Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5022.

THE ZERO FORCING NUMBER OF EXPANDED PATHS AND CYCLES

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Abstract. The zero forcing number is defined as the minimum size of a zero forcing set, and features as an upper bound for the graph nullity. An expanded path $P_{m_1,m_2,\dots,m_k}$ (resp. expanded cycle $C_{m_1,m_2,\dots,m_k}$) is obtained from the k-vertex path (cycle) by replacing the ith vertex with an independent set of m_i vertices. We prove that the zero forcing number of $P_{m_1,m_2,\dots,m_k}$ (resp. $C_{m_1,m_2,\dots,m_k}$) belongs to $\{n-k, n-k+1\}$ ($\{n-k+1, n-k+2\}$), where n is the number of vertices. It is also decided for which expanded paths and expanded cycles the zero forcing number is $n-k+1$. As an application, we offer a new proof of the result of Liang, Li and Xu that gives a characterization of triangle-free graphs with zero forcing number $n-3$. We also show that the zero forcing number of a cycle-spliced graph (i.e., a connected graph whose every block is a cycle) is $c+1$, where c is the cyclomatic number. This result induces an upper bound for the nullity of a cycle-spliced graph and extends the result of Wong, Zhou and Tian concerning the bipartite case.

1. Introduction

We consider finite, undirected graphs $G = G(V, E)$ without loops or multiple edges. The number of vertices n is called the *order*. Dynamic colourings of vertices in a graph have been well investigated in many branches. A particular colouring based on the concept of a zero forcing procedure, together with the related zero forcing number, was introduced in [1] to study the problem of the maximum nullity in the family of prescribed symmetric graph matrices. Independently, the zero forcing number was abbreviated the infection number in the framework of controllability of quantum systems [6]. As highlighted in [9, 12], the zero forcing routine exemplifies propagation processes on graphs and finds numerous applications across various fields including mathematics, computer science and physics. In particular, this invariant appears in the domain of logic circuits [4], dynamic systems [5] and power domination [16].

2020 *Mathematics Subject Classification.* 05C75, 05C50.

Key words and phrases. zero forcing set, graph nullity, expanded path, expanded cycle, cyclic-spliced graph.

This research is partially supported by (a) the Natural Science Foundation of Guangdong Province (No. 2022A1515011786 and 2024A1515011899), (b) the National Natural Science Foundation of China (No. 12271182), (c) the Guangdong Basic and Applied Basic Research Foundation (No. 2023A1515110456) and (d) the Science Fund of the Republic of Serbia; grant number 7749676: Spectrally Constrained Signed Graphs with Applications in Coding Theory and Control Theory – SCSG-ctct.

Submitted: October 26, 2024
Accepted: February 24, 2025
Published (early view): February 25, 2025

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The zero forcing is a deterministic iterative graph colouring procedure in which vertices are initially coloured either black or white. From the initial colouring, vertices change colour according to the following rule: If a black vertex u has a unique white neighbour v, then v switches the colour to black. We say that u *forces* v, and designate this by writing $u \to v$. The initial set of black vertices is denoted by S, and the *derived set* $\mathcal{F}(S)$ is the set of vertices coloured black after the colour-change rule is applied until no more changes are possible. The vertices of $\mathcal{F}(S) \setminus S$ are arranged as $u_1, u_2, \dots, u_k$, in such a way that for every i there is a vertex v_i in $S \cup \{u_j : 1 \leq j \leq i-1\}$ for which u_i is the unique neighbour of v_i in $V(G) \setminus (S \cup \{u_j : 1 \leq j \leq i-1\})$. The sequence $v_1 \to u_1, v_2 \to u_2, \dots, v_k \to u_k$ is called a *forcing sequence* for S. The set S itself is a *zero forcing set* in G provided $\mathcal{F}(S) = V(G)$. The *zero forcing number* $Z(G)$ is the minimum size of a zero forcing set in G.

The inequality

$$\eta(G) \leq Z(G) \tag{1}$$

relates the zero forcing number to a spectral invariant known as the graph nullity, that is the multiplicity of zero in the spectrum of the standard $\{0,1\}$-adjacency matrix.

