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THE NATURAL OPERATORS LIFTING q-FORMS TO p-FORMS ON WEIL BUNDLES

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Abstract. Let q, p, k, m be positive integers with $m \geq k+p+1$ and let A be a Weil algebra with k generators. The (not necessarily regular) natural operators lifting q-forms on m-dimensional manifolds M into p-forms on the Weil bundle T^AM are completely described be means of the so called excellent maps. As a consequence, we derive that any natural operator lifting q-forms on m-dimensional manifolds M into p-forms on T^AM is regular, i.e. it sends smoothly parametrized families into smoothly ones. We apply our general results in the case when T^A is the r-order tangent bundle $T^rM = J^r_0(\mathbf{R}, M)$ (in particular the tangent bundle).

1. Introduction

All manifolds and maps considered in the paper are assumed to be smooth (of class C^∞).

The theory of Weil functors and the concept of natural operators can be found in [5].

Let $\mathcal{M}f$ be the category of manifolds and maps, $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their submersions and let $\mathcal{FM}$ be the category of fibred manifolds and their fibred maps. Let A be a Weil algebra and $T^A : \mathcal{M}f \to \mathcal{FM}$ be its Weil functor.

First, in Section 2, we study all (not necessarily regular) $\mathcal{M}f_m$-natural operators $C : \wedge^q T^* \rightsquigarrow \wedge^0 T^A$ transforming q-forms ω on m-manifolds M into maps $C(\omega) : T^AM \to \mathbf{R}$. Namely, using A we define the finite dimensional real vector bundle Q^q_A and prove that any C in question is of the form $C(\omega) = \omega^{<h_C>}$ for some map $h_C : Q^q_A \to \mathbf{R}$. Consequently, we obtain that any such C is regular, i.e. it transforms smoothly parametrized families of q-forms into smoothly parametrized families of maps.

Next, in Section 3 we study all (not necessarily regular) $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ transforming q-forms ω on m-manifolds M into p-forms $D(\omega)$ on T^AM. Using the well-known fact that $TT^AM \times_{T^AM} \cdots \times_{T^AM} TT^AM = T^BM$ for some new Weil algebra depending on A we can treat such operators D in question as $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$. By Section 2, they are regular and of the form $D(\omega) = \omega^{<h_D>}$ for some $h : Q^q_B \to \mathbf{R}$. But for some h such obtained $D(\omega) = \omega^{<h>}$ cannot be fiber skew p-linear, i.e. cannot be p-form on

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T^AM. That is why, we define the concept of so called excellent maps $h : Q^q_B \to \mathbf{R}$. We deduce that $D(\omega) = \omega^{<h>}$ is a p-form on T^AM if and only if h is excellent.

Next, in Section 4, we try to estimate the space of the excellent maps $Q^q_B \to \mathbf{R}$. We introduce the bigger space of the so called semi-excellent maps $Q^q_B \to \mathbf{R}$ and reduce the description of semi-excellent maps $Q^q_B \to \mathbf{R}$ to solutions of systems of linear equations with coefficients from $\{0, 1\}$ with unknown from $\{0, 1\}$. Consequently, we essentially reduce the problem of the description of all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^p T^*T^A$ to a rather combinatoric problem.

In the last section we apply practically our general results when T^A is the r-th order tangent bundle $T^rM = J^r_0(\mathbf{R}, M)$.

We inform that the linear $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^p T^*T^A$ are described by J. Debecki [2, 3]. In the present paper, we describe all (not necessarily linear) such natural operators.

The present paper is also a generalization of [6], where regular $\mathcal{M}f_m$-natural operators $T^* \rightsquigarrow T^*T^A$ are described.

From now on, let k and m and q and p and r be positive integers.

Let $x^1, \dots, x^m$ be the usual coordinates on $\mathbf{R}^m$.

Let $y^1, \dots, y^k$ be the usual coordinates on $\mathbf{R}^k$ and let $y^1, \dots, y^k, u^1, \dots, u^p$ be the usual coordinates on $\mathbf{R}^{k+p} = \mathbf{R}^k \times \mathbf{R}^p$.

2. Lifting q-forms on manifolds to maps on Weil bundles

Let $C^\infty_0(\mathbf{R}^k)$ be the local algebra of germs at 0 of smooth maps $\mathbf{R}^k \to \mathbf{R}$ and let $\mathbf{m}$ be its maximal ideal. For the simplicity of notations for any map $f : \mathbf{R}^k \to \mathbf{R}$, we will denote the germ at $0 \in \mathbf{R}^k$ of f by the same letter f.

Let $\Omega^q_0(\mathbf{R}^k)$ be the $C^\infty_0(\mathbf{R}^k)$-module of germs at 0 of q-forms on $\mathbf{R}^k$. Similarly, for any q-form ω on $\mathbf{R}^k$, we will denote the germ at $0 \in \mathbf{R}^k$ of ω by the same letter ω.

Let $\underline{A}$ be a ideal in $C^\infty_0(\mathbf{R}^k)$ such that $\mathbf{m}^2 \supset \underline{A} \supset \mathbf{m}^{r+1}$ and let

$$A = C^\infty_0(\mathbf{R}^k)/\underline{A} \text{ (the factor algebra).}$$

This factor algebra A is a Weil algebra (of order r).

Given a manifold M, two maps $\gamma, \gamma_1 : \mathbf{R}^k \to M$ have the same A-jet

$$j^A\gamma = j^A\gamma_1 \ \text{ if } g \circ \gamma - g \circ \gamma_1 \in \underline{A} \text{ for any map } g : M \to \mathbf{R}.$$

Let

$$T^AM = \{j^A\gamma \mid \ \gamma : \mathbf{R}^k \to M\}$$

be the space of all A-jets of maps $\mathbf{R}^k \to M$. Then T^AM is a fibred manifold with the base M and the projection $j^A\gamma \mapsto \gamma(0)$. Any map $f : M \to M_1$ of two manifolds induces a fibred map

$$T^Af : T^AM \to T^AM_1 \ , \ T^Af(v) := j^A(f \circ \gamma) \ , \ v = j^A\gamma.$$

So, we have the bundle functor $T^A : \mathcal{M}f \to \mathcal{FM}$, the Weil functor of A-jets, see [4, 5, 8].

We have the sub-modules

$$\underline{A} \cdot \Omega^q_0(\mathbf{R}^k) \ \text{ and } \ d\underline{A} \wedge \Omega^{q-1}_0(\mathbf{R}^k)$$

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in the $C^\infty_0(\mathbf{R}^k)$-module $\Omega^q_0(\mathbf{R}^k)$, the one spanned (over $C^\infty_0(\mathbf{R}^k)$) by all $\eta\sigma$ for $\eta\in\underline{A}$ and $\sigma\in\Omega^q_0(\mathbf{R}^k)$ and the one spanned (over $C^\infty_0(\mathbf{R}^k)$) by all $d\eta\wedge\sigma$ for $\eta\in\underline{A}$ and $\sigma\in\Omega^{q-1}_0(\mathbf{R}^k)$, respectively, where d is the exterior derivative. ($\Omega^0(\mathbf{R}^k):=C^\infty_0(\mathbf{R}^k)$.)

Let

$$Q^q_A:=\Omega^q_0(\mathbf{R}^k)/\underline{Q}^q_A \text{ (the factor module) },\quad \underline{Q}^q_A:=\underline{A}\cdot\Omega^q_0(\mathbf{R}^k)+d\underline{A}\wedge\Omega^{q-1}_0(\mathbf{R}^k)\ ,$$

where $\underline{Q}^q_A$ is the sub-module in the $C^\infty_0(\mathbf{R}^k)$-module $\Omega^q_0(\mathbf{R}^k)$, the one consisting of all elements $\sigma_1+\sigma_2$ for $\sigma_1\in\underline{A}\cdot\Omega^q_0(\mathbf{R}^k)$ and $\sigma_2\in d\underline{A}\wedge\Omega^{q-1}_0(\mathbf{R}^k)$.

One can easily see that the real vector space Q^q_A is finite dimensional. Any element of $w\in Q^q_A$ is of the form $w=[\sigma]_{\underline{Q}^q_A}$ for a $\sigma\in\Omega^q_0(\mathbf{R}^k)$.

Example 2.1. Let $h:Q^q_A\to\mathbf{R}$ be a map. Given a q-form $\omega\in\Omega^q(M)$ on an m-dimensional manifold M we define a map $\omega^{<h>}:T^AM\to\mathbf{R}$ by

$$\omega^{<h>}(v):=h([\gamma^*\omega]_{\underline{Q}^q_A})\ ,\ v\in T^AM\ ,$$

where $\gamma:\mathbf{R}^k\to M$ is a map such that $v=j^A\gamma$ and where $\gamma^*\omega\in\Omega^q_0(\mathbf{R}^k)$ is the pull-back of ω with respect to γ.

