \Gamma_r$

and as initial data

$$u_0(x) = 0.$$

Now, we can generalize and verify the following general formula that describes the solution for time $t_j \leq j$, for any given integer j. We have

$$u_\infty(x,t) = \begin{cases} t, & x \in B_{1+\delta}(0) \setminus B_1(0),\ t \in [t_{j-1}, t_j) \\ t-1, & x \in B_1(0) \setminus B_{1-\delta}(0),\ t \in [t_{j-1}, t_j) \\ t-2, & x \in B_{1-\delta}(0) \setminus B_{1-2\delta}(0),\ t \in [t_{j-1}, t_j) \\ \vdots & \vdots \\ t-(j-1), & x \in B_{1-(j-2)\delta}(0) \setminus B_{1-(j-1)\delta}(0),\ t \in [t_{j-1}, t_j) \\ 0, & \text{otherwise.} \end{cases}$$

where $\delta < \frac{1}{j-1}$. It is clear that $u_\infty(x,t) \in A_\infty$ and (16) holds.

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