The zero forcing number has been computed for many particular classes of graphs. For example, it is easily verified that $Z(P_n) = 1$ holds for every path P_n, with an endvertex in the role of S. Similarly, for every cycle C_n, we have $Z(C_n) = 2$, where the corresponding zero forcing set is comprised of any pair of adjacent vertices. Moreover, it follows from definition that $Z(G) = 1$ (resp. $Z(G) = n-1$) holds if and only if G is a path (complete graph with at least 2 vertices). The graphs with forcing number 2 or $n-2$ are also known, see [14]. In the same reference, Row proposed a problem of characterizing graphs with zero forcing number $n-3$. A partial answer is reported in [13] where the authors characterized all connected subcubic or triangle-free graphs with the desired forcing number.

To formulate our main results, we need to introduce certain graphs; the terminology is consistent with [10]. Let $P_k \coloneqq v_1 v_2 \cdots v_k$ be a path with vertices $v_1, v_2, \dots, v_k$ and edges $v_i v_{i+1}$ for $1 \leq i \leq k-1$. By replacing each vertex v_i with an edgeless graph of order m_i ($m_i \geq 1$), denoted by O_{m_i}, and adding edges between every vertex of O_{m_i} and every vertex of $O_{m_{i+1}}$, we obtain an *expanded path* denoted by $P_{m_1, m_2, \dots, m_k}$. In a similar way, from a cycle C_k we obtain an *expanded cycle* denoted by $C_{m_1, m_2, \dots, m_k}$. The expanded path and the expanded cycle have $\sum_{i=1}^{k} m_i$ vertices, each. In both cases, k is referred to as the *length* of the corresponding expanded graph.

In this paper we prove the following.

Theorem 1.1. *Let $G \cong P_{m_1, m_2, \dots, m_k}$ be an expanded path with n vertices and $k \geq 3$. Then*

$$n - k \leq Z(G) \leq n - k + 1,$$

where the first equality holds if and only if k is even and G has one of the following properties:

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Accepted: February 24, 2025
Published (early view): February 25, 2025

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(i) $m_i \geq 2$ *for all odd integers* $1 \leq i \leq k$;
(ii) $m_i \geq 2$ *for all even integers* $1 \leq i \leq k$;
(iii) $m_1, m_3, \dots, m_s, m_{s+3}, m_{s+5}, \dots, m_k \geq 2$, *for some odd integer* s $(1 \leq s \leq k-3)$.

The case $k = 2$ is excluded since there G reduces to a complete bipartite graph K_{m_1,m_2} with zero forcing number $m_1 + m_2 - 2$, unless $m_1 = m_2 = 1$ when it is 1 [1, 11].

Theorem 1.2. *Let* $G \cong C_{m_1,m_2,\dots,m_k}$ *be an expanded cycle with* n *vertices and* $k \geq 3$. *Then*

$$n - k + 1 \leq Z(G) \leq n - k + 2,$$

where the first equality holds if and only if k *is odd and* $m_i \geq 2$ *for all* i *with the same parity.*

On the basis of these results we offer an alternative proof of the result of [13] that gives a characterization of triangle-free graphs with zero forcing number $n-3$.

Corollary 1.3. [13] *Let* G *be a connected triangle-free graph with* n $(n \geq 4)$ *vertices. Then* $Z(G) = n-3$ *holds if and only if* $G \in \{P_{1,m_1,m_2,1}, P_{1,1,m_1,m_2}, C_{1,1,1,m_1,m_2} : m_1, m_2 \geq 1\}$.

A *block* in a connected graph is a maximal connected subgraph with no cut-vertex. Accordingly, a *cycle-spliced graph* is a connected graph whose every block is a cycle. It can also be seen as a cactus in which every block is a cycle. We write $c(G) = |E(G)| - n + 1$ for the *cyclomatic number* of a connected graph G.

Theorem 1.4. *For a cycle-spliced graph* G *with cyclomatic number* $c(G)$, $Z(G) = c(G) + 1$ *holds.*

Taking into account that the nullity of G is never larger that $Z(G)$, we immediately arrive at the following consequence.

Corollary 1.5. *For a cycle-spliced graph* G *with nullity* $\eta(G)$ *and cyclomatic number* $c(G)$, $\eta(G) \leq c(G) + 1$ *holds.*

This corollary extends a result of [15] where the same inequality is proved for bipartite cycle-spliced graphs.