Lemma 2.2. *The value $\omega^{<h>}(v)$ is well defined.*

Proof. Suppose $v=j^A\gamma=j^A\gamma_1$, where $\gamma,\gamma_1:\mathbf{R}^k\to M$. We have to prove that

$$h([\gamma^*\omega]_{\underline{Q}^q_A})=h([\gamma_1^*\omega]_{\underline{Q}^q_A}).$$

We may assume $M=\mathbf{R}^m$ and $\gamma(0)=\gamma_1(0)=0$ and $\omega=\sum\omega_{i_1,\dots,i_q}dx^{i_1}\wedge\dots\wedge dx^{i_q}$, where $\sum$ is over all integers with $1\le i_1<\dots<i_q\le m$.

Let $\gamma=(\gamma^1,\dots,\gamma^m)$ and $\gamma_1=(\gamma^1_1,\dots,\gamma^m_1)$. Since $j^A\gamma=j^A\gamma_1$, then

$$\omega_{i_1,\dots,i_q}\circ\gamma-\omega_{i_1,\dots,i_q}\circ\gamma_1\in\underline{A} \text{ and } \gamma^i-\gamma^i_1\in\underline{A}$$

for all integers $i_1,\dots,i_q,i$ with $1\le i_1<\dots<i_q\le m$ and $i=1,\dots,m$. On the other hand we have

$$\begin{aligned}\gamma^*\omega-\gamma_1^*\omega&=\sum\omega_{i_1,\dots,i_q}\circ\gamma\cdot d\gamma^{i_1}\wedge\dots\wedge\gamma^{i_q}-\sum\omega_{i_1,\dots,i_q}\circ\gamma_1\cdot d\gamma_1^{i_1}\wedge\dots\wedge d\gamma_1^{i_q}\\&=\sum(\omega_{i_1,\dots,i_q}\circ\gamma-\omega_{i_1,\dots,i_q}\circ\gamma_1)\cdot d\gamma^{i_1}\wedge\dots\wedge d\gamma^{i_q}\\&\quad+\sum_{s=1}^q\sum(-1)^{s+1}\omega_{i_1,\dots,i_q}\circ\gamma_1\cdot d(\gamma^{i_s}-\gamma_1^{i_s})\wedge d\gamma^{i_1}\wedge\dots\wedge\widehat{d\gamma^{i_s}}\wedge\dots\wedge d\gamma^{i_q}\ ,\end{aligned}$$

where $\widehat{d\gamma^{i_s}}$ means that $d\gamma^{i_s}$ is dropped. Then $\gamma^*\omega-\gamma_1^*\omega\in\underline{Q}^q_A$. So, $[\gamma^*\omega]_{\underline{Q}^q_A}=[\gamma_1^*\omega]_{\underline{Q}^q_A}$. Consequently, $h([\gamma^*\omega]_{\underline{Q}^q_A})=h([\gamma_1^*\omega]_{\underline{Q}^q_A})$. □

Lemma 2.3. *The map $\omega^{<h>}:T^AM\to\mathbf{R}$ is smooth.*

Proof. Let $v_t\in T^AM$ be a smooth curve. Then there exist a smoothly parameter family of maps $\gamma_t:\mathbf{R}^k\to M$ such that $v_t=j^A\gamma_t$ for $t\in\mathbf{R}$. Then $[\gamma_t^*\omega]_{\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)}$ is a smooth curve in the finite dimensional real vector space $\Omega^q_0(\mathbf{R}^k)/\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)$, the factor space of the $C^\infty_0(\mathbf{R}^k)$-module $\Omega^q_0(\mathbf{R}^k)$ by its

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sub-module $\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)$. Then $[\gamma_t^*\omega]_{\underline{Q}^q_A}$ is a smooth curve in Q^q_A because $[\gamma_t^*\omega]_{\underline{Q}^q_A}$ is the image of $[\gamma_t^*\omega]_{\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)}$ with respect to the obvious projection $\Omega^q_0(\mathbf{R}^k)/\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)\to Q^q_A(\mathbf{R}^k)$. Then $\omega^{<h>}(v_t)$ depends smoothly on t. So, $\omega^{<h>}:T^AM\to\mathbf{R}$ is smooth because of the Boman theorem ([1]). □

Lemma 2.4. *If ω_τ is a smoothly parametrized family of q-forms on M then $(\omega_\tau)^{<h>}$ is a smoothly parametrized family of maps $T^AM\to\mathbf{R}$.*

Proof. The proof of the lemma is a simple modification of the proof of the previous one. □

Proposition 2.5. *The correspondence $(-)^{<h>}:\wedge^qT^*\rightsquigarrow\wedge^0T^A$ given by $\omega\mapsto\omega^{<h>}$ is a regular $\mathcal{M}f_m$-natural operator in the sense of* [5].

Remark 2.6. The concept of (not necessarily regular) natural operators can be found in [5]. In our situation, the $\mathcal{M}f_m$-naturality (invariance) of $(-)^{<h>}$ means that for any $\mathcal{M}f_m$-map $f:M\to M_1$ and q-forms $\omega\in\Omega^1(M)$ and $\omega_1\in\Omega^q(M_1)$ on M and M_1 respectively if ω and ω_1 are f-related, then so are $\omega^{<h>}$ and $\omega_1^{<h>}$. The regularity of $(-)^{<h>}$ means that $(-)^{<h>}$ transforms smoothly parametrized families of q-forms into smoothly parametrized families of maps.

Proof. The proposition is clear. In particular, because of the canonical character of the construction of $\omega^{<h>}$ it follows the $\mathcal{M}f_m$-invariance of $(-)^{<h>}$. The regularity of $(-)^{<h>}$ is exactly Lemma 2.4. □

The main result of this section is

Theorem 2.7. *If $m\geq k+1$, then any (not necessarily regular) $\mathcal{M}f_m$-natural operator $C:\wedge^qT^*\rightsquigarrow\wedge^0T^A$ sending q-forms ω on m-manifolds M into maps $C(\omega):T^AM\to\mathbf{R}$ is of the form $C=(-)^{<h>}$ for a uniquely determined (by C) map $h=h_C:Q^q_A\to\mathbf{R}$. In particular, any such C is regular.*

The proof of Theorem 2.7 will occupy the rest of this section. We need several lemmas.

Let $C:\wedge^qT^*\rightsquigarrow\wedge^0T^A$ be a (not necessarily regular) $\mathcal{M}f_m$-natural operator. (Of course, the $\mathcal{M}f_m$-naturality and the regularity of C are explained in Remark 2.6 with C instead of $(-)^{<h>}$.)

Assume $m\geq k+1$. Define $\Phi_C:\Omega^q_0(\mathbf{R}^m)\to\mathbf{R}$ by

$$\Phi_C(\omega)=C(\omega)(\kappa^A)\ ,\ \omega\in\Omega^q_0(\mathbf{R}^m)\ ,$$

where $\kappa^A:=j^A(\iota)\in T^A_0\mathbf{R}^m$, $\iota:\mathbf{R}^k\to\mathbf{R}^m$, $\iota(y_1,\dots,y_k)=(y_1,\dots,y_k,0,\dots,0)$, $y_1,\dots,y_k\in\mathbf{R}$.

Lemma 2.8. *The operator C is determined by Φ_C, i.e. if C_1 is an another operator in question such that $\Phi_{C_1}=\Phi_C$ then $C_1=C$.*

Proof. By the rank theorem, κ^A has dense $\mathcal{M}f_m$-orbit in $T^A\mathbf{R}^m$. So, the lemma is clear. □

Lemma 2.9. *If φ is an $\mathcal{M}f_m$-map preserving κ^A, then $\Phi_C(\varphi^*\omega)=\Phi_C(\omega)$ for any $\omega\in\Omega^q_0(\mathbf{R}^m)$.*

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Proof. It is a immediate consequence of the invariance of C with respect to φ. □

Lemma 2.10. *For any $\omega \in \Omega^q_0(\mathbf{R}^m)$, we have*

$$\Phi_C(\omega) = \Phi_C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega).$$

(We remember that $x^1, \ldots, x^m$ denote the usual coordinates on $\mathbf{R}^m$ and $(-)^*$ denote the pullback.)

