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NEW CHARACTERIZATION OF (b,c)-INVERSES THROUGH POLARITY

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ABSTRACT. Given any ring R with unity 1 and any $a,b,c \in R$, a is called (b,c)-*polar* if there exist two idempotents $p,q \in R$ such that $p \in bRca$, $q \in abRc$, $pb = b$, $cq = c$, $cap = ca$ and $qab = ab$. These p and q are shown to be unique whenever they exist. The existence of $a^{\|(b,c)}$ the (b,c)-inverse of a is shown to be equivalent to that a is (b,c)-polar, and hence that $a^{\|(b,c)}$ is itself unique and expressed in terms of p and q. Generalizing results of Koliha–Patrício and Song–Zhu–Mosić, further connections between the (b,c)-polar and (b,c)-invertible properties are found. Applying these results to linear bounded operators on a Banach space we also generalize some known results in this setting.

1. INTRODUCTION

Throughout this paper, R will denote an associative ring with unity 1. An element $a \in R$ is *regular* if $a \in aRa$ i.e., $a = axa$ for some $x \in R$. Any such x is called an *inner inverse* of a. An inner inverse of a will be denoted by a^-. We denote the set of all inner invertible elements in R by R^-, while the group of units in R is denoted by R^{-1} and the set of all left invertible (respectively right invertible) elements in R by R_l^{-1} (respectively R_r^{-1}). For any $a \in R$ we define the *commutant* and *double commutant* of a respectively by

$$\operatorname{comm}(a) = \{x \in R : ax = xa\}$$

$$\operatorname{comm}^2(a) = \{x \in R : xy = yx, \text{ for all } y \in \operatorname{comm}(a)\}.$$

An element a is *quasinilpotent* if $1 + xa \in R^{-1}$ for all $x \in \operatorname{comm}(a)$ [8]. Let R^{nil} and R^{qnil} denote, respectively, the set of all nilpotent and quasinilpotent elements in R.

Following Drazin ([3]) an element $a \in R$ is said to be *Drazin invertible* if there exists $x \in R$ such that

$$x \in \operatorname{comm}(a),\ xax = x \text{ and } a^{k+1}x = a^k$$

for some nonnegative integer k. The element x is unique if it exists and is called the *Drazin inverse* of a and is denoted by a^D. The smallest nonnegative integer k

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satisfying the above conditions is called the Drazin index of a, and denoted by $ind(a)$. The set of all Drazin invertible elements in R is denoted by R^D. If $ind(a) \leq 1$, then x is called the *group inverse* of a. It is denoted by $a^\sharp$. We denote the set of all group invertible elements in R by $R^\sharp$.

Koliha and Patrício [10] extended the notion of Drazin inverse to generalized Drazin inverse: an element $a \in R$ is *generalized Drazin invertible* if there exists $b \in R$ such that

$$b \in \mathrm{comm}^2(a),\ ab^2 = b \text{ and } a^2b - a \in R^{\mathrm{qnil}}. \tag{1.1}$$

Any element $b \in R$ satisfying those conditions in (1.1) is unique and is called the *g-Drazin inverse* of a, and it is denoted by a^{gD}. The set of all g-Drazin invertible elements in R is denoted by R^{gD}. Koliha and Patrício gave a characterization for (generalized) Drazin invertibility via idempotents by introducing the notion of polar and quasipolar elements. An element $a \in R$ is *quasipolar* (resp. *polar*) if there exists an idempotent $p \in R$ such that

$$p \in \mathrm{comm}^2(a),\ a + p \in R^{-1} \text{ and } ap \in R^{\mathrm{qnil}} \,(\text{resp. } ap \in R^{\mathrm{nil}}). \tag{1.2}$$

The idempotent p is unique and is called the *spectral idempotent* of a and is denoted by a^π. It is proved that a is generalized Drazin invertible if and only if it is quasipolar, also a is Drazin invertible if and only if a is polar. In this case, $a^{gD} = (a+p)^{-1}(1-p)$.

Based on this approach, Wang and Chen in [16] introduced the notion of pseudopolarity. An element $a \in R$ is said to be *pseudopolar* if there exists an idempotent $p \in R$ such that

$$p \in \mathrm{comm}^2(a),\ a + p \in R^{-1} \text{ and } ap \in R^{\mathrm{rad}}; \tag{1.3}$$

where R^{rad} denotes the Jacobson radical of R. Also the idempotent p is unique if it exists. They also introduced the notion of pseudo Drazin invertibility, which lying between the Drazin invertibility and generalized Drazin invertibility: an element a is *pseudo Drazin invertible* if there exists $b \in R$ such that

$$b \in \mathrm{comm}^2(a), bab = b \text{ and } a^k - a^{k+1}b \in R^{\mathrm{rad}} \tag{1.4}$$

for some nonnegative integer k. Such an element is unique if it exists and is called the pseudo Drazin inverse of a. Moreover, a is pseudo Drazin invertible if and only if a is pseudopolar, [16].

Mary [12] introduced a generalized inverse using Green's relations. An element $a \in R$ will be said to be invertible along $d \in R$ if there exists $y \in R$ such that

$$yad = d = day,\ yR \subseteq dR,\ Ry \subseteq Rd.$$

Such a y is unique if it exists and called *the inverse of a along d*, denoted by $a^{\|d}$. Moreover, if a is invertible along d then d is regular. The set of all elements in R that are invertible along d is denoted by $R^{\|d}$.