For undefined notions, we refer the reader to [2, 3]. Section 2 can be seen as a preparatory containing a mixture of known results and several simple but useful lemmas. The proofs of Theorems 1.1, 1.2, 1.4 and Corollary 1.3 are given in Section 3.

2. Preliminaries

We start with a lemma concerning the zero forcing number of a graph containing a fixed path as an induced subgraph.

Lemma 2.1. *Let* G *be a graph of order* n *containing an induced path of length* k. *Then* $Z(G) \leq n - k$.

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Accepted: February 24, 2025
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Proof. Suppose that $P = v_0v_1 \cdots v_k$ is an induced path of G, and let $S = (V(G) \setminus V(P)) \cup \{v_0\}$. Then $v_0 \to v_1, v_1 \to v_2, \ldots, v_{k-1} \to v_k$ is a forcing sequence for S. Thus, we have $\mathcal{F}(S) = V(G)$, which implies that S is a zero forcing set of G. From $|S| = n - k$, we get $Z(G) \leq |S| = n - k$. □

We proceed with an induced cycle.

Lemma 2.2. *Let G be a graph of order n containing an induced cycle of length k. Then $Z(G) \leq n - k + 2$.*

Proof. If $C = v_1v_2 \cdots v_kv_1$ is an induced cycle of G, we set $S = (V(G) \setminus V(C)) \cup \{v_1, v_2\}$. As in the previous proof, $v_2 \to v_3, v_3 \to v_4, \ldots, v_{k-1} \to v_k$ is a forcing sequence for S, and the desired result follows. □

We now quote the two known results.

Lemma 2.3. [10] *Let G be an expanded path of order n and length k $(k \geq 2)$. For the nullity $\eta(G)$, we have*

$$\eta(G) = \begin{cases} n-k & \text{if } k \text{ is even}, \\ n-k+1 & \text{if } k \text{ is odd}. \end{cases}$$

Lemma 2.4. [10] *Let G be an expanded cycle C of order n and length k $(k \geq 3)$. For the nullity $\eta(G)$, we have*

$$\eta(G) = \begin{cases} n-k+2 & \text{if } k \equiv 0 \pmod 4, \\ n-k & \text{otherwise}. \end{cases}$$

Henceforth, $N_G(u)$ denotes the neighbourhood of a vertex u in a graph G.

Lemma 2.5. *Let S be a zero forcing set of a graph G. If $|V(G) \setminus S| \geq 2$, then $N_G(u) \neq N_G(v)$ holds for every pair $u, v \in V(G) \setminus S$.*

Proof. Suppose that there is a pair of vertices $u, v \in V(G) \setminus S$, such that $N_G(u) = N_G(v)$. Then for $w \in V(G) \setminus \{u, v\}$, we have either $\{u, v\} \subseteq N_G(w)$ or $\{u, v\} \cap N_G(w) = \emptyset$. This implies that there is no vertex that would force u or v. Consequently, $\mathcal{F}(S) \neq V(G)$, which means that S is not a zero forcing set. □

Remark 2.6. On the basis of the previous lemmas, one may deduce several known results on the zero forcing number of connected graphs.

(a) Lemma 2.1 implies $Z(G) \leq n - \mathrm{diam}(G)$, where diam stands for the diameter. (This result originates from [14].)
(b) Lemma 2.2 implies $Z(G) \leq n - \mathrm{gr}(G) + 2$, where gr is the girth and G is not a tree. (See [8].)
(c) From (a), (b) and the inequality (1), we deduce $\eta(G) \leq n - \mathrm{diam}(G)$ and $\eta(G) \leq n - \mathrm{gr}(G) + 2$. (See [7].)

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3. Proofs

We go straight to the proof of the first result.

Proof of Theorem 1.1. It follows from definition that $G \cong P_{m_1,m_2,\dots,m_k}$ contains an induced path of length $k-1$. By Lemma 2.1, we have $Z(G) \le n-k+1$. This, together with Lemma 2.3 and the inequality (1), yields

$$\begin{cases} Z(G) = n-k+1 & \text{if } k \text{ is odd,} \\ n-k \le Z(G) \le n-k+1 & \text{if } k \text{ is even.} \end{cases}$$

This proves the desired inequalities, and it remains to consider the equality cases.