Proof. Let t_n, ϵ_n be two sentences of real number such that $0 < t_n < \exp(-n)$ and $0 < \epsilon_n < \exp(-n)$ for any n. Let $y_n = (0, \ldots, 0, \frac{1}{n}) \in \mathbf{R}^m$. By the Whitney extension theorem ([9]), there exists a smooth map $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ such that

$$\varphi|D(y_{2n+1}, \epsilon_{2n+1}) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

and

$$\varphi|D(y_{2n}, \epsilon_{2n}) = (x^1, \ldots, x^k, t_{2n}x^{k+1}, \ldots, t_{2n}x^m)$$

for sufficiently large n, where $D(y, \epsilon)$ is the disk $\{x \in \mathbf{R}^m \ : \ ||x - y|| < \epsilon\}$. Then

$$C(\varphi^*\omega)(j^A(\iota + y_{2n+1})) = C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega)(j^A(\iota + y_{2n+1}))$$

and

$$\begin{aligned} C(\varphi^*\omega)(j^A(\iota + y_{2n})) &= C((x^1, \ldots, x^k, t_{2n}x^{k+1}, \ldots, t_{2n}x^m)^*\omega)(j^A(\iota + y_{2n})) \\ &= C(\omega)(j^A(\iota + t_{2n}y_{2n})) \end{aligned}$$

for sufficiently large n. The last equality follows from the invariance of C with respect to the $\mathcal{M}f_m$-map $(x^1, \ldots, x^k, t_{2n}x^{k+1}, \ldots, t_{2n}x^m)$. If $n \to \infty$, we derive

$$C(\omega)(j^A\iota) = C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega)(j^A\iota)\ ,$$

i.e. $\Phi_C(\omega) = \Phi_C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega)$, as well. □

Lemma 2.11. *If $\sigma_t \in \Omega^q(\mathbf{R}^k)$, $t \in \mathbf{R}$ is a smoothly parametrized family, i.e. the resulting map $\sigma : \mathbf{R}^k \times \mathbf{R} \to \wedge^q T^*\mathbf{R}^k$ is smooth, then the map*

$$\mathbf{R} \ni t \mapsto \Phi_C((x^1, \ldots, x^k)^*\sigma_t) \in \mathbf{R}$$

is smooth.

Proof. Define $\omega \in \Omega^q(\mathbf{R}^m)$ by

$$\omega(x_1, \ldots, x_m) := ((x^1, \ldots, x^k)^*\sigma_{x_m})(x_1, \ldots, x_m) \in \wedge^q T^*_{(x_1,\ldots,x_m)}\mathbf{R}^m\ ,$$

where $(x_1, \ldots, x_m) \in \mathbf{R}^m$. Then

$$(x^1, \ldots, x^k, 0, \ldots, 0)^*(\tau_{(0,\ldots,0,t)})^*\omega = (x^1, \ldots, x^k)^*\sigma_t\ ,$$

where $\tau_{(0,\ldots,0,t)} : \mathbf{R}^m \to \mathbf{R}^m$ is the translation by $(0, \ldots, 0, t) \in \mathbf{R}^m$, $t \in \mathbf{R}$. Then using the previous lemma and the invariance of C with respect to $\tau_{(0,\ldots,0,t)}$, we get

$$\begin{aligned} \Phi_C((x^1, \ldots, x^k)^*\sigma_t) &= \Phi_C((x^1, \ldots, x^k, 0, \ldots, 0)^*(\tau_{(0,\ldots,0,t)})^*\omega) \\ &= \Phi_C((\tau_{(0,\ldots,0,t)})^*\omega) = C((\tau_{(0,\ldots,0,t)})^*\omega)(\kappa^A) \\ &= C(\omega) \circ T^A\tau_{(0,\ldots,0,t)}(\kappa^A). \end{aligned}$$

Since $C(\omega) \circ T^A\tau_{(0,\ldots,0,t)}(\kappa^A)$ depends smoothly on t, the proof is complete. □

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Lemma 2.12. *For any $\omega \in \Omega^q_0(\mathbf{R}^m)$ of the form $\omega = \omega(x^1,\dots,x^k,dx^1,\dots,dx^k)$ and any $\sigma \in \Omega^{q-1}_0(\mathbf{R}^m)$ of the form $\sigma = \sigma(x^1,\dots,x^k,dx^1,\dots,dx^k)$ and any $\eta \in \underline{A}$, we have*

$$\Phi_C(\omega) = \Phi_C(\omega + d\eta \wedge \sigma) \ ,$$

where $\eta \circ (x^1,\dots,x^k)$ is (for simplicity) denoted by η.

Proof. Using Lemma 2.10 and next using Lemma 2.9 with $\varphi = (x^1,\dots,x^{m-1},x^m + \eta(x^1,\dots,x^k))$ and next using again Lemma 2.10, we get

$$\Phi_C(\omega) = \Phi_C(\omega + dx^m \wedge \sigma) = \Phi_C(\omega + dx^m \wedge \sigma + d\eta \wedge \sigma) = \Phi_C(\omega + d\eta \wedge \sigma).$$

The proof of the lemma is complete. □

Lemma 2.13. *For any $\omega \in \Omega^q_0(\mathbf{R}^m)$ of the form $\omega = \omega(x^1,\dots,x^k,dx^1,\dots,dx^k)$ and any $\omega_1 \in \Omega^q_0(\mathbf{R}^m)$ of the form $\omega_1 = \omega_1(x^1,\dots,x^k,dx^1,\dots,dx^k)$ and any $\eta \in \underline{A}$, we have*

$$\Phi_C(\omega) = \Phi_C(\omega + \eta \cdot \omega_1) \ ,$$

where η denotes $\eta \circ (x^1,\dots,x^k)$.

Proof. By the same arguments as in the proof of Lemma 2.12, we have

$$\Phi_C(\omega) = \Phi_C(\omega + x^m \cdot \omega_1) = \Phi_C(\omega + x^m \cdot \omega^1 + \eta \cdot \omega_1) = \Phi_C(\omega + \eta \cdot \omega_1).$$

The proof of the lemma is complete. □

We are now in position to prove Theorem 2.7.

Proof. We (must) define $h_C : Q^q_A \to \mathbf{R}$ by

$$h_C(w) := \Phi_C(\sigma) \ , \ w = [\sigma]_{\underline{Q}^q_A} \ , \ \sigma \in \Omega^q_0(\mathbf{R}^k) \subset \Omega^q_0(\mathbf{R}^m)$$

(the inclusion is given by the pull-back with respect to the projection $(x^1,\dots,x^k) : \mathbf{R}^m \to \mathbf{R}^k$). By Lemmas 2.12 and 2.13, h_C is well defined. Clearly, h_C is smooth because of Lemma 2.11 and the Boman theorem. Further, by Lemmas 2.8 and 2.10, C is determined by h_C. On the other hand, if $C_1 := C(-)^{<h_C>}$ then $h_{C_1} = h_C$. Consequently, $C = C_1 = (-)^{<h_C>}$. The proof of Theorem 2.7 is complete. □

Corollary 2.14. *Assume additionally $q > k$. If $m \geq k+1$, then any $\mathcal{M}f_m$-natural operator $C : \wedge^q T^* \rightsquigarrow \wedge^0 T^A$ sending q-forms ω on m-manifolds M into maps $C(\omega) : T^A M \to \mathbf{R}$ is a real constant one.*

Proof. If $q > k$, then $\Omega^q_0(\mathbf{R}^k) = (0)$, and then $Q^q_A = (0)$. Then any map $h : Q^q_A \to \mathbf{R}$ is constant. Now, the corollary is clear. □

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3. Lifting q-forms on manifolds to p-forms on Weil bundles

Let k, m, q, p, r be the positive integers $y^1, \dots, y^k, u^1, \dots, u^p$ be the usual coordinates on $\mathbf{R}^{k+p} = \mathbf{R}^k \times \mathbf{R}^p$.

Let $A = C_0^\infty(\mathbf{R}^k)/\underline{A}$ be the Weil algebra as in the previous section.

Using A we can produce new Weil algebra B (depending on A) by

$$B := C_0^\infty(\mathbf{R}^{k+p})/\underline{B}\ ,\quad \underline{B} =< \underline{A} \circ (y^1, \dots, y^k)\ ,\ u^i u^j \mid i, j = 1, \dots, p >,$$

where $\underline{B}$ is the ideal in $C_0^\infty(\mathbf{R}^{k+p})$ spanned by the collection consisting of $\eta \circ (y^1, \dots, y^k)$ for all $\eta \in \underline{A}$ and all $u^i u^j$ for $i, j = 1, \dots, p$.