Recently, to give a new characterization of the invertibility along an element via idempotents elements, Song, Zhu and Mosić [15] provided a definition for the

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concept of the polarity along an element in R. Let $a, d \in R$, we say that a is *polar along* d if there exists some $p \in R$ such that

$$p = p^2 \in \mathrm{comm}(da),\ pd = d \text{ and } 1 + da - p \in R^{-1}.$$

Which is equivalent to

$$p = p^2 \in \mathrm{comm}(da),\ pd = d \text{ and } p \in daRda.$$

In this case p is unique and is denoted by $a^{d\pi}$. It is also proved that a is invertible along d if and only if a is polar along d. In this case, the inverse of a along d is given by $a^{\| d} = (1 + da - p)^{-1}d$ and p is also established via $p = a^{\| d}a$. Also a is invertible along d if and only if a is dually polar along d. Recall that a is *dually polar along* d if exists some $q \in R$ such that

$$q = q^2 \in \mathrm{comm}(ad),\ dq = d \text{ and } 1 + ad - q \in R^{-1}.$$

Which is equivalent to

$$q = q^2 \in \mathrm{comm}(ad),\ dq = d \text{ and } q \in adRad.$$

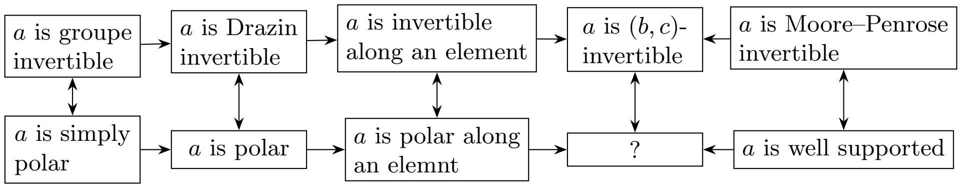
In this case q is unique and is denoted by $a_{d\pi}$.

In 2012, Drazin introduced a class of outer inverses [4] which extend inverses along elements and so Drazin inverses and Moore–Penrose inverses. For any $a, b, c \in R$, a is said to be (b, c)*-invertible* if there exists $y \in R$ such that

$$y \in bR \cap Rc\ ,\ yab = b\ ,\ cay = c. \tag{1.5}$$

If such a y exists, it is unique and called the (b, c)*-inverse* of a and denoted by $a^{\|(b,c)}$. Also if a is (b, c)-invertible, then b, c and cab are regular. The set of all (b, c)-invertible elements in R is denoted by $R^{\|(b,c)}$. In the case where $b = e$ and $c = f$ such that e and f are idempotents, we say that a is (e, f)*-Bott–Duffin invertible* if a is (e, f)-invertible [4]. Moreover the inverse along an element is a special case of the more general class of the (b, c)-inverse which occurs when $b = c$, consequently we hold $a^{\| d} = a^{\|(d,d)}$ and $a^D = a^{\|(a^k, a^k)}$, where k is the index of a; and in particular $a^\sharp = a^{\|(a,a)}$.

The approach of introducing new generalized inverses via polarities were used also in [6, 11]. So it is natural to ask if there exists a kind of polarity which extends polarity and polarity along an element and also characterizes the (b, c)-invertibility (see also [6]). More precisely, the motivation for this paper arises from the following incomplete diagram of the related concepts:



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We introduce in this paper the notion of (b,c)-polarity (Definition 2.1). We show that when $b=c$ then (b,b)-polarity coincides with polarity along b, which extends then the polarity along an element. Moreover, we show that an element a is (b,c)-polar if and only if a is (b,c)-invertible. We give then a new characterization of (b,c)-invertible elements. In Section 3, we introduce the concept of dually (b,c)-polar elements as an extension of dually polar along an element introduced in [15]. Among other things, we show that a is dually (b,c)-polar if and only if a is (c,b)-invertible. The last section is devoted to illustrating (b,c)-polarity in the context of bounded linear operators.

2. The (b,c)-Polarity

We start by introducing the concept of (b,c)-polarity.

Definition 2.1. Let a, b, $c\in R$, we say that a is (b,c)-*polar* if there exist p, $q\in R$ such that

(1) $p^2=p\in bRca$;
(2) $q^2=q\in abRc$;
(3) $pb=b$, $cq=c$;
(4) $cap=ca$, $qab=ab$.

Any idempotent p (respectively q) satisfying the above conditions is called a *left (b,c)-spectral idempotent* of a (respectively a *right (b,c)-spectral idempotent* of a).

In the following we show the uniqueness of the left and right (b,c)-spectral idempotents of a (b,c)-polar element.

Theorem 2.2. *Let a, b, $c\in R$ such that a is (b,c)-polar. Then a has a unique left (b,c)-spectral idempotent and a unique right (b,c)-spectral idempotent.*

Proof. Suppose that p and p' are two left (b,c)-spectral idempotents of a and q and q' are two right (b,c)-spectral idempotents of a.
As $p\in bRca$, then $p=btca$ for some $t\in R$. It follows that

$$p-p'p=btca-p'btca=(b-p'b)tca=0 \quad (\text{since } b=pb=p'b)\ .$$

So we obtain

$$p=p'p.$$

Similary $p'-pp'=0$, and we get

$$p'=pp'.$$

In other side, we have $cap=ca=cap'$, so $p-pp'=btca-btcap'=btca-btca=0$. Hence $p=pp'$ and thus $p=p'$.

Similarly we show that $q=q'$. ☐

If a is (b,c)-polar then we denote the left (b,c)-spectral idempotent p by $a_l^{(b,c)\pi}$ and the right (b,c)-spectral idempotent q by $a_r^{(b,c)\pi}$.

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Example 2.3. Let $R=\mathcal{M}_2(\mathbb{Z})$, and $a,b,c\in R$ such that

$$a=\begin{pmatrix}0&0\\1&0\end{pmatrix},\ b=\begin{pmatrix}1&-1\\0&0\end{pmatrix},\ c=\begin{pmatrix}0&1\\0&1\end{pmatrix}.$$

Then a is (b,c)-polar with $p=a_l^{(b,c)\pi}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $q=a_r^{(b,c)\pi}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Indeed, a quick check, we obtain

i) $p^2=p$, $q^2=q$.
ii) $p=b\begin{pmatrix}0&0\\-2&1\end{pmatrix}ca$ and $q=ab\begin{pmatrix}1&0\\0&0\end{pmatrix}c$.
iii) $pb=b$, $cq=c$.
iv) $cap=ca$, $qab=ab$.

