Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

LEFT AND RIGHT W-WEIGHTED G-DRAZIN INVERSES AND NEW MATRIX PARTIAL ORDERS

D. E. FERREYRA, D. MOSIĆ, A. N. PRIORI, AND M. L. LLANES

Abstract. This paper investigates a way to define left and right versions of the class of G-Drazin inverses for complex rectangular matrices. More precisely, the concepts of W-weighted left (resp. right) G-Drazin inverses are introduced and characterized by means of a simultaneous core-nilpotent decomposition as well as by certain system of matrix equations. Then, new partial orders associated with these weighted generalized inverses are presented and studied.

1. Introduction

The set of complex rectangular matrices of size $m\times n$ will be denoted by $\mathbb{C}^{m\times n}$. For $A\in\mathbb{C}^{m\times n}$, the symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\operatorname{rank}(A)$, and $\operatorname{Ind}(A)$ denote the column space, the null space, the rank, and the index ($m=n$) of A, respectively.

By considering a fixed nonzero complex matrix W of size $n\times m$ and an arbitrary complex matrix A of size $m\times n$, Cline and Greville [3] showed the existence and uniqueness of a solution X to the system of equations

$$XWAWX=X,\quad AWX=XWA,\quad XW(AW)^{k+1}=(AW)^k,$$

for some integer positive k.

This solution is called the W-weighted Drazin inverse of A and is written as $A^{d,W}$. When $m=n$ and $W=I_n$, we recover the classical Drazin inverse, that is, $A^{d,I_n}=A^d$.

It is well known that $A^{d,W}=A[(WA)^d]^2=[(AW)^d]^2A$, $WA^{d,W}=(WA)^d$, and $A^{d,W}W=(AW)^d$.

Using the W-weighted Drazin inverse, the weighted Drazin pre-order [7] was defined as follows

$$A\preceq^{d,W}B \;\Leftrightarrow\; AWA^{d,W}=BWA^{d,W} \text{ and } A^{d,W}WA=A^{d,W}WB.$$

2020 *Mathematics Subject Classification.* 15A09, 06A06, 15A24

Key words and phrases. Weighted generalized inverses, Partial orders, W-weighted Drazin inverse, W-weighted G-Drazin inverse, Drazin pre-order.

The research has been supported by Universidad Nacional de Río Cuarto (Grant PPI 18/C634), by CONICET (Grant PIBAA 28720210100658CO), and by Universidad Nacional de La Pampa, Facultad de Ingeniería (Grant RCD 172/24). The second author is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, Grant 451-03-65/2024-03/200124.

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

Note that $\preceq^{d,W}$ is an extension to the rectangular case of the pre-order $\preceq^{d}$ defined and studied in depth by Mitra, Bhimasankaram and Malik in [9]. More interesting results related to pre-orders introduced by the W-weighted Drazin inverse can be found in [8, 12, 14].

Recently, Wang and Liu [18] studied the class of G-Drazin inverses of a square matrix. Recall that X is an inner inverse of $A \in \mathbb{C}^{m\times n}$ if it satisfies $AXA = A$ [1].

Definition 1.1. [18] Let $A \in \mathbb{C}^{n\times n}$ and $k = \text{Ind}(A)$. Then a matrix X is called a G-Drazin inverse of A if X is an inner inverse of A such that $XA^{k+1} = A^k$ and $A^{k+1}X = A^k$.

The authors observed that, in general, the G-Drazin inverse, is not unique. Thus, the class of G-Drazin inverses of A is denoted by $A\{GD\}$. Clearly, $A\{GD\} \subseteq A\{1\} = \{A^- \in \mathbb{C}^{n\times m} : AA^-A = A\}$. Using the G-Drazin inverses, the authors also introduced a new partial order on the set of square matrices called the G-Drazin partial order.

Definition 1.2. [18] Let $A, B \in \mathbb{C}^{n\times n}$. The matrix A is said to be below the matrix B under the G-Drazin partial order if there exist $X_1, X_2 \in A\{GD\}$ such that $X_1A = X_1B$ and $AX_2 = BX_2$, and it is denoted by $A \leq^{GD} B$.