Clearly, $B = A \otimes \mathbf{D}_p^1$, where $\mathbf{D}_p^1 = \mathbf{D} \oplus_{\mathbf{R}} \dots \oplus_{\mathbf{R}} \mathbf{D}$ (p-times of $\mathbf{D}$), where $\mathbf{D}$ is the algebra of dual numbers. Then (by the theory of Weil functors) given manifold M we have

$$T^B M = TT^A M \times_{T^A M} \dots \times_{T^A M} TT^A M \ (p\text{-times of } TT^A M)\ ,$$

where T^B is the Weil functor corresponding to the Weil algebra B.

We are going to describe all (not necessarily regular) $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ sending q-forms ω on m-manifolds M into p-forms $D(\omega)$ on $T^A M$.

Clearly, any such $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A M$ can be treated (in obvious way) as an excellent $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$ in the sense of Definition 3.1, and vice-versa. So, in particular, any such D is regular because of the previous section.

Definition 3.1. A $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$ is excellent if for any m-manifold M and any point $u \in T^A M$ and any q-form ω on M the map $D(\omega)_{|T_u T^A M \times \dots \times T_u T^A M} : T_u T^A M \times \dots \times T_u T^A M \to \mathbf{R}$ is skew-symmetric p-linear.

So, we are going to describe all excellent $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$, where B is as above.

For, we consider an excellent $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$.

Similarly as in the previous section, let

$$Q_B^q := \Omega_0^q(\mathbf{R}^{k+p})/\underline{Q}_B^q\ ,\quad \underline{Q}_B^q = \underline{B} \cdot \Omega_0^q(\mathbf{R}^{k+p}) + d\underline{B} \wedge \Omega_0^{q-1}(\mathbf{R}^{k+p}).$$

By Theorem 2.7 for B instead of A and D instead of C, if $m \geq k + p + 1$, then $D = (-)^{<h_D>}$ for a uniquely determined (by D) smooth map $h_D : Q_B^q \to \mathbf{R}$. By the proof of Theorem 2.7,

$$h_D(w) := D(\sigma)(\kappa^B)\ ,\ w = [\sigma]_{\underline{Q}_B^q}\ ,\ \sigma \in \Omega_0^q(\mathbf{R}^{k+p}) \subset \Omega_0^q(\mathbf{R}^m)$$

(the inclusion is given by the pull-back with respect to the obvious projection $\mathbf{R}^m = \mathbf{R}^{k+p} \times \mathbf{R}^{m-(k+p)} \to \mathbf{R}^{k+p}$), where $\kappa^B := j^B(\iota) \in T_0^B \mathbf{R}^m$, $\iota : \mathbf{R}^{k+p} \to \mathbf{R}^m$, $\iota(y_1, \dots, y_{k+p}) = (y_1, \dots, y_{k+p}, 0, \dots, 0)$, $y_1, \dots, y_{k+p} \in \mathbf{R}$.

Let $\mathbf{R}_+ = (\mathbf{R}_+, \cdot)$ be the multiplicative group of positive real numbers. We have the action of $(\mathbf{R}_+)^p$ on Q_B^q given by

$$t \diamond w := [(a_t)^* \sigma]_{\underline{Q}_B^q}\ ,$$

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where $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$, $w = [\sigma]_{\underline{Q}^q_B} \in Q^q_B$, $\sigma \in \Omega^q_0(\mathbf{R}^{k+r})$ and $a_t := (y^1, \dots, y^k, t_1 u^1, \dots, t_p u^p) : \mathbf{R}^{k+p} \to \mathbf{R}^{k+p}$, where $y^1, \dots, y^k, u^1, \dots, u^p : \mathbf{R}^{k+p} \to \mathbf{R}$ are the usual coordinates on $\mathbf{R}^{k+p}$. The action is well defined because $\underline{Q}^q_B$ is $(a_t)^*$-invariant.

Let S_p be the group of permutations of p-elements $\{1, \dots, p\}$. We have the action of S_p on Q^q_B given by

$$s \Box w := [(a_{s^{-1}})^* \sigma]_{\underline{Q}^q_B} \ , \ s \in S_p \ , \ w = [\sigma]_{\underline{Q}^q_B} \in Q^q_B \ , \ \sigma \in \Omega^q_0(\mathbf{R}^{k+p}) \ ,$$

where $a_s := (y^1, \dots, y^k, u^{s(1)}, \dots, u^{s(p)})$. The action is well defined because $\underline{Q}^q_B$ is $(a_{s^{-1}})^*$-invariant.

Definition 3.2. A map $h : Q^q_B \to \mathbf{R}$ is excellent if

$$h(t \diamond w) = t_1 \cdot \dots \cdot t_p \cdot h(w) \quad \text{and} \quad h(s \Box w) = \mathrm{sign}(s) \cdot h(w)$$

for any $w \in Q^q_B$, any $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$ and any $s \in S_p$.

Lemma 3.3. *Assume $m \geq k + p + 1$. Let D be as above. Then the map $h_D : Q^q_B \to \mathbf{R}$ is excellent.*

Proof. The first formula from Definition 3.2 (for h_D instead of h) is a consequence of the invariance of D with respect to

$$\tilde{a}_t := (x^1, \dots, x^k, t_1 x^{k+1}, \dots, t_p x^{k+p}, x^{k+p+1}, \dots, x^m)$$

for $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$ and the p-linearity of $D(\omega)_{|T_{\kappa^A} T^A \mathbf{R}^m \times \dots \times T_{\kappa^A} T^A \mathbf{R}^m}$ for any q-form ω on $\mathbf{R}^m$. More detailed, we can proceed as follows. Consider $w \in Q^q_B$ and $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$. We can write $\kappa^B = (v_1, \dots, v_p) \in T_{\kappa^A} T^A \mathbf{R}^m \times \dots \times T_{\kappa^A} T^A \mathbf{R}^m$ and $w = [\sigma]_{\underline{Q}^q_B}$, where $\sigma \in \Omega^q_0(\mathbf{R}^{k+p}) \subset \Omega^q_0(\mathbf{R}^m)$. Then $T^B \tilde{a}_t(\kappa^B) = (t_1 v_1, \dots, t_p v_p)$. Then

$$h_D(t \diamond w) = h_D([(a_t)^* \sigma]_{\underline{Q}^q_B}) = D((\tilde{a}_t)^* \sigma)(\kappa^B) = D(\sigma)(T^B \tilde{a}_t(\kappa^B))$$

$$= D(\sigma)(t_1 v_1, \dots, t_p v_p) = t_1 \cdot \dots \cdot t_p \cdot D(\sigma)(v_1, \dots, v_p) = t_1 \cdot \dots \cdot t_p \cdot h_D(w) \ ,$$

i.e. $h_D(t \diamond w) = t_1 \cdot \dots \cdot t_p \cdot h_D(w)$, as well.

Similarly, the second formula from Definition 3.2 (for h_D instead of h) follows from the invariance of D with respect to

$$\tilde{a}_s := (x^1, \dots, x^k, x^{k+s(1)}, \dots, x^{k+s(p)}, x^{k+p+1}, \dots, x^m)$$

for $s \in S_p$ and the skew-symmetry of $D(\omega)_{|T_{\kappa^A} T^A \mathbf{R}^m \times \dots \times T_{\kappa^A} T^A \mathbf{R}^m}$ for any q-form ω on $\mathbf{R}^m$. More detailed, we can proceed as follows. Consider $w \in Q^q_B$ and $s \in S_p$. We can write $\kappa^B = (v_1, \dots, v_p) \in T_{\kappa^A} T^A \mathbf{R}^m \times \dots \times T_{\kappa^A} T^A \mathbf{R}^m$ and $w = [\sigma]_{\underline{Q}^q_B}$, where $\sigma \in \Omega^q_0(\mathbf{R}^{k+p}) \subset \Omega^q_0(\mathbf{R}^m)$. Then $T^B \tilde{a}_s(\kappa^B) = (v_{s(1)}, \dots, v_{s(p)})$. Then

$$h_D(s \Box w) = h_D([(a_{s^{-1}})^* \sigma]_{\underline{Q}^q_B}) = D((\tilde{a}_{s^{-1}})^* \sigma)(\kappa^B) = D(\sigma)(T^B \tilde{a}_{s^{-1}}(\kappa^B))$$

$$= D(\sigma)(v_{s^{-1}(1)}, \dots, v_{s^{-1}(p)}) = \mathrm{sign}(s) \cdot D(\sigma)(v_1, \dots, v_p) = \mathrm{sign}(s) \cdot h_D(w) \ ,$$

i.e. $h_D(s \Box w) = \mathrm{sign}(s) \cdot h_D(w)$, as well. ☐

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Lemma 3.4. *Assume $m \geq k+p+1$. Let $h : Q^q_B \to \mathbf{R}$ be excellent. Then $(-)^{<h>} : \wedge^q T^* \rightsquigarrow \wedge^0 T^B$ can be treated as a natural operator $\wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$.*