Here we show that the (b,c)-polarity is an extension of the polarity along an element.

Proposition 2.4. *Let a and $b\in R$. Then a is (b,b)-polar if and only if a is polar along b.*

Proof. If a is (b,b)-polar then we have

$$p=p^2\in bRba,\ q=q^2\in abRb,\ pb=b=bq\ \text{ and }\ bap=ba=pba,$$

which implies $p=p^2\in \mathrm{comm}(ba)$ and $pb=b$. Also, $p=bxba$ for some $x\in R$, then $p=bqxba\in babRbxba\subseteq baRba$. Hence, a is polar along b by [15, Theorem 2.4].

Conversely, if a is polar along b then there exists a unique $p=p^2\in R$ such that $p\in\mathrm{comm}(ba)$, $pb=b$ and $p\in baRba$. Then we have

$$\begin{cases} p & \in & baRba\subseteq bRba\\ pb &=& b\\ bap &=& pba=ba\end{cases} \tag{2.1}$$

On the other hand, we have a is polar along b if and only if a is dually polar along b, then there exists a unique $q=q^2\in R$ such that $q\in\mathrm{comm}(ab)$, $bq=b$ and $q\in abRab$. So

$$\begin{cases} q & \in & abRab\subseteq abRb\\ bq &=& b\\ qab &=& abq=ab\end{cases} \tag{2.2}$$

Now from (2.1) and (2.2), we can see that a is (b,b)-polar. □

Lemma 2.5. *Let $a,b,c\in R$. If a is (b,c)-polar, then a, c and cab are regular.*

Proof. Suppose that a is (b,c)-polar. We have $b=a_l^{(b,c)\pi}b\in bRcab\subseteq bRb$, also $c=ca_r^{(b,c)\pi}\in cabRc\subseteq cRc$ and $cab=ca_r^{(b,c)\pi}aa_l^{(b,c)\pi}b\in cabRcabRcab\subseteq cabRcab$. Which means that b, c and cab are regular and admit respectively inner inverses denoted by b^-, c^- and $(cab)^-$. □

The following theorem shows the equivalence between the (b,c)-polarity and the (b,c)-invertibility.

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Theorem 2.6. *Let $a, b, c \in R$. Then a is (b,c)-polar if and only if a is (b,c)-invertible.*

In this case we have

i) $p = a_l^{(b,c)\pi} = a^{\|(b,c)}a$.
ii) $q = a_r^{(b,c)\pi} = aa^{\|(b,c)}$.
iii) $a^{\|(b,c)} = (1+p-bb^-)_l^{-1}b(cab)^-c = b(cab)^-c(1+q-c^-c)_r^{-1}$. *With $(1+p-bb^-)_l^{-1}$ is a left inverse of $1+p-bb^-$ and $(1+q-c^-c)_r^{-1}$ is a right inverse of $1+q-c^-c$.*

Proof. Suppose that a is (b,c)-polar. To prove that a is (b,c)-invertible, it suffices to prove that $b \in Rcab$ and $c \in cabR$ by [1, Lemma 1]. We have

$$a_l^{(b,c)\pi} \in bRca,\ a_r^{(b,c)\pi} \in abRc,\ ca_r^{(b,c)\pi} = c \text{ and } a_l^{(b,c)\pi}b = b.$$

It follows that

$$a_l^{(b,c)\pi}b \in bRcab \Rightarrow b \in bRcab \subseteq Rcab,$$

and

$$ca_r^{(b,c)\pi} \in cabRc \Rightarrow c \in cabRc \subseteq cabR.$$

So a is (b,c)-invertible.

By Lemma 2.5 we know that b, c and cab are regular with b^-, c^- and $(cab)^-$ as inner inverses of b, c and cab, respectively. Moreover we have $1+a_l^{(b,c)\pi}-bb^-$ is left invertible and $1+a_r^{(b,c)\pi}-c^-c$ is right invertible. Indeed, since $a_l^{(b,c)\pi} \in bRca \subseteq bR$, $a_l^{(b,c)\pi} = bt$ for some $t \in R$ and we write $a_l^{(b,c)\pi} = bt = bb^-bt = bb^-a_l^{(b,c)\pi}$, so

$$(bb^- + 1 - a_l^{(b,c)\pi})(1 + a_l^{(b,c)\pi} - bb^-) = 1$$

which means that $1+a_l^{(b,c)\pi}-bb^-$ is left invertible, and we denote a left inverse of $1+a_l^{(b,c)\pi}-bb^-$ by $(1+a_l^{(b,c)\pi}-bb^-)_l^{-1}$. Similarly for $1+a_r^{(b,c)\pi}-c^-c$, as $a_r^{(b,c)\pi} \in abRc \subseteq Rc$ we write $a_r^{(b,c)\pi} = xc$ for some $x \in R$, so $a_r^{(b,c)\pi} = xc = xcc^-c = a_r^{(b,c)\pi}c^-c$ which implies

$$(1 + a_r^{(b,c)\pi} - c^-c)(c^-c + 1 - a_r^{(b,c)\pi}) = 1.$$

Hence $1+a_r^{(b,c)\pi}-c^-c$ is right invertible, we denote a right inverse of $1+a_r^{(b,c)\pi}-c^-c$ by $(1+a_r^{(b,c)\pi}-c^-c)_r^{-1}$.

Now set $y = (1+a_l^{(b,c)\pi}-bb^-)_l^{-1}b(cab)^-c = b(cab)^-c(1+a_r^{(b,c)\pi}-c^-c)_r^{-1}$. Then $yab = b$, $cay = c$ and $y \in bR \cap Rc$. Therefore, y is the (b,c)-inverse of a.