Suppose that $Z(G) = n-k$. By the previous part of this proof, k must be even. We next show that one of the conditions (i), (ii) or (iii) holds. Let S be a zero forcing set in G with $|S| = n-k$, and let $V(G) \setminus S = \{v_1, v_2, \dots, v_k\}$. Observe that for every i $(1 \le i \le k)$, the vertices in $V(O_{m_i}) \setminus S$ share the same neighbourhood in G. By employing Lemma 2.5, we obtain $0 \le |V(O_{m_i}) \setminus S| \le 1$, for every i. Since $Z(G) = n-k$, we get $|V(O_{m_i}) \setminus S| = 1$. Without loss of generality, we assume that $v_i \in V(O_{m_i}) \setminus Z$ for $1 \le i \le k$, and denote by u_i a vertex in $V(O_{m_i}) \cap S$ when $m_i \ge 2$.

The desired conclusion follows whenever $m_i \ge 2$ holds for either all odd i or all even i. Therefore, we suppose that ℓ is the smallest odd integer and t is the largest even integer, such that $m_\ell = m_t = 1$. By the choice of ℓ and t, we have $m_1, m_3, \dots, m_{\ell-2} \ge 2$ and $m_{t+2}, m_{t+4}, \dots, m_k \ge 2$. Thus, we have $u_i \to v_{i+1}$ for $i \in \{1, 3, \dots, \ell-2\}$ and $u_j \to v_{j-1}$ for $j \in \{k, k-2, \dots, t+2\}$. Let $S_1 = S \cup \{v_2, v_4, \dots, v_{\ell-1}\} \cup \{v_{t+1}, v_{t+3}, \dots, v_{k-1}\}$. This yields $S_1 \subseteq \mathcal{F}(S)$. If $t > \ell$, then every vertex of S_1 has at least two neighbours outside S_1, which implies $S_1 = \mathcal{F}(S)$. However, this contradicts $\mathcal{F}(S) = V(G)$, because $v_\ell, v_t \notin S_1$. Hence, $t < \ell$. Again, by the choice of ℓ and t, we have $3 \le \ell \le k-1$ and $2 \le t \le k-2$. Then $m_1, m_3, \dots, m_{\ell-2} \ge 2$ and $m_{\ell+1}, m_{\ell+3}, \dots, m_k \ge 2$, and so (iii) holds by setting $s = \ell - 2$.

Conversely, suppose that one of the conditions (i), (ii) and (iii) holds. Note that (i) is equivalent to (ii) by reordering $m_i = m_{k-i+1}$, for every i $(1 \le i \le k)$. Hence, it is sufficient to consider the case in which one of (i) or (iii) holds. We conclude the proof by explicit constructions of the corresponding forcing sequences.

If the condition (i) holds (i.e., $m_i \ge 2$ for all odd i $(1 \le i \le k)$), then on the basis of the assumptions on u_i and v_i, we obtain a forcing sequence

$$u_1 \to v_2, u_3 \to v_4, \dots, u_{k-1} \to v_k, v_k \to v_{k-1}, v_{k-2} \to v_{k-3}, \dots, v_2 \to v_1.$$

This implies that S is a zero forcing set in G, which gives $Z(G) = n-k$.

If (iii) holds, then we obtain a forcing sequence

$$u_1 \to v_2, u_3 \to v_4, \dots, u_s \to v_{s+1}, u_k \to v_{k-1}, u_{k-2} \to v_{k-3}, \dots, u_{s+3} \to v_{s+2},$$
$$v_{s+1} \to v_s, v_{s-1} \to v_{s-2}, \dots, v_2 \to v_1, v_{s+2} \to v_{s+3}, v_{s+4} \to v_{s+5}, \dots, v_{k-1} \to v_k.$$

This implies that S is a zero forcing set in G, along with $Z(G) = n-k$. □

We proceed with the next proof.

Submitted: October 26, 2024
Accepted: February 24, 2025
Published (early view): February 25, 2025

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Proof of Theorem 1.2. An expanded cycle $G \cong C_{m_1,m_2,\dots,m_k}$ contains an induced cycle of length k. By Lemma 2.2, we have $Z(G) \leq n-k+2$. This, together with Lemma 2.4 and (1), immediately gives $n-k \leq Z(G) \leq n-k+2$.