Proof. By the first formula from Definition 3.2 we get that $\omega^{<h>}(w_1, \dots, w_p)$ is p-linear in $w_1, \dots, w_p \in T_u T^A M$ for any q-form ω on M and any $u \in T^A M$. More detailed, we can proceed as follows. Since the $\mathcal{M}f_m$-orbit of κ^A in $T^A M$ is dense (as $m \geq k$) and $(-)^{<h>}$ is $\mathcal{M}f_m$-invariant, we can assume $M = \mathbf{R}^m$ and $u = \kappa^A \in T^A\mathbf{R}^m$. Let $\kappa^B = (v_1, \dots, v_p)$. Let $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$ and let a_t and $\tilde{a}_t$ be as in the proof of the previous lemma and let ω be a q-form on $\mathbf{R}^m$. Then

$$\begin{aligned}\omega^{<h>}(t_1 v_1, \dots, t_p v_p) &= \omega^{<h>}(T^B \tilde{a}_t(v_1, \dots, v_p)) = ((\tilde{a}_t)^*\omega)^{<h>})(v_1, \dots, v_p) \\ &= h([\iota^*(\tilde{a}_t)^*\omega]_{\underline{Q}^q B}) = h([(a_t)^*\iota^*\omega]_{\underline{Q}^q_B}) = h(t \diamond [\iota^*\omega]_{\underline{Q}^q_B}) \\ &= t_1 \cdot \dots \cdot t_p \cdot h([\iota^*\omega]_{\underline{Q}^q_B}) = t_1 \cdot \dots \cdot t_p \cdot \omega^{<h>}(v_1, \dots, v_p) \ ,\end{aligned}$$

i.e. $\omega^{<h>}(t_1 v_1, \dots, t_p v_p) = t_1 \cdot \dots \cdot t_p \cdot \omega^{<h>}(v_1, \dots, v_p)$. Then by the $\mathcal{M}f_m$-invariance of $(-)^{<h>}$ and the fact that the $\mathcal{M}f_m$-orbit of κ^B is dense in $(T^B M)_{\kappa^A}$ (as $m \geq k+p$), we get that

$$\omega^{<h>}(t_1 w_1, \dots, t_p w_p) = t_1 \cdot \dots \cdot t_p \cdot \omega^{<h>}(w_1, \dots, w_p)$$

for any q-form ω on $\mathbf{R}^m$ and any $w_1, \dots, w_p \in T_{\kappa^A} T^A \mathbf{R}^m$ and any $t_1 > 0, \dots, t_p > 0$. Now, the homogeneous function theorem implies that $\omega^{<h>}(w_1, \dots, w_p)$ is p-linear in $w_1, \dots, w_p \in T_{\kappa^A} T^A \mathbf{R}^m$ for any q-form ω on $\mathbf{R}^m$, as well.

The second equality implies $\omega^{<h>}(w_1, \dots, w_p)$ is skew-symmetric in $w_1, \dots, w_p \in T_u T^A M$ for any q-form ω on M and any $u \in T^A M$. More detailed, we can proceed as follows. We can assume $M = \mathbf{R}^m$ and $u = \kappa^A \in T^A \mathbf{R}^m$. Let $\kappa^B = (v_1, \dots, v_p)$. Let $s \in S_p$ and let $\tilde{a}_s$ be as in the proof of the previous lemma and let ω be a q-form on $\mathbf{R}^m$. Then

$$\begin{aligned}\omega^{<h>}(v_{s(1)}, \dots, v_{s(p)}) &= \omega^{<h>}(T^B \tilde{a}_s(v_1, \dots, v_p)) = ((\tilde{a}_s)^*\omega)^{<h>}(v_1, \dots, v_p) \\ &= h([\iota^*(\tilde{a}_s)^*\omega]_{\underline{Q}^q B}) = h([(a_s)^*\iota^*\omega]_{\underline{Q}^q_B}) = h(s^{-1}\Box[\iota^*\omega]_{\underline{Q}^q_B}) \\ &= \operatorname{sign}(s) \cdot h([\iota^*\omega]_{\underline{Q}^q_B}) = \operatorname{sign}(s) \cdot \omega^{<h>}(v_1, \dots, v_p) \ ,\end{aligned}$$

i.e. $\omega^{<h>}(v_{s(1)}, \dots, v_{s(p)}) = \operatorname{sign}(s) \cdot \omega^{<h>}(v_1, \dots, v_p)$. Then by the $\mathcal{M}f_m$-invariance of $(-)^{<h>}$ and the fact that the $\mathcal{M}f_m$-orbit of κ^B is dense in $(T^B M)_{\kappa^A}$, we get that

$$\omega^{<h>}(w_{s(1)}, \dots, w_{s(p)}) = \operatorname{sign}(s) \cdot \omega^{<h>}(w_1, \dots, w_p)$$

for any q-form ω on $\mathbf{R}^m$ and any $w_1, \dots, w_p \in T_{\kappa^A} T^A \mathbf{R}^m$ and any $s \in S_p$, as well.

Then $\omega^{<h>}$ is a p-form on $T^A M$ for any q-form ω on an m-manifold M. $\square$

Summing up, we have proved

Theorem 3.5. *Assume $m \geq k+p+1$. The described above correspondence $D \mapsto h_D$ between the (not necessarily regular) $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ and the excellent maps $h_D : Q^q_B \to \mathbf{R}$ is one to one. The inverse correspondence is $h \mapsto (-)^{<h>}$. In particular, any (not necessarily regular) $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ is regular.*

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Corollary 3.6. *Assume additionally $q > k+p$. If $m \geq k+p+1$, then any $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ sending q-forms ω on m-manifolds M into p-forms $D(\omega) : T^A M \to \mathbf{R}$ is the 0 one.*

Proof. If $q > k+p$, then $\Omega^q_0(\mathbf{R}^{k+p}) = (0)$, and then $Q^q_B = (0)$. Then any excellent map $Q^q_B \to \mathbf{R}$ is 0. Now, the corollary is clear. □

Definition 3.7. A map $H : Q^q_B \to \mathbf{R}$ is semi-excellent if $H(t \diamond w) = t_1 \cdot \dots \cdot t_p \cdot H(w)$ for any $w \in Q^q_B$, any $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$.

Quite similarly one can prove (in fact we have proved) the following

Theorem 3.8. *Assume $m \geq k+p+1$. There exists one-to-one correspondence between the $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ and the semi-excellent maps $Q^q_B \to \mathbf{R}$.*

Corollary 3.9. *Assume additionally $q > k+p$. If $m \geq k+p+1$, then any $\mathcal{M}f_m$-natural operator $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ is the 0 one.*

Remark 3.10. Theorems 3.5 and 3.8 show that to describe all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ (resp. $\wedge^q T^* \rightsquigarrow \wedge^q T^* T^A$) it is sufficient to describe all semi-excellent (resp.excellent) maps $Q^q_B \to \mathbf{R}$. Such description in question will be presented in the next section.

4. On semi-excellent maps

We will use the notations as in the previous sections. In particular, let B and Q^q_B be as in the previous section.

We are going to present the full description of all semi-excellent maps $Q^q_B \to \mathbf{R}$.

Let $\mathbf{N}$ be the set of non-negative integers.

Given $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbf{N}^p$ and $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$, we denote $t^\alpha := (t_1)^{\alpha_1} \dots (t_p)^{\alpha_p}$.

Definition 4.1. An element $w \in Q^q_B$ is homogeneous of weight $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbf{N}^p$ if
$$t \diamond w = t^\alpha \cdot w$$
for any $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$, where $\diamond$ and Q^q_B are as in the previous section.

Definition 4.2. An element $w \in Q^q_B$ is adapted if it is homogeneous of the weight $\alpha(w)$ satisfying $\alpha(w) \in \{0,1\}^p$.

Definition 4.3. A basis $\mathcal{B} = \{w_1, \dots, w_K\}$ of the real vector space Q^q_B is called adapted if $w_1, \dots, w_K$ are adapted.