Conversely, suppose that a is (b,c)-invertible with y as the (b,c)-inverse of a. Set $p = ya$ and $q = ay$, then p is the left (b,c)-spectral idempotent of a and q is the right (b,c)-spectral idempotent of a. Indeed, we have

$$p^2 = yaya = ya = p \text{ and } q^2 = ayay = ay = q.$$

In other hand, as y is the (b,c)-inverse of a then $y \in bR \cap Rc$. Thus

$$p = ya \in bRa \subseteq bR \text{ and } p = ya \in Rca \text{ so, } p = p^2 \in bRca,$$

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and

$$q = ay \in abR \text{ and } q = ay \in aRc \subseteq Rc \text{ so, } q = q^2 \in abRc.$$

We have $pb = yab = b$, $cq = cay = c$, $cap = caya = ca$ and $qab = ayab = ab$. Finally, a is (b,c)-polar. □

Combining Proposition 2.4 and Theorem 2.6 we retrieve the main result of [15].

Corollary 2.7. *Let $a,b \in R$. Then a is polar along b if and only if a is invertible along b if and only if a is (b,b)-invertible if and only if a is (b,b)-polar.*

Definition 2.8. Given $a,e,f \in R$, we say a is (e,f)-*Bott–Duffin polar* if a is (e,f)-polar and e and f are idempotents.

Proposition 2.9. *Let $a,b,c \in R$. If a is (b,c)-polar then a is (e,f)-Bott–Duffin polar with $e = a_l^{(b,c)\pi}$ and $f = a_r^{(b,c)\pi}$.*

Proof. Set $a_l^{(e,f)} = a_l^{(b,c)}$ and $a_r^{(e,f)} = a_r^{(b,c)}$. Then we obtain the result. □

Corollary 2.10. *Let $a,e,f \in R$. Then a is (e,f)-Bott–Duffin polar if and only if a is (e,f)-Bott–Duffin invertible.*

3. The dual (b,c)-polarity

Definition 3.1. Let a, b and $c \in R$. We say that a *is dually (b,c)-polar* if there exist $r,s \in R$ such that

(1) $r^2 = r \in acRb$;
(2) $s^2 = s \in cRba$;
(3) $br = b$ and $sc = c$;
(4) $rac = ac$ and $bas = ba$.

Any idempotent r (respectively s) satisfying the above conditions is called a *dual right (b,c)-spectral idempotent* of a (respectively a *dual left (b,c)-spectral idempotent* of a).

Theorem 3.2. *Let $a,b,c \in R$ such that a is dually (b,c)-polar. Then a has a unique dual right (b,c)-spectral idempotent and a unique dual left (b,c)-spectral idempotent.*

Proof. Suppose that r and r' are two dual right (b,c)-spectral idempotent of a. Then we have

$$r - r'r = actb - r'actb = actb - actb = 0,$$

for some $t \in R$. Then $r = r'r$. Also

$$r' - rr' = acxb - racxb = acxb - acxb = 0,$$

for some $x \in R$. Hence $r' = rr'$. Consequently, we have

$$r'r - r' = acxbr - acxb = acx(br - b) = 0,$$

so $r'r = r'$. Therefore we get $r' = r$.

By the same way we prove the uniqueness of the dual left (b,c)-spectral idempotent of a. □

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We denote the dual right (b,c)-spectral idempotent of a by $r = a^r_{(b,c)\pi}$ and the dual left (b,c)-spectral idempotent of a by $s = a^l_{(b,c)\pi}$.

Lemma 3.3. *Let $a, b, c \in R$. If a is dually (b,c)-polar then b, c and bac are regular.*

Proof. It is similar to the proof of the Lemma 2.5. □

Theorem 3.4. *Let $a, b, c \in R$. Then a is dually (b,c)-polar if and only if a is (c,b)-invertible.*

In this case we have

i) $r = a^r_{(b,c)\pi} = aa^{\|(c,b)}$.
ii) $s = a^l_{(b,c)\pi} = a^{\|(c,b)}a$.
iii) $a^{\|(c,b)} = (1 + a^l_{(b,c)\pi} - cc^-)^{-1}_l c(bac)^- b = c(bac)^- b(1 + a^r_{(b,c)\pi} - b^- b)^{-1}_r$.

Proof. Suppose that a is dually (b,c)-polar, then there exist $r, s \in R$ such that

$$br = b \quad \text{and} \quad r \in acRb,$$
$$sc = c \quad \text{and} \quad s \in cRba.$$

Then $br \in bacRb$, which implies that $b \in bacRb \subseteq bacR$. Also $sc \in cRbac$, which means that $c \in cRbac \subseteq Rbac$. Thus a is (c,b)-invertible .

To obtain the formulas of the (c,b)-inverse of a, we follow the same procedure of the proof of Theorem 2.6.

Conversely, suppose that a is (c,b)-invertible, then we set $r = aa^{\|(c,b)}$ and $s = a^{\|(c,b)}a$. Follow the same procedure of the proof of Theorem 2.6, we obtain that a is dually (b,c)-polar with r (respectively s) its dual right (b,c)-spectral idempotent (dual left (b,c)-spectral idempotent). □

It maybe that a is dually (b,c)-polar but it is not (b,c)-polar as shown by the following example.

Example 3.5. Let $R = \mathcal{M}_2(\mathbb{Z})$, and $a, b, c \in R$ such that

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then a is dually (b,c)-polar with $r = a^r_{(b,c)\pi} = s = a^l_{(b,c)\pi} = a$. Indeed, a quick check, we obtain

- $r^2 = r$;
- $r = ac \begin{pmatrix} x_1 & x_2 \\ x_3 & 1 \end{pmatrix} b$, with x_1, x_2 and x_3 are arbitrary elements of $\mathbb{Z}$;
- $s^2 = s$; $s = c \begin{pmatrix} t_1 & t_2 \\ t_3 & 1 \end{pmatrix} ba$, with t_1, t_2 and t_3 are arbitrary elements of $\mathbb{Z}$;
- $br = b$ and $sc = c$;
- $rac = ac$ and $bas = ba$.

Notice that in this example, a is not (b,c)-polar because a is not (b,c)-invertible since $ab = 0$.