The notions of left and right invertibility as extensions of invertibility, motivated many authors to investigate similar problem for various kinds of generalized invertibility. As weaker versions of the G-Drazin inverse, the concepts of left and right G-Drazin inverses were proposed in [16].

Definition 1.3. [16] Let $A \in \mathbb{C}^{n\times n}$ and $k = \text{Ind}(A)$. Then a matrix X is called a left (resp. right) G-Drazin inverse of A if X is an inner inverse of A such that $XA^{k+1} = A^k$ (resp. $A^{k+1}X = A^k$).

The sets of all left (or right) G-Drazin inverse of A will be denoted by $A\{l, GD\}$ ($A\{r, GD\}$). Replacing $X_1, X_2 \in A\{GD\}$ in Definition 1.2 with $X_1, X_2 \in A\{l, GD\}$ (or $X_1, X_2 \in A\{r, GD\}$), a left (or right) G-Drazin partial order was defined in [16] on the set of square matrices. More details related to the G-Drazin inverses can be found in [5, 6, 10, 11].

Later, Coll et al. [4] extended the G-Drazin inverses to the rectangular case using an appropriate weighted W in Definition 1.1. More precisely, a matrix $X \in \mathbb{C}^{m\times n}$ is a W-weighted G-Drazin inverse of $A \in \mathbb{C}^{m\times n}$ if satisfies the three equations

$$WAWXWAW = WAW, \quad (AW)^{k+1}XW = (AW)^k, \quad WX(WA)^{k+1} = (WA)^k, \tag{1.1}$$

where $k = \max\{\text{Ind}(AW), \text{Ind}(WA)\}$. The set of W-weighted G-Drazin inverses of A is denoted by $A\{W - GD\}$. Moreover, the following binary relation on $\mathbb{C}^{m\times n}$ was considered in order to extend Definition 1.2

$$A \preceq^{GD}_{W} B \;\Leftrightarrow\; WAWX_1 = WBWX_1 \text{ and } X_2WAW = X_2WBW, \tag{1.2}$$

for some $X_1, X_2 \in A\{GD - W\}$.
The relation $\preceq^{GD}_{W}$ is a pre-order on $\mathbb{C}^{m\times n}$ however is not a partial order because it is not antisymmetric. The concept of the W-weighted G-Drazin inverse was studied

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

for operators between two Banach spaces in [13], and the definition of the strong W-weighted G-Drazin inverse was given in [15]. In particular, a matrix $X \in \mathbb{C}^{m\times n}$ is a strong W-weighted G-Drazin inverse of $A \in \mathbb{C}^{m\times n}$ if it is a solution to the system of two equations

$$AWXWA = A, \quad (WA)^d WAWXW = WXW(AW)^d AW.$$

The definitions of left and right G-Drazin inverses as well as the W-weighted G-Drazin inverse and the strong W-weighted G-Drazin inverse, motivated us to consider weighted versions of left and right G-Drazin inverses. The main goal of this article is to present a variant of the system (1.1) and therefore an alternative binary relation to the one given in (1.2) in order to obtain a weighted partial order on the set of complex rectangular matrices. Precisely, as solutions of corresponding systems of two equations, we introduce the W-weighted left (resp. right) G-Drazin inverse for rectangular matrices. Also, the concept of the W-weighted left-right G-Drazin inverse is given as a both left and right G-Drazin inverse. Since left (or right) G-Drazin inverses and G-Drazin inverses are particular cases of our new inverses, we define three new wider classes of already existing generalized inverses. Especially, these new inverses can be applied to propose three new kinds of partial orders on the set of rectangular matrices. As we previously mentioned, based on weighted generalized inverses many pre-orders were defined, but we present partial orders using the W-weighted left (or right) G-Drazin inverse and the W-weighted left-right G-Drazin inverse.

The structure of this article is organized as follows. In Section 2, we introduce and characterize three new type of generalized inverses for rectangular matrices, the W-weighted left (resp. right) G-Drazin inverses and the W-weighted left-right G-Drazin inverses. Using a simultaneous decomposition, we obtain a canonical form for these generalized inverses. In Section 3, parameterized representations of W-weighted left (resp. right) G-Drazin inverses are obtained. Section 4 is developed to study new binary relations involving the W-weighted left (resp. right) G-Drazin inverses which result partial orders on the set of rectangular matrices.