The description of all semi-excellent maps $Q^q_B \to \mathbf{R}$ is presented in the following

Theorem 4.4. *(i) We can choose the basis $\mathcal{B} = \{w_1, \dots, w_K\}$ in the real vector space Q^q_B and $K_o \leq K$ such that $w_1, \dots, w_K$ are adapted and $w_1, \dots, w_{K_o}$ are all elements from $\mathcal{B}$ of weight equal to $(0, \dots, 0)$.*

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(ii) Given $w \in \mathcal{B}$, let $\alpha(w)$ be the weight of w, where $\mathcal{B}$ is the adapted basis as above (in the part (i)). Let $\lambda_1, \ldots, \lambda_K$ be the dual basis to $\mathcal{B}$. Then any semi-excellent map $H : Q^q_B \to \mathbf{R}$ is the linear combination of monomials

$$(\lambda_{K_o+1})^{\gamma_{K_o+1}} \cdot \ldots \cdot (\lambda_K)^{\gamma_K}$$

for all $\gamma_{K_o+1}, \ldots, \gamma_K \in \{0,1\}$ with $\sum_{j=K_o+1}^{K} \gamma_j \alpha(w_j) = (1, \ldots, 1)$ with (uniquely determined) coefficients being smooth maps depending on $\lambda_1, \ldots, \lambda_{K_o}$. And vice-versa, any such linear combination (in question) is semi-excellent.

Proof. Clearly, because of the definition of Q^q_B we can choose elements $v_1, \ldots, v_L \in Q^q_B$ such that $Q^q_B = span_{\mathbf{R}}\{v_1, \ldots, v_L\}$ and any of v_j for $j = 1, \ldots, L$ is of the form

$$[y^\alpha u^\beta dy^{\mu_1} \wedge \cdots \wedge dy^{\mu_{\tilde{k}}} \wedge du^{\nu_1} \wedge \cdots \wedge du^{\nu_{\tilde{p}}}]_{\underline{Q}^q_B}$$

for some $\alpha \in \mathbf{N}^k$ and $\beta = (\beta_1, \ldots, \beta_p) \in \mathbf{N}^p$ and some $\tilde{k}, \tilde{p}, \mu_1, \ldots, \mu_{\tilde{k}}, \nu_1, \ldots, \nu_{\tilde{p}} \in \mathbf{N}$ satisfying $1 \le \mu_1 < \cdots < \mu_{\tilde{k}} \le k$ and $1 \le \nu_1 < \cdots < \nu_{\tilde{p}} \le p$ and $\tilde{k} + \tilde{p} = q$ and $|\beta| = \beta_1 + \cdots + \beta_p \le 1$ and (if $\beta_s = 1$ then $s \in \{1, \ldots, p\} \setminus \{\nu_1, \ldots, \nu_{\tilde{p}}\}$). From these generators we can choose the basis

$$\mathcal{B} = \{w_1, \ldots, w_K\}$$

of the real vector space Q^q_B. Then $\mathcal{B}$ is adapted. Of course, we may assume that $w_1, \ldots, w_{K_o}$ are all elements from $\mathcal{B}$ with weight equal to $(0, \ldots, 0)$.

The proof of part (i) of the theorem is complete.

Let $H : Q^q_B \to \mathbf{R}$ be a semi-excellent map.

Given $w \in \mathcal{B}$, let $\alpha(w)$ be the weight of w.

Any element $w \in Q^q_B$ is of the form

$$w = \sum_{j=1}^{K} \lambda_j(w) w_j \quad \text{where } \lambda_j : Q^q_B \to \mathbf{R} \text{ are the functionals dual to } w_1, \ldots, w_K.$$

Then for any $w = \sum_{j=1}^{K} \lambda_j(w) w_j \in Q^q_B$ and $t = (t_1, \ldots, t_p) \in (\mathbf{R}_+)^p$ we have

$$t_1 \cdot \ldots \cdot t_p \cdot H(\sum_{j=1}^{K} \lambda_j(w) w_j) = H(t \diamond \sum_{j=1}^{K} \lambda_j(w) w_j) = H(\sum_{j=1}^{K} \lambda_j(w) t^{\alpha(w_j)} w_j).$$

Then because of the homogeneous function theorem ([5]), H is the linear combination of monomials

$$(\lambda_1)^{\gamma_{K_o}+1} \cdot \ldots \cdot (\lambda_K)^{\gamma_K} \text{ for all } \gamma_{K_o+1}, \ldots, \gamma_K \in \mathbf{N} \text{ with } \sum_{j=K_o+1}^{K} \gamma_j \alpha(w_j) = (1, \ldots, 1)$$

with coefficients being smooth maps depending on $\lambda_1, \ldots, \lambda_{K_o}$. Then $\gamma_{K_o+1}, \ldots, \gamma_K \in \{0,1\}$ because $\alpha(w_j) \neq (0, \ldots, 0)$ for $j = K_o + 1, \ldots, K$.

The proof of part (ii) of the theorem is complete. □

Remark 4.5. The condition $\sum_{j=K_o+1}^{K} \gamma_j \alpha(w_j) = (1, \ldots, 1)$ from the part (ii) of the theorem is a system of p linear equations with coefficients from $\{0,1\}$ with

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$(K - K_o)$ unknown γ_j from $\{0,1\}$. So, we have reduced the description of regular $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ to a rather combinatoric problem.

5. Applications

For start, applying our obtained (in the previous sections) general results, if $m \geq q+2 \geq 4$, we find explicitly all $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^q T^* T$ lifting q-forms ω on a m-manifold M into q-forms $D(\omega)$ on the tangent bundle TM. We prove

Proposition 5.1. *If $m \geq q+2 \geq 4$ then any $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^q T^* T$ is of the form*

$$\omega \mapsto a\omega^C + b\omega^V + cdi_L\omega^C$$

for (uniquely determined by D) real numbers a, b, c, where ω^C is the complete lift of ω from M to TM and ω^V is the vertical lift of ω from M to TM and i_L is the inner derivative with respect to the Liouville vector field L on TM and d is the exterior derivative.

Proof. We prove this proposition for $m \geq q+2 \geq 5$. We are going to apply the obtained previous general results in the case $k=1$, $p=q$, $A = C^\infty(\mathbf{R})/((u^0)^2)$ and $\underline{B} =< (u^0)^2, u^i u^j \mid i,j = 1,\dots,q >$, where $u^0 := y^1$. The collection consisting of $K = q+3$ classes

$$v_i^{(1)} := [du^0 \wedge du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^q]_{Q^q_{\underline{B}}} \ , \ v^{(3)} := [u^1 du^0 \wedge du^2 \wedge \dots \wedge du^q]_{Q^q_{\underline{B}}}$$

$$v^{(1)} := [du^1 \wedge \dots \wedge du^q]_{Q^q_{\underline{B}}} \ , \ v^{(2)} := [u^o du^1 \wedge \dots \wedge du^q]_{Q^q_{\underline{B}}}$$

for $i = 1,\dots,q$ generate (over $\mathbf{R}$) the real vector space Q^q_B. This will be proved in a more general situation in Lemma 5.4. So, we have the basis $w_1,\dots,w_{L_1},\dots,w_L$ of Q^q_B such that $w_{L_1+1},\dots,w_L \in \{v^{(3)}, v^{(1)}, v^{(2)}\}$ and $w_1,\dots,w_{L_1} \in \{v_i^{(1)} \mid i = 1,\dots,q\}$. We can see that this basis is adapted. Namely, $v_i^{(1)}$ is homogeneous of weight $(1,\dots,1,0,1,\dots,1) \in \{0,1\}^q$ (0 in i-th position) for $i = 1,\dots,q$ and the other elements are of weight $(1,\dots,1,\dots,1) \in \{0,1\}^q$. Let $\lambda_1,\dots,\lambda_L$ be the dual basis. Using Theorem 4.4, we can immediately see that any semi-excellent map $H : Q^q_B \to \mathbf{R}$ is of the form

$$H = a_{L_1+1}\lambda_{L_1+1} + \dots + a_L\lambda_L \ ,$$

where the coefficients are real numbers. Thus the vector space of all excellent maps $H : Q^q_B \to \mathbf{R}$ is of dimension $\leq L - L_1 \leq 3$. Thus the vector space of all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^q T^* T$ is of dimension ≤ 3 because of Theorem 3.5. On the other hand we have three linearly independent natural operators in question. Namely, ω^C and ω^V and $di_L\omega^C$. So, using the dimension argument we end the proof of the proposition in this case. That the operators ω^C and ω^V and $di_L\omega^C$ are linearly independent it will be proved later in Lemma 5.6 (in some more general situation).

So, using the dimension argument we end the proof of the proposition.