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Corollary 3.6. *Let $a,b,c \in R$. Then a is (b,c)-polar if and only if a is dually (c,b)-polar.*

Proof. It follows by Theorems 2.6 and 3.4. □

Theorem 3.7. *Let $a,b,c \in R$. If a is both (b,c) and (c,b)-invertible such that $aba \in \mathrm{comm}(c)$ and $aca \in \mathrm{comm}(b)$, then we have*

(1) $a_l^{(b,c)\pi} = a^r_{(b,c)\pi}$.
(2) $a_r^{(b,c)\pi} = a^l_{(b,c)\pi}$.
(3) $a^{\|(b,c)} = ba(caba + 1 - (aba)^{c\pi})^{-1}c = (baca + 1 - (aca)^{c\pi})^{-1}bac$.
(4) $a^{\|(c,b)} = (caba + 1 - (aba)^{c\pi})^{-1}cab = ca(baca + 1 - (aca)^{b\pi})^{-1}b$.

Proof. To prove that, we should write at first the formula of $a^{\|(b,c)}$ and $a^{\|(c,b)}$.

We have $a \in R^{\|(b,c)} \cap R^{\|(c,b)}$. And by [14, Theorem 1] we get

$$a^{\|(b,c)} = ba(aba)^{\|c} = (aca)^{\|b}ac, \tag{3.1}$$

and

$$a^{\|(c,b)} = (aba)^{\|c}ab = ca(aca)^{\|b}. \tag{3.2}$$

So we obtain

$$\begin{aligned} a_l^{(b,c)\pi} &= a^{\|(b,c)}a &&= (aca)^{\|b}aca;\\ a_r^{(b,c)\pi} &= aa^{\|(b,c)} &&= aba(aba)^{\|c};\\ a^r_{(b,c)\pi} &= aa^{\|(c,b)} &&= aca(aca)^{\|b};\\ a^l_{(b,c)\pi} &= a^{\|(c,b)}a &&= (aba)^{\|c}aba. \end{aligned}$$

As $aba \in \mathrm{comm}(c)$ and $aca \in \mathrm{comm}(b)$, then $(aba)^{\|c}aba = aba(aba)^{\|c}$ and $aca(aca)^{\|b} = (aca)^{\|b}aca$ by [12, Theorem 10] and it follows that

$$a_l^{(b,c)\pi} = a^r_{(b,c)\pi} \quad \text{and} \quad a_r^{(b,c)\pi} = a^l_{(b,c)\pi}.$$

Using [15, Theorem 2.8], we obtain that

$$(aba)^{\|c} = (caba + 1 - (aba)^{c\pi})^{-1}c.$$

$$(aca)^{\|b} = (baca + 1 - (aca)^{b\pi})^{-1}b.$$

Substitute in (3.1) and (3.2), we obtain the result of (3) and (4). □

Remark 3.8. (1) We can also write $a^{\|(b,c)}$ and $a^{\|(c,b)}$ by using the result of [17, Theorem 2.6] or [14, Proposition 5] as follow:

$$a^{\|(b,c)} = bac(abac)^{\sharp} = ba(caba)^{\sharp}c = b(acab)^{\sharp}ac = (baca)^{\sharp}bac,$$

$$a^{\|(c,b)} = cab(acab)^{\sharp} = ca(baca)^{\sharp}b = c(abac)^{\sharp}ab = (caba)^{\sharp}cab.$$

(2) If a is only (c,b)-polar, this does not allow us to have the equalities in the previous theorem, because we may not have the right and the left (b,c)-spectral idempotents of a since a may not be (b,c)-polar, as we showed in Example 3.5.

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An *involution* $*$ is a bijection $x \mapsto x^*$ on R, which satisfies the following conditions for all $a, b \in R$:

i) $(a^*)^* = a$;

ii) $(ab)^* = b^*a^*$;

iii) $(a+b)^* = a^* + b^*$.

We say that R is a $*$-ring if there is an involution on R.

Proposition 3.9. *Let R be a $*$-ring, and let $a, b, c \in R$. Then a is (b,c)-polar if and only if a^* is dually (b^*, c^*)-polar.*

In this case we have

$$(a_l^{(b,c)\pi})^* = (a^*)^r_{(b^*,c^*)\pi}.$$
$$(a_r^{(b,c)\pi})^* = (a^*)^l_{(b^*,c^*)\pi}.$$

Proof. We have a is (b,c)-polar if and only if a is (b,c)-invertible by theorem 2.6. Suppose that $y = a^{\|(b,c)}$ so $y \in bR \cap Rc$, $yab = b$ and $cay = c$. By involution we get $y^* \in c^*R \cap Rb^*$, $b^*a^*y^* = b^*$ and $y^*a^*c^* = c^*$ which means that a^* is (c^*, b^*)-invertible with inverse y^* i.e., $y^* = (a^{\|(b,c)})^* = a^{*\|(c^*,b^*)}$. And a^* is (c^*, b^*)-invertible equivalent to a^* is dually (b^*, c^*)-polar. And we have

$$(a_l^{(b,c)\pi})^* = (a^{\|(b,c)}a)^* = a^*(a^{\|(b,c)})^* = a^*a^{*\|(c^*,b^*)} = (a^*)^r_{(b^*,c^*)\pi},$$
$$(a_r^{(b,c)\pi})^* = (aa^{\|(b,c)})^* = (a^{\|(b,c)})^*a^* = a^{*\|(c^*,b^*)}a^* = (a^*)^l_{(b^*,c^*)\pi}.$$

□

Proposition 3.10. *Let $a, b, c \in R$ such that a is (b,c)-polar and let $k \geq 1$. If $a \in \operatorname{comm}(b) \cap \operatorname{comm}(c)$ and $ba = ca$, then we have*

(1) *a is polar along b and $a^{b\pi} = a_l^{(b,c)\pi}$.*

(2) *a is dually polar along c and $a_{c\pi} = a_r^{(b,c)\pi}$.*

(3) *a^k is (b^k, c^k)-polar, with $(a^k)_l^{(b^k,c^k)\pi} = a_l^{(b,c)\pi}$ and $(a^k)_r^{(b^k,c^k)\pi} = a_r^{(b,c)\pi}$.*