In what follows, we improve the lower bound on $Z(G)$ to the bound given in the formulation of the theorem. Let S be a zero forcing set in G with $|S| = Z(G)$. Observe that for i ($1 \leq i \leq k$), the vertices in $V(O_{m_i}) \setminus S$ share the same neighbourhood in G. Using this, by Lemma 2.5, we obtain $0 \leq |V(O_{m_i}) \setminus S| \leq 1$ for each i. Now, assuming that $Z(G) = n-k$, we obtain $|V(O_{m_i}) \setminus S| = 1$ for each i. Thus, every vertex in S has exactly two neighbours outside S, which implies $S = \mathcal{F}(Z)$. From $S \subsetneqq V(G)$, we have $\mathcal{F}(S) \subsetneqq V(G)$, contradicting $\mathcal{F}(S) = V(G)$. Hence, $n-k+1 \leq Z(G)$, as desired.

Therefore, there are exactly two possibilities for $Z(G)$, and we need to decide whether each of them occurs.

Suppose that $Z(G) = n-k+1$, and let $V(G) \setminus S = \{v_1, v_2, \dots, v_{k-1}\}$. The inequality $|V(O_{m_i}) \setminus Z| \leq 1$ (for each i), allows us to assume that $u_i \in V(O_{m_i}) \cap S$ (if $m_i \geq 2$) and $v_i \in V(O_{m_i}) \setminus S$ for $1 \leq i \leq k-1$.

We first show that k must be odd. Namely, if k is even and $m_i \geq 2$ holds for each odd i ($1 \leq i \leq k$), then by applying the forces $u_i \to v_{i+1}$ for $i \in \{1, 3, \dots, k-3\}$, we obtain $\mathcal{F}(S) = S \cup \{v_2, v_4, \dots, v_{k-2}\}$, but this contradicts $\mathcal{F}(S) = V(G)$. Hence, the set $I = \{i : m_i = 1, i \text{ is odd}\}$ is non-empty. By fixing ℓ and t to be the smallest and the largest element of I, we obtain $\mathcal{F}(S) = S \cup \{v_2, v_4, \dots, v_{\ell-1}\} \cup \{v_{k-2}, v_{k-4}, \dots, v_{t+1}\}$. In particular, $v_1 \notin \mathcal{F}(S)$, which is impossible. Hence, k is odd.

In what follows, we assume that there exist an odd i and an even j, such that $m_i = m_j = 1$. We can do this because if this assumption does not hold, then we would have $m_i \geq 2$ for either all even i or all odd i, which is a conclusion given the formulation of the theorem.

As before, let ℓ be the smallest odd integer and t the largest even integer, such that $m_\ell = m_t = 1$. This implies $m_1, m_3, \dots, m_{\ell-2} \geq 2$ and $m_{t+2}, m_{t+4}, \dots, m_{k-1} \geq 2$. Thus, we have $u_i \to v_{i+1}$ for every $i \in \{1, 3, \dots, \ell-2\}$ and $u_j \to v_{j-1}$ for every $j \in \{k-1, k-3, \dots, t+2\}$. Denote $S_1 = S \cup \{v_2, v_4, \dots, v_{\ell-1}\} \cup \{v_{t+1}, v_{t+3}, \dots, v_{k-2}\}$. It holds $S_1 \subseteq \mathcal{F}(S)$.

If $t > \ell$, then every vertex of S_1 has at least two neighbours outside S_1, which implies $S_1 = \mathcal{F}(S)$. However, this contradicts $\mathcal{F}(S) = V(G)$, because $v_\ell, v_t \notin S_1$. Therefore, we have $t < \ell$ and, by the choice of ℓ and t, $3 \leq \ell \leq k-2$ and $2 \leq t \leq k-3$, giving $m_1, m_3, \dots, m_{\ell-2} \geq 2$ and $m_{\ell+1}, m_{\ell+3}, \dots, m_{k-1} \geq 2$. By reordering the indices from $\{1, 2, \dots, k\}$ to $\{\ell, \ell+1, \dots, k, 1, 2, \dots, k-1\}$, we obtain $m_i \geq 2$ for each even i, which concludes this part of the proof.