Throughout this paper, we will fix a nonzero matrix $W \in \mathbb{C}^{n\times m}$ and utilize it as a weight.

2. Weighted left and right G-Drazin inverses

We start with the following definitions.

Definition 2.1. Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. A matrix $X \in \mathbb{C}^{m\times n}$ is

(i) a W-weighted left G-Drazin inverse of A if the following equalities hold:

$$AWXWA = A, \quad XW(AW)^{k+1} = (AW)^k;$$

(ii) a W-weighted right G-Drazin inverse of A if the following equalities hold:

$$AWXWA = A, \quad (WA)^{k+1}WX = (WA)^k.$$

(iii) a W-weighted left-right G-Drazin inverse if X is both W-weighted left and right G-Drazin inverse of A.

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

Denoted by $A\{l, W-GD\}$, $A\{r, W-GD\}$, and $A\{l, r, W-GD\}$ the sets of all W-weighted left, right, and left-right G-Drazin inverses of A, respectively.

Remark 2.2. When $m = n$ and $W = I_n$, then Definition 2.1 recovers the concepts of left and right G-Drazin inverses [16, Definition 3.1] and G-Drazin inverses [18, Definition 3.1].

Necessary and sufficient conditions for a matrix to be a W-weighted left G-Drazin inverse are investigate below.

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the following conditions are equivalent:*
(a) $X \in A\{l, W-GD\}$;
(b) $AWXWA = A$ and $XWAW(AW)^d = (AW)^d$;
(c) $AWXWA = A$ and $XWAW(AW)^d AW = (AW)^d AW$;
(d) $AWXWA = A$ and $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(XWAW)$.

Proof. $(a) \Rightarrow (b)$ It is sufficient to prove the second condition. In fact, from $XW(AW)^{k+1} = (AW)^k$, we have $XW(AW)^{k+1}[(AW)^d]^{k+1} = (AW)^k[(AW)^d]^{k+1}$. As $AW(AW)^d$ is idempotent, we get $XWAW(AW)^d = (AW)^d$.
$(b) \Rightarrow (c)$ Trivial.
$(c) \Rightarrow (a)$ Multiplying $XWAW(AW)^d AW = (AW)^d AW$ from the right side by $(AW)^d(AW)^{k+1}$, we obtain $XW(AW)^{k+1} = (AW)^k$. Since $AWXWA = A$, we deduce that $X \in A\{l, W-GD\}$.
$(b) \Leftrightarrow (d)$ Note that $AWXWA = A$ implies that $XWAW$ is idempotent. Thus, $XWAW(AW)^d = (AW)^d$ is equivalent to $\mathcal{R}((AW)^k) = \mathcal{R}((AW)^d) \subseteq \mathcal{N}(I_m - XWAW) = \mathcal{R}(XWAW)$. □

In a similar manner as the above theorem, we can characterize the W-weighted right G-Drazin inverses.

Theorem 2.4. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the following conditions are equivalent:*
(a) $X \in A\{r, W-GD\}$;
(b) $AWXWA = A$ and $(WA)^d WAWX = (WA)^d$;
(c) $AWXWA = A$ and $WA(WA)^d WAWX = WA(WA)^d$;
(d) $AWXWA = A$ and $\mathcal{N}(WAWX) \subseteq \mathcal{N}((WA)^k)$.

As a consequence of Theorem 2.3 and Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the following conditions are equivalent:*
(a) $X \in A\{l, r, W-GD\}$;
(b) $AWXWA = A$, $XWAW(AW)^d = (AW)^d$, and $(WA)^d WAWX = (WA)^d$;
(c) $AWXWA = A$, $XWAW(AW)^d AW = (AW)^d AW$, and $WA(WA)^d WAWX = WA(WA)^d$;
(d) $AWXWA = A$, $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(XWAW)$, and $\mathcal{N}(WAWX) \subseteq \mathcal{R}((WA)^k)$.