In the case $m \geq q+2 = 4$, the proof is quite similar. □

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Remark 5.2. In the presented in Proposition 5.1 description of $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^q T^*T$ we cannot see

$$\omega \mapsto i_L d\omega^C .$$

Why? Because of $\omega^C = \mathcal{L}_L \omega^C = i_L d\omega^C + d i_L \omega^C$.

In the rest of this section, we generalize Proposition 5.1. Namely, if $m \geq q+2 \geq 4$, we find explicitly all $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^q T^* T^r$ lifting q-forms ω on a m-manifold M into q-forms $D(\omega)$ on the r-tangent bundle $T^r M = J^r_0(\mathbf{R}, M)$.

At first, we consider the case $m \geq q+2 \geq 5$. We prove

Theorem 5.3. *If $m \geq q+2 \geq 5$ then any $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^q T^* T^r$ is of the form*

$$\omega \mapsto \sum_{s=0}^{r} a_s \omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L \omega^{(\sigma)}$$

for (uniquely determined by D) real numbers a_s, b_σ, where the $\omega^{(s)}$ for $s = 0, \dots, r$ are the classical (s)-lifts of ω to $T^r M$ in the sense of A. Morimoto ([7]) and L is the canonical vector field on $T^r M$ with the flow given by $Exp(\tau L)(v) := j^r_0(\gamma \circ a_\tau)$, $\tau \in \mathbf{R}$, $v = j^r_0 \gamma \in T^r M$, where $a_\tau : \mathbf{R} \to \mathbf{R}$, $a_\tau(t) = \exp(\tau)t$.

To prove Theorem 5.3 we need a preparation. Now, $k = 1$, $p = q$, $A = C^\infty(\mathbf{R})/((u^0)^{r+1})$ and $\underline{B} =< (u^0)^{r+1}, u^i u^j \mid i,j = 1, \dots, q >$, where $u^0 := y^1$. We prove some lemmas.

Lemma 5.4. *The collection consisting of $K = qr + 1 + r + r$ classes*

$$v_i^{(1,s)} := [d(u^0)^s \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}} \ , \ v^{(3,s)} := [u^1 d(u^0)^s \wedge du^2 \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}} \ ,$$

$$v^{(1)} := [du^1 \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}} \ , \ v^{(2,s)} := [(u^o)^s du^1 \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}}$$

for $i = 1, \dots, q$ and $s = 1, \dots, r$ generate (over $\mathbf{R}$) the vector space $Q^q_{\underline{B}}$. All elements of this collection are adapted. Namely $v_i^{(1,s)}$ is homogeneous of weight $(1, \dots, 1, 0, 1, \dots, 1) \in \{0,1\}^q$ (0 in i-th position) for $i = 1, \dots, q$ and the other elements are of weight $(1, \dots, 1, \dots, 1) \in \{0,1\}^q$.

Proof. Clearly, the classes $[u^\beta du^0 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}}$ for all $\beta \in \mathbf{N}^{q+1}$ and $j = 0, \dots, q$ generate the vector space $Q^q_{\underline{B}}$. Because of the construction of $Q^q_{\underline{B}}$, all these classes are 0 except (eventually) of $[(u^0)^k du^1 \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}}$ and

$$[(u^0)^{k_1} du^0 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}} \text{ and } [(u^0)^{k_1} u^i du^0 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}}$$

for $k = 0, \dots, r$ and $k_1 = 0, \dots, r-1$ and $i = 1, \dots, q$. Moreover,

$$[(u^0)^{k_1} du^0 \wedge d(u^1 u^i) \wedge du^2 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}} = 0 \ ,$$

so $[(u^0)^{k_1} u^1 du^0 \wedge du^2 \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}} = -[(u^0)^{k_1} u^i du^0 \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q]_{Q^q_{\underline{B}}}$. Consequently, the classes $v_i^{(1,s)}$, $v^{(3,s)}$, $v^{(1)}$, $v^{(2,s)}$ for $i = 1, \dots, q$ and $s = 1, \dots, r$ generate (over $\mathbf{R}$) the vector space $Q^q_{\underline{B}}$.

The rest of this lemma is a simple observation. ☐

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Lemma 5.5. *Let $w_1, \dots, w_L$ be the adapted basis in Q^q_B such that $w_1, \dots, w_{L_1} \in \{v^{(1,s)}_i \mid s = 1, \dots, r\ ,\ i = 1, \dots, q\}$ and $w_{L_1+1}, \dots, w_L \in \{v^{(3,s)}, v^{(2,s)}, v^{(1)} \mid s = 1, \dots, r\}$. Let $\lambda_1, \dots \lambda_L$ be the dual basis. Then any semi-excellent map $Q^q_B \to \mathbf{R}$ is*

$$a_{L_1+1}\lambda_1 + \cdots + a_L\lambda_L$$

for any $a_{L_1+1}, \dots, a_L \in \mathbf{R}$. Thus the space of excellent maps $Q^q_B \to \mathbf{R}$ is of dimension $\leq L - L_1 \leq 2r + 1$.

Proof. It follows immediately from Theorem 4.4. □

Lemma 5.6. *If $m \geq q + 2 \geq 4$, the collection of $\mathcal{M}f_m$-natural operators $\omega^{(s)}$ and $di_L\omega^{(\sigma)}$ for $s = 0, 1, \dots, r$ and $\sigma = 1, \dots, r$ is linearly independent.*

Proof. Suppose we have

$$\eta = \sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma di_L\omega^{(\sigma)} = 0$$

for any $\omega \in \Omega^q(M)$. Then

$$\sum_{s=0}^{r} a_s(d\omega)^{(s)} = \sum_{s=0}^{r} a_s d(\omega^{(s)}) = d\eta = 0$$

for any $\omega \in \Omega^q(\mathbf{R}^m)$ because of $d(\omega^{(s)}) = (d\omega)^{(s)}$. Putting $\omega = (x^1)^\lambda x^2 dx^3 \wedge \cdots \wedge dx^{q+2}$ we get

$$0 = \sum_{s=0}^{r} a_s(d\omega)^{(s)}(\partial_2{}^{(r)}, \dots, \partial^{(r)}_{q+2}) = \sum_{s=0}^{r} a_s(d\omega(\partial_2, \dots, \partial_{q+2}))^{(s)} = \sum_{s=0}^{r} a_s((x^1)^\lambda)^{(s)}$$

for $\lambda = 0, 1, \dots, r$, where $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, \dots, m$. Evaluating at $j^r_0(t, 0 \dots, 0) \in T^r\mathbf{R}^m$ we get

$$0 = \sum_{s=0}^{r} a_s \frac{1}{s!}\frac{d^s t^\lambda}{dt^s}_{|t=0} = a_\lambda$$

for $\lambda = 0, 1, \dots, r$. Then $a_0 = a_1 = \cdots = a_r$ *and then

$$\sum_{\sigma=1}^{r} b_\sigma di_L\omega^{(\sigma)} = 0$$

for any $\omega \in \Omega^q(\mathbf{R}^m)$. Since the flow of L is a natural transformation $T^r \to T^r$ and the flow of $X^{(r)}$ is $\{T^r(\varphi_\tau)\}$ if $\{\varphi_\tau\}$ is the flow of X, then the Lie derivative $\mathcal{L}_L X^{(r)} = 0$ for any vector field X on $\mathbf{R}^m$. Then

$$(\mathcal{L}_L\omega^{(\sigma)})(X^{(r)}_1, \dots, X^{(r)}_q) = L\omega^{(\sigma)}(X^{(r)}_1, \dots, X^{(r)}_q) = L\omega(X_1 \dots, X_q)^{(\sigma)}\ ,$$

and then

$$(di_L\omega^{(\sigma)})(X^{(r)}_1, \dots, X^{(r)}_q) = L\omega(X_1 \dots, X_q)^{(\sigma)}$$

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for any closed q-form ω on $\mathbf{R}^m$ and any vector fields $X_1, \dots, X_q$ on $\mathbf{R}^m$ (because $i_L d\omega^{(\sigma)} = 0$ if $d\omega = 0$) for $\sigma = 1, \dots, r$. Then putting $\omega = (x^1)^\lambda dx^1 \wedge \dots \wedge dx^q$ (it is closed) and $X_j = \partial_j$ for $j = 1, \dots, q$ and $\lambda = 0, \dots, r$ we get

$$\sum_{\sigma=1}^{r} b_\sigma L((x^1)^\lambda)^{(\sigma)} = 0$$

for $\lambda = 1, \dots, r$. Evaluating at $j_0^r(t, 0, \dots, 0) \in T^r M$ we get

$$0 = \sum_{\sigma=1}^{r} b_\sigma \frac{1}{\sigma!} \frac{d}{d\tau}_{|\tau=0} \frac{d^\sigma \exp(\tau\lambda) t^\lambda}{dt^\sigma}_{|t=0} = \lambda b_\lambda$$

for any $\lambda = 1, \dots, r$. Then $b_1 = \dots = b_r = 0$. That is why the operators $\omega^{(s)}$ and $d i_L \omega^{(\sigma)}$ for $s = 0, \dots, r$ and $\sigma = 1, \dots, r$ are linearly independent, as well.