Suppose now that k is odd and $m_i \geq 2$ holds for each i. By taking $v_i \in V(O_{m_i})$, for $1 \leq i \leq k-1$, we create the set $S_0 = V(G) \setminus \{v_1, v_2, \dots, v_{k-1}\}$. For each even i, we may also take $u_i \in V(O_{m_i}) \cap S, u_i \neq v_i$, as we have $m_i \geq 2$. This leads to the forcing sequence

$$u_{k-1} \to v_{k-2}, u_{k-3} \to v_{k-4}, \dots, u_2 \to v_1, v_1 \to v_2, v_3 \to v_4, \dots, v_{k-2} \to v_{k-1}.$$

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Therefore, S_0 is a zero forcing set in G, and so $Z(G) \leq n-k+1$. Together with already obtained $n-k+1 \leq Z(G)$, this leads to $Z(G) = n-k+1$, and we are done. □

We now prove a corollary, actually a known result. For a graph G, we say that a vertex dominates a subset of $V(G)$ if it is adjacent to every vertex of this subset.

Proof of Corollary 1.3 If $G \in \{P_{1,m_1,m_2,1}, P_{1,1,m_1,m_2}, C_{1,1,1,m_1,m_2} : m_1, m_2 \geq 1\}$, Then $Z(G) = n-3$ by Theorems 1.1 and 1.2.

In the remainder of the proof, we assume that $Z(G) = n-3$. Lemma 2.1 ensures that G is P_5-free. Together with the assumption that G is triangle-free, this yields that G is either bipartite or contains C_5 as an induced subgraph.

Case 1: G is bipartite. This immediately implies that G is not $C_{1,1,1,m_1,m_2}$. Suppose that

$$G \notin \{P_{1,m_1,m_2,1}, P_{1,1,m_1,m_2} : m_1, m_2 \geq 1\}, \tag{2}$$

and let X and Y be the colour classes of G. Also, let $X' \subseteq X$ be the set of vertices that do not dominate Y and $Y' \subseteq Y$ the set of vertices do not dominate X.

Denote $s_1 = |X'|$, $s_2 = |Y'|$, $t_1 = |X \setminus X'|$ and $t_2 = |Y \setminus Y'|$. If either $s_1 = 0$ or $s_2 = 0$, then G is a complete bipartite graph K_{t_1,t_2} with $Z(G) \neq n-3$, which contradicts the initial assumption. Hence, $s_1, s_2 \geq 1$. However, we claim a stronger restriction: $s_1, s_2 \geq 2$. Namely, if $s_1 = 1$, then $G \cong P_{1,t_2,t_1,s_2}$ by definition of Y'. Together with (2), this leads to $s_2 \geq 2$ and $t_2 \geq 2$. By Theorem 1.1, we have $Z(G) = Z(P_{1,t_2,t_1,s_2}) = n-4$, which is impossible. The case $s_2 = 1$ is symmetric, and so $s_1, s_2 \geq 2$, as claimed.

Let $X' = \{x_1, x_2, \dots, x_{s_1}\}$ and $Y(x_i) = Y \setminus N(x_i)$, for $1 \leq i \leq s_1$. By definition of X', we have $Y(x_i) \neq \emptyset$. Since G is connected, we also have $Y(x_i) \neq Y$. If $Y(x_1) = Y(x_2) = \cdots = Y(x_{s_1})$, then $Y' = Y(x_1)$, and so $G \cong P_{s_1,t_2,t_1,s_2}$ (with $s_1, s_2 \geq 2$). By employing Theorem 1.1, we obtain $Z(G) = n-4$, which is a contradiction. Thus there exist two vertices in X', say x_1 and x_2, such that $Y(x_1) \neq Y(x_2)$. If $Y(x_1) \nsubseteq Y(x_2)$ and $Y(x_2) \nsubseteq Y(x_1)$, then there are two vertices $y_1, y_2 \in Y$, such that $y_1 \in N(x_1) \setminus N(x_2)$ and $y_2 \in N(x_2) \setminus N(x_1)$. This tells us that G contains an induced path $P_k, k \geq 5$, with $x_1, x_2, y_1, y_2 \in V(P_k)$. By Lemma 2.1, we have $Z(G) \leq n-4$, a contradiction as before.