Proof. Since a is (b,c)-polar then there exist $p = a_l^{(b,c)\pi}$ and $q = a_r^{(b,c)\pi}$ such that

$$p = p^2 \in bRca,\ q = q^2 \in abRc,\ pb = b,\ cq = c,\ cap = ca \ \text{ and } \ qab = ab.$$

(1) and (2): As $ba = ca = ac = ab$ then $bap = ba = pba$ which means that $p \in \operatorname{comm}(ba)$ and $acq = ac = qac$ which means that $q \in \operatorname{comm}(ac)$. Also $p \in bRba \subseteq Rba$ and $q \in abRc = acRc \subseteq acR$. Then $p = tba$ for some $t \in R$ and $q = acx$ for some $x \in R$. Then

$$\begin{aligned}(ptp + 1 - p)(ba + 1 - p) &= ptpba + ptp(1-p) + (1-p)ba + 1 - p \\ &= ptba + 0 + 0 + 1 - p = p^2 + 1 - p \\ &= 1.\end{aligned}$$

Hence $(ba + 1 - p) \in R_l^{-1}$. And

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$$\begin{aligned}(ac+1-q)(qxq+1-q) &= acqxq+ac(1-q)+(1-q)qxq+1-q\\ &= acxq+0+0+1-q\\ &= q^2+1-q\\ &= 1.\end{aligned}$$

Hence $ac+1-q\in R_r^{-1}$.

By Jacobson's lemma and the expression of $p=a^{\|(b,c)}a$ and $q=aa^{\|(b,c)}$, we have

$$1+ba-p\in R_l^{-1}\Longleftrightarrow ac+1-q\in R_l^{-1},$$

and

$$1+ac-q\in R_r^{-1}\Longleftrightarrow ba+1-p\in R_r^{-1}.$$

Thus we obtain $ba+1-p\in R^{-1}$ and $ac+1-q\in R^{-1}$. Consequently a is polar along b with $a^{b\pi}=p=a_l^{(b,c)\pi}$ and a is dually polar along c with $a_{c\pi}=q=a_r^{(b,c)\pi}$.

(3): Since p and q are idempotents and $a\in\operatorname{comm}(b)\cap\operatorname{comm}(c)$ we have

$$c^ka^kp=(ca)^kp^k=(ca)^k=c^ka^k,$$

and

$$qa^kb^k=q^k(ab)^k=(ab)^k=a^kb^k.$$

In other side, we have $pb^k=pbb^{k-1}=bb^{k-1}=b^k$ and $c^kq=c^{k-1}cq=c^{k-1}c=c^k$.

Using (1) and (2) we have

$$\begin{aligned}ba+1-p\in R^{-1} &\Longrightarrow (ba+1-p)^k\in R^{-1}\\ &\Longleftrightarrow (ba)^k+1-p\in R^{-1}\\ &\Longleftrightarrow p\in (ba)^kR(ba)^k \text{ by [15, Theorem 2.4]}\\ &\Longleftrightarrow p\in b^ka^kR(ca)^k\subseteq b^kRc^ka^k.\end{aligned}$$

Similarly

$$\begin{aligned}ac+1-q\in R^{-1} &\Longrightarrow (1+ac-q)^k\in R^{-1}\\ &\Longleftrightarrow (ac)^k+1-q\in R^{-1}\Longleftrightarrow q\in (ac)^kR(ac)^k.\\ &\Longleftrightarrow q\in (ab)^kRa^kc^k\subseteq a^kb^kRc^k\end{aligned}$$

Finally a^k is (b^k,c^k) polar with $(a^k)_l^{(b^k,c^k)\pi}=p=a_l^{(b,c)\pi}$ and $(a^k)_r^{(b^k,c^k)\pi}=q=a_r^{(b,c)\pi}$. □

Theorem 3.11. *Let a,b,c and $d\in R$ such that a is (b,c)-polar. Then the following conditions are equivalent :*

(1) *d is (b,c)-polar such that $a_l^{(b,c)\pi}=d_l^{(b,c)\pi}$ and $a_r^{(b,c)\pi}=d_r^{(b,c)\pi}$.*

(2) $\begin{cases} cda_l^{(b,c)\pi}=cd,\ a_l^{(b,c)\pi}\in bRcd \text{ and } a_l^{(b,c)\pi}b=b\\ a_r^{(b,c)\pi}db=db,\ a_r^{(b,c)\pi}\in dbRc \text{ and } ca_r^{(b,c)\pi}=c\end{cases}$

(3) $\begin{cases} cda_l^{(b,c)\pi}=cd,\ a_l^{(b,c)\pi}\in bRcd\cap bRca \text{ and } a_l^{(b,c)\pi}b=b\\ a_r^{(b,c)\pi}db=db,\ a_r^{(b,c)\pi}\in abRc\cap dbRc \text{ and } ca_r^{(b,c)\pi}=c\end{cases}$

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(4) *d is (b,c)-polar, $cda_l^{(b,c)\pi} = cd$ and $a_r^{(b,c)\pi}db = db$.*

Proof. (1) ⇒ (2): It is clear.

(2) ⇒ (1): By Definition 2.1 and Theorem 2.2

(2) ⇒ (3): We have $a_l^{(b,c)\pi} \in bRca$ and $a_r^{(b,c)\pi} \in abRc$, also we have by hypothesis $a_l^{(b,c)\pi} \in bRcd$ and $a_r^{(b,c)\pi} \in dbRc$, so $a_l^{(b,c)\pi} \in bRcd \cap bRca$ and $a_r^{(b,c)\pi} \in abRc \cap dbRc$.

(3) ⇒ (2): It is obvious

(1) ⇒ (4): It is clear.