The proof of the lemma is complete. □

We are now in position to prove Theorem 5.3.

Proof. By Lemma 5.5, the vector space of excellent maps $Q_B^q \to \mathbf{R}$ is of dimension $\leq 2r + 1$. Then (by Theorem) the vector space of all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^q T^* T^r$ is of dimension $\leq 2r + 1$, too. Then using Lemma 5.6 and the dimension argument we complete the proof of the theorem. □

Now, we pass to the case $m \geq q + 2 = 4$. We prove

Lemma 5.7. *If $m \geq q + 2 = 4$ then the vector space of $\mathcal{M}f_m$-natural operators $\wedge^2 T^* \rightsquigarrow \wedge^2 T^* T^r$ is of dimension $\leq (2r + 1 + \frac{r(r-1)}{2})$.*

Proof. Clearly, Lemma 5.4 for $m \geq q + 2 = 4$ is true, too. Then we have the following version of Lemma 5.5. Let $w_1, \dots, w_L$ be the adapted basis in Q_B^q such that $w_1, \dots, w_{L_1} \in \{v_1^{(1,s)} \mid s = 1, \dots, r\}$ and $w_{L+1}, \dots, w_{L_2} \in \{v_2^{(1,s)} \mid s = 1, \dots, r\}$ and $w_{L_2+1}, \dots, w_L \in \{v^{(3,s)}, v^{(2,s)}, v^{(1)} \mid s = 1, \dots, r\}$. Let $\lambda_1, \dots . \lambda_L$ be the dual basis. Then (because of Theorem 4.4) any semi-excellent map $Q_B^q \to \mathbf{R}$ is of the form

$$H = \sum_{k_1=1}^{L_1} \sum_{k_2=L_1+1}^{L_2} a_{k_1 k_2} \lambda_{k_1} \lambda_{k_2} + a_{L_2+1} \lambda_{L_2+1} + \dots + a_L \lambda_L$$

for a (uniquely determined by H) real coefficients. Then there is a linear monomorphism sending any H (in question) into the collection of the sequence $(a_{L_1+1}, \dots, a_L)$ (corresponding to H) and the $r \times r$-matrix $[b_{s_1 s_2}]$ such that $b_{s_1 s_2} := H(v_1^{(1,s_1)} + v_2^{(1,s_2)})$ for $s_1, s_2 = 1, \dots, r$. If H is excellent, then

$$-b_{s_1 s_2} = -H(v_1^{(1,s_1)} + v_2^{(1,s_2)}) = H(s\Box(v_1^{(1,s_1)} + v_2^{(1,s_2)}))$$
$$= H(v_1^{(1,s_2)} + v_2^{(1,s_1)}) = b_{s_2 s_1}$$

for any $s_1, s_2 = 1, \dots, r$, where $s = (1, 2) \in S_2$ is the cycle. Now, the lemma is clear. □

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Theorem 5.8. *If $m \geq q+2=4$, then any $\mathcal{M}f_m$-natural operator $D : \wedge^2 T^* \rightsquigarrow \wedge^2 T^* T^r$ is of the form*

$$\omega \mapsto \sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} + \sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L\omega^{(\sigma)}$$

for (uniquely determined by D) real numbers $a_{\sigma_1\sigma_2}, a_s, b_\sigma$.

Proof. Suppose we have

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} + \sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L\omega^{(\sigma)} = 0$$

for any q-form ω on $\mathbf{R}^m$. Putting $t\omega$ instead of ω we get

$$t^2 \sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} + t(\sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L\omega^{(\sigma)}) = 0$$

for any $t \in \mathbf{R}$. Then

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} = 0 \text{ and } \sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L\omega^{(\sigma)} = 0$$

for any q-form ω on $\mathbf{R}^m$. Then $a_0 = \cdots = a_r = 0$ and $b_1 = \cdots = b_r = 0$ (because of Lemma 5.6) and

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} = 0$$

for any q-form ω on $\mathbf{R}^m$. Put $\omega = dx^1 \wedge dx^2$. It is $L = \sum_{i=1}^{m}\sum_{\mu=1}^{r} \mu(x^i)^{(\mu)}\partial_i^{(r-\mu)}$ (because of $(L((x^i)^{(\mu)}))_{|j_0^r\gamma} = \frac{1}{\mu!}\frac{d}{d\tau}_{|\tau=0}\frac{d^\mu}{dt^\mu}_{|t=0}(x^i(\gamma(\exp(\tau)t)) = \mu(x^i)^{(\mu)}_{|j_0^r\gamma})$. Then

$$(i_L\omega^{(\sigma)})(\partial_1^{(r)}) = \sum_{i=1}^{m}\sum_{\mu=1}^{r} \mu(x^i)^{(\mu)}\omega^{(\sigma)}(\partial_i^{(r-\mu)}, \partial_1^{(r)})$$

$$= \sum_{i=1}^{m}\sum_{\mu=1}^{r} \mu(x^i)^{(\mu)}\omega(\partial_i, \partial_1)^{(\sigma-\mu)} = -\sigma(x^2)^{(\sigma)}.$$

Similarly,

$$(i_L\omega^{(\sigma)})(\partial_2^{(r)}) = \sigma(x^1)^{(\sigma)}.$$

Then

$$(i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)})(\partial_1^{(r)}, \partial_2^{(r)}) = \sigma_1\sigma_2(-(x_2)^{(\sigma_1)}(x^1)^{(\sigma_2)} + (x^1)^{(\sigma_1)}(x^2)^{(\sigma_2)}).$$

Then

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2}\sigma_1\sigma_2(-(x_2)^{(\sigma_1)}(x^1)^{(\sigma_2)} + (x^1)^{(\sigma_1)}(x^2)^{(\sigma_2)}) = 0.$$

Evaluating at $j_0^r(t^{s_1}, t^{s_2}, 0, \ldots, 0) \in T^r\mathbf{R}^m$, we get $a_{s_1s_2} = 0$ for $1 \leq s_1 < s_2 \leq r$. Then the dimension argument ends the proof of our theorem. □

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Remark 5.9. If $m \geq q+2 = 3$, one can see that any semi-excellent (then excellent) map $H : Q^1_B \to \mathbf{R}$ is of the form

$$a_1(\lambda_1^{(1,1)}, \ldots, \lambda_1^{(1,s)})\lambda^{(1)} + \sum_{s=1}^{r} a_{2,s}(\lambda_1^{(1,1)}, \ldots, \lambda_1^{(1,s)})\lambda^{(2,s)}$$

$$+\sum_{s=1}^{r} a_{3,s}(\lambda_1^{(1,1)}, \ldots, \lambda_1^{(1,s)})\lambda^{(3,s)},$$

where $\lambda_1^{(1,1)}, \ldots$ is the dual basis to the adapted basis $v_1^{(1,1)}, \ldots$ (the collection of Lemma 5.4), where $a_1, a_{2,s}, a_{3,s} : \mathbf{R}^r \to \mathbf{R}$ are smooth maps. So, we reobtained the result of [6] saying that if $m \geq 3$, then the space of all $\mathcal{M}f_m$-natural operators $T^* \rightsquigarrow T^*T^r$ is the free $(2r+1)$-dimensional $C^\infty(\mathbf{R}^r)$-module. In [6], we presented the (rather complicated) basis of this module. It seems that any $\mathcal{M}f_m$-natural operator $D : T^* \rightsquigarrow T^*T^r$ is of the form

$$\omega \mapsto \sum_{\nu=0}^{r} h_\nu(i_L\omega^{(1)}, \ldots, i_L\omega^{(r)})\omega^{(\nu)} + \sum_{\sigma=1}^{r} g_\sigma(i_L\omega^{(1)}, \ldots, i_L\omega^{(r)})di_L\omega^{(\sigma)}$$

for (uniquely determined by D) smooth maps $h_\nu, g_\sigma : \mathbf{R}^r \to \mathbf{R}$, where L is the (mentioned above) canonical vector field on T^r. We leave this problem to be open.

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