By symmetry, we may assume that $\emptyset \neq Y(x_1) \subset Y(x_2) \subset Y$. Let $y_1 \in Y(x_1)$, $y_2 \in Y(x_2) \setminus Y(x_1) = N(x_1) \setminus N(x_2)$ and $y_3 \in Y \setminus Y(x_2) = N(x_2)$. Since G is connected, there exists $x_3 \in X \setminus \{x_1, x_2\}$, such that x_3 is adjacent to y_1. We set $S = V(G) \setminus \{x_1, x_3, y_2, y_3\}$. Note that the set $\{x_1, x_2, y_1, y_2\}$ induces the edge x_1y_2. By applying the forces

$$y_1 \to x_3, x_2 \to y_3, y_3 \to x_1, x_1 \to y_2,$$

we obtain that S features as a zero forcing set in G, and so $Z(G) \leq n-4$. This contradiction denies (2) and concludes this case.

Case 2: G contains C_5. In this case (2) holds true, and we need to show that G is isomorphic to $C_{1,1,1,m_1,m_2}$. Let C be an induced cycle of G with $V(C) = \{v_1, v_2, \dots, v_5\}$ and $S_i(C) = \{x \in V(G) \setminus V(C) : N_G(x) \cap V(C) = N_G(v_i) \cap V(C)\}$, for $1 \leq i \leq 5$.

Submitted: October 26, 2024
Accepted: February 24, 2025
Published (early view): February 25, 2025

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5022.

We claim that

$$V(G) \setminus V(C) = \bigcup_{i=1}^{5} S_i(C). \tag{3}$$

Indeed, by taking $x \in V(G) \setminus V(C)$, since G is triangle-free and P_5-free, we deduce that x is adjacent to exactly two non-adjacent vertices v_j, v_k of C, that is $x \in S_i(C)$, where v_i is the common neighbour of v_j and v_k in C.

We proceed with the following structural examinations. First, since G is triangle-free, every $S_i(C)$ is an independent set. Secondly, by taking any two vertices $u_i \in S_i(C)$ and $u_j \in S_j(C)$, for $i \neq j$, we arrive at the following implications: $v_iv_j \in E(G) \implies u_iu_j \in E(G)$ (otherwise, $\{s_i, s_j\} \cup V(C) \setminus \{v_i, v_j\}$ induces P_5) and $v_iv_j \notin E(G) \implies u_iu_j \notin E(G)$ (otherwise, there exists a triangle $C_3 : v_ks_is_jv_k$, where v_k is the unique common neighbour of v_i and v_j in C).

Now, the previous implications, together with (3) and the fact that $S_i(C)$ is an independent set for each i, give the structure of G, i.e., lead to the conclusion that G is an expanded cycle. By Theorem 1.2, we have $G \cong C_{1,1,1,m_1,m_2}$. □

It remains to prove the result on cycle-spliced graphs.

Proof of Theorem 1.4 We use the induction on $c(G)$. If $c(G) = 1$, then G is a cycle, and so $Z(G) = 2 = c(G) + 1$. Next, we set $c = c(G) \geq 2$ and suppose that the result holds for every cycle-spliced graph with cyclomatic number $c - 1$. Consider a pendant block C of G, which contains exactly one cut-vertex u_1. Let G_1 be a subgraph of G induced by $(G \backslash V(C)) \cup \{u_1\}$, and let $C = u_1u_2 \cdots u_ru_1$.

It holds $c(G_1) = c - 1$ and, by the induction hypothesis, there exists a zero forcing set S_1 of G_1 with $|S_1| = c$. Then $S = S_1 \cup \{u_2\}$ is a zero forcing set of G, which implies $Z(G) \leq c + 1$.

We next show that $Z(G) \geq c + 1$. Let S_2 be a minimum zero forcing set of G. Since $N_G(u_1) \cap V(C) = \{u_2, u_r\}$, we have $1 \leq |S_2 \cap V(C)| \leq 2$.

If $|S_2 \cap V(C)| = 1$, then $u_1 \notin S_2$, and so $S_2 \setminus V(C)$ is a zero forcing set of G_1. Then we have $c = Z(G_1) \leq |S_2| - 1$, that is $Z(G) \geq c + 1$.

If $|S_2 \cap V(C)| = 2$, then $(S_2 \setminus V(C)) \cup \{u_1\}$ is a zero forcing set of G_1. Hence, $c = Z(G_1) \leq |S_2| - 1$, which implies $Z(G) \geq c + 1$. □

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Submitted: October 26, 2024
Accepted: February 24, 2025
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Revista de la Unión Matemática Argentina **Accepted article · Early view version**

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Submitted: October 26, 2024
Accepted: February 24, 2025
Published (early view): February 25, 2025