(4) ⇒ (1): We prove that $a_l^{(b,c)\pi} = d_l^{(b,c)\pi}$ and $a_r^{(b,c)\pi} = d_r^{(b,c)\pi}$. At first we have $d_l^{(b,c)\pi} \in bRcd$ so $d_l^{(b,c)\pi} = bxcd$ for some $x \in R$; and $a_l^{(b,c)\pi} \in bRca$ so $a_l^{(b,c)\pi} = btca$ for some $t \in R$. Moreover $cdd_l^{(b,c)\pi} = cd = cda_l^{(b,c)\pi}$ and $a_l^{(b,c)\pi}b = b = d_l^{(b,c)\pi}b$, thus

$$d_l^{(b,c)\pi} - d_l^{(b,c)\pi}a_l^{(b,c)\pi} = bxcd - bxcda_l^{(b,c)\pi} = bxcd - bxcd = 0.$$

Hence

$$d_l^{(b,c)\pi} = d_l^{(b,c)\pi}a_l^{(b,c)\pi}.$$

And

$$a_l^{(b,c)\pi} - d_l^{(b,c)\pi}a_l^{(b,c)\pi} = btca - d_l^{(b,c)\pi}btca = btca - btca = 0.$$

So

$$a_l^{(b,c)\pi} = d_l^{(b,c)\pi}a_l^{(b,c)\pi}.$$

Consequently, we get $a_l^{(b,c)\pi} = d_l^{(b,c)\pi}$.

Similarly we show that $d_r^{(b,c)\pi} = a_r^{(b,c)\pi}$. □

4. The (B,C)-polarity for bounded linear operators

Let X be a complex Banach space and $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators. Let $A \in \mathcal{B}(X)$, we will denote by $\mathcal{N}(A) = \{x \in X \ : \ Ax = 0\ \}$ the null space of A and by $\mathcal{R}(A) = \{Ax \ : \ x \in X\ \}$ the range of A and we write $I \in \mathcal{B}(X)$ for the identity operator.

Let A, B, $C \in \mathcal{B}(X)$, then A is (B,C)-polar if there exist two projections P and $Q \in \mathcal{B}(X)$ such that

(1) $P \in B\mathcal{B}(X)CA$;
(2) $Q \in AB\mathcal{B}(X)C$;
(3) $PB = B$, $CQ = C$;
(4) $CAP = CA$, $QAB = AB$.

A closed subspace M of X is a complemented subspace of X if there exits a closed subspace N of X such that $X = M \oplus N$.

Recall that an operator $A \in \mathcal{B}(X)$ is regular if and only if $\mathcal{R}(A)$ is closed and a complemented subspace of X and $\mathcal{N}(A)$ is a complemented subspace of X (see [13, Proposition 13.1]).

Theorem 4.1. *Let A, B, $C \in \mathcal{B}(X)$ such that B, C and CAB are regular. Then the following assertions are equivalent :*

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(1) *A is (B,C)-invertible;*
(2) *A is (B,C)-polar;*
(3) *There exist projections P, $Q \in \mathcal{B}(X)$ such that*
  i) $\mathcal{R}(P) = \mathcal{R}(B)$;
  ii) $\mathcal{N}(Q) = \mathcal{N}(C)$;
  iii) $\mathcal{R}(Q) = \mathcal{R}(AB)$;
  iv) $\mathcal{N}(P) = \mathcal{N}(CA)$.

Proof. (1) $\Leftrightarrow$ (2): By Theorem 2.6.
(2) $\Rightarrow$ (3):
i) From $P \in B\mathcal{B}(X)CA$ we have

$$\mathcal{R}(P) \subseteq \mathcal{R}(B).$$

Also from $PB = B$, we see $\mathcal{R}(B) \subseteq \mathcal{R}(P)$. Hence

$$\mathcal{R}(P) = \mathcal{R}(B).$$

ii) $CQ = C \Rightarrow \mathcal{N}(Q) \subseteq \mathcal{N}(C)$.
Now let $x \in X$ such that $Cx = 0$. Since $Q \in AB\mathcal{B}(X)C$, then $Qx = 0$ and so $\mathcal{N}(C) \subseteq \mathcal{N}(Q)$, thus

$$\mathcal{N}(Q) = \mathcal{N}(C).$$

iii) $Q \in AB\mathcal{B}(X)C \Rightarrow \mathcal{R}(Q) \subseteq \mathcal{R}(AB)$. Also $QAB = AB \Rightarrow \mathcal{R}(AB) \subseteq \mathcal{R}(Q)$.

$$\Rightarrow \mathcal{R}(Q) = \mathcal{R}(AB).$$

iv) $CAP = CA \Rightarrow \mathcal{N}(P) \subseteq \mathcal{N}(CA)$ and $P \in B\mathcal{B}(X)CA \Rightarrow \mathcal{N}(CA) \subseteq \mathcal{N}(P)$

$$\Rightarrow \mathcal{N}(P) = \mathcal{N}(CA).$$

(3) $\Rightarrow$ (1): To show that A is (B,C)-invertible, it suffices to prove that $\mathcal{N}(B) = \mathcal{N}(CAB)$ and $\mathcal{R}(C) = \mathcal{R}(CAB)$ by virtue of [2, Theorem 4.1].

At first we can see that $PB = B$ and $CQ = C$. Indeed, let $x \in X$. As $Bx \in \mathcal{R}(B) = \mathcal{R}(P)$ then $P(Bx) = Bx$ and so $PB = B$.
We have $Cx = C(x - Q(x)) + CQx) = 0 + CQx = CQx$ as $x - Qx \in \mathcal{N}(Q) = \mathcal{N}(C)$. Hence $C = CQ$.

Obviously we have $\mathcal{R}(CAB) \subseteq \mathcal{R}(C)$. Let $y \in \mathcal{R}(C)$ then there exists some $x \in X$ such that $y = Cx$. As $CQ = C$ we get $y = CQx$ and we can write $y = Cz$ for some $z = Qx \in \mathcal{R}(Q) = \mathcal{R}(AB)$, so $z = ABt$ for some $t \in X$. Thus we obtain $y = Cz = CABt$ which implies that $y \in \mathcal{R}(CAB)$. Hence $\mathcal{R}(C) \subseteq \mathcal{R}(CAB)$ and consequently

$$\mathcal{R}(C) = \mathcal{R}(CAB).$$

In the other side, we have obviously $\mathcal{N}(B) \subseteq \mathcal{N}(CAB)$. Suppose that $s \in \mathcal{N}(CAB)$, then $CABs = 0$, set $z = Bs$, so we get $CAz = 0$ which means that $z \in \mathcal{N}(CA) = \mathcal{N}(P)$, which implies $Pz = 0 = PBs$. Since $PB = B$, we obtain $Bs = 0$ which gives $s \in \mathcal{N}(B)$. Hence $\mathcal{N}(CAB) \subseteq \mathcal{N}(B)$, and we conclude that

$$\mathcal{N}(B) = \mathcal{N}(CAB).$$

Therefore A is (B,C)-invertible. □

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Remark 4.2. Assume that A is (B,C)-invertible. Then with respect to the decomposition $X=\mathcal{R}(P)\oplus\mathcal{N}(P)$ and $X=\mathcal{R}(Q)\oplus\mathcal{N}(Q)$ we have the following matrix representation of A:

$$A=\begin{bmatrix} QA & QA \\ (I-Q)A & (I-Q)A \end{bmatrix} : \begin{array}{ccc} \mathcal{R}(P)\oplus\mathcal{N}(P) & \longrightarrow & \mathcal{R}(Q)\oplus\mathcal{N}(Q) \\ x_1+x_2 & \longmapsto & A(x_1+x_2) \end{array}$$

Indeed, we respect to the Priece decomposition we have

$$A=\begin{bmatrix} QAP & QA(I-P) \\ (I-Q)AP & (I-Q)A(I-P) \end{bmatrix}.$$

Then for $x_1\in\mathcal{R}(P)$, $x_2\in\mathcal{N}(P)$ we have

- $\begin{array}{rcl} QAP:\ \mathcal{R}(P) & \to & \mathcal{R}(Q) \\ x_1 & \mapsto & QAPx_1=QAx_1; \end{array}$
- $\begin{array}{rcl} (I-Q)AP:\ \mathcal{R}(P) & \to & \mathcal{N}(Q) \\ x_1 & \mapsto & (I-Q)APx_1=(I-Q)Ax_1; \end{array}$
- $\begin{array}{rcl} QA(I-P):\ \mathcal{N}(P) & \to & \mathcal{R}(Q) \\ x_2 & \mapsto & QAx_2; \end{array}$
- $\begin{array}{rcl} (I-Q)A(I-P):\ \mathcal{N}(P) & \to & \mathcal{N}(Q) \\ x_2 & \mapsto & (I-Q)Ax_2. \end{array}$

Corollary 4.3. *Let A and $B\in\mathcal{B}(X)$ such that B is regular. Then the following assertions are equivalent:*

(1) *A is invertible along B;*
(2) *A is polar along B;*
(3) *There exists a projection $P\in\mathcal{B}(X)$ such that*
  i) $\mathcal{N}(P)=\mathcal{N}(BA)=\mathcal{N}(B)$;
  ii) $\mathcal{R}(P)=\mathcal{R}(AB)=\mathcal{R}(B)$.
(4) *$\mathcal{R}(B)$ is closed and a complemented subspace of X, $\mathcal{R}(AB)$ is closed such that $X=\mathcal{R}(AB)\oplus\mathcal{N}(B)$ and the operator $A|\mathcal{R}(B):\mathcal{R}(B)\to\mathcal{R}(AB)$ is invertible.*

Proof. The equivalence between 1), 2) and 3) follows from Theorem 4.1. The equivalence between 1) and 4) is [9, Theorem 2], however we can give another proof by showing that 2) or 3) is equivalent to 4).

Indeed, assume that A is polar along B, then by (3) $\mathcal{R}(B)=\mathcal{R}(P)$ which is closed and complemented in X since P is a bounded projection. Also $\mathcal{R}(AB)=\mathcal{R}(P)$ is closed and $X=\mathcal{R}(P)\oplus\mathcal{N}(P)=\mathcal{R}(AB)\oplus\mathcal{N}(B)$.

The operator $A|\mathcal{R}(B)$ is surjective by construction. So let $x\in\mathcal{R}(B)$ such that $ABx=0$, then $BABx=0$ and hence $Bx\in\mathcal{N}(BA)=\mathcal{N}(P)$ by (3). Thus $0=PBx=Bx$. Therefore, $A|\mathcal{R}(B)$ is injective.

Conversely, assume that (4) holds. Let P be the bounded projection onto $\mathcal{R}(AB)$. Let $x\in X$ with $x=x_1+x_2$ such that $x_1\in\mathcal{R}(AB)=\mathcal{R}(P)$ and

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$x_2 \in \mathcal{N}(B) = \mathcal{N}(P)$. Then

$$BPx = Bx_1 = B(x_1 + x_2) = Bx.$$

So $BP = B$. Also

$$ABPx = ABx = ABx_1 = PABx_1 = PABx.$$

Hence $P \in \mathrm{comm}(AB)$.

Now to deduce that A is dually polar along B, it remains to show that $AB+I-P$ is invertible. We have $(AB + I - P)x = ABx_1 + x_2$. Then if $(AB + I - P)x = 0$, we get $ABx_1 = 0$ and $x_2 = 0$. Since the operator $A|\mathcal{R}(B)$ is invertible, we deduce that $x_1 = 0$. Thus $AB + I - P$ is injective.

Let $y = y_1 + y_2$ such that $y_1 \in \mathcal{R}(AB)$ and $y_2 \in \mathcal{N}(B)$. Then $y_1 = ABx = ABx_1$ for some $x \in X$. Set $z = x_1 + y_2$, then $(AB + I - P)z = ABx_1 + y_2 = y$. Hence $AB + I - P$ is surjective. Therefore A is dually polar along B and so A is polar along B. □

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