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

In order to obtain more properties and representations of $A^{d,W}$, the following simultaneous decomposition of A and W (called weighted core-nilpotent decomposition) was introduced by Wei [17]

$$A = P \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} Q^{-1} \quad \text{and} \quad W = Q \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} P^{-1}, \tag{2.1}$$

where $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, $A_1, W_1 \in \mathbb{C}^{t\times t}$ are nonsingular matrices, $A_2 \in \mathbb{C}^{(m-t)\times(n-t)}$ and $W_2 \in \mathbb{C}^{(n-t)\times(m-t)}$ are rectangular matrices such that A_2W_2 and W_2A_2 are nilpotent of indices $\operatorname{Ind}(AW)$ and $\operatorname{Ind}(WA)$, respectively. In particular, if $m = n$ and $AW = WA$ then $Q = P$. In this case, if $W = I_n$, then $W_1 = I_t$ and $W_2 = I_{n-t}$.

Now, we provide a characterization of a W-weighted left G-Drazin inverse by means of the weighted core-nilpotent decomposition.

Theorem 2.6. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\operatorname{Ind}(AW), \operatorname{Ind}(WA)\}$. If A and W are written as in* (2.1)*, then $X \in A\{l, W-GD\}$ if and only if*

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} Q^{-1}, \tag{2.2}$$

where $X_3W_2A_2 = 0$ and $W_2X_2W_2 \in A_2\{1\}$.

Proof. Consider $X \in A\{l, W-GD\}$ written as

$$X = P \begin{bmatrix} X_1 & X_3 \\ X_4 & X_2 \end{bmatrix} Q^{-1}, \tag{2.3}$$

according to the size of blocks in A. From (2.1) and (2.3), we have

$$AW = P \begin{bmatrix} A_1W_1 & 0 \\ 0 & A_2W_2 \end{bmatrix} P^{-1}, \quad XW = P \begin{bmatrix} X_1W_1 & X_3W_2 \\ X_4W_1 & X_2W_2 \end{bmatrix} P^{-1}. \tag{2.4}$$

As A_2W_2 is nilpotent of index k, from (2.4), we get

$$(AW)^k = P \begin{bmatrix} (A_1W_1)^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1},$$

and therefore

$$XW(AW)^{k+1} = (AW)^k \Leftrightarrow X_4W_1(A_1W_1)^{k+1} = 0 \Leftrightarrow X_4 = 0. \tag{2.5}$$

Now, from (2.4) and (2.5), we obtain

$$AWXWA = A \;\Leftrightarrow\; X_1 = (W_1A_1W_1)^{-1}, \quad X_3W_2A_2 = 0, \quad A_2W_2X_2W_2A_2 = A_2. \tag{2.6}$$

From (2.5) and (2.6), it follows (2.2). □

Similarly, we can establish the following result concerning W-weighted right G-Drazin inverses.

Theorem 2.7. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\operatorname{Ind}(AW), \operatorname{Ind}(WA)\}$. If A and W are written as in* (2.1)*, then $X \in A\{r, W-GD\}$ if and only if*

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ X_4 & X_2 \end{bmatrix} Q^{-1},$$

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

where $A_2W_2X_4 = 0$ *and* $W_2X_2W_2 \in A_2\{1\}$.

As a consequence of Theorem 2.6 and Theorem 2.7, we get a canonical form for the W-weighted left-right G-Drazin inverses.

Corollary 2.8. *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ *and* $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. *If* A *and* W *are written as in* (2.1), *then* $X \in A\{l, r, W - GD\}$ *if and only if*

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & X_2 \end{bmatrix} Q^{-1},$$

where $W_2X_2W_2 \in A_2\{1\}$.

It is clear that if $m = n$ and $W = I_n$ in (2.1), we recovers the classical core-nilpotent decomposition of a matrix $A \in \mathbb{C}^{n\times n}$ of index k

$$A = P \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} P^{-1}, \tag{2.7}$$

where $t = \mathrm{rank}(A^k)$, $A_1 \in \mathbb{C}^{t\times t}$ is nonsingular and $A_2 \in \mathbb{C}^{(n-t)\times(n-t)}$ is nilpotent of index k.

Corollary 2.9. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$. *If* A *is written as in* (2.7), *then the following hold:*

(a) $X \in A\{l, GD\}$ *if and only if* $X = P \begin{bmatrix} A_1^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} P^{-1}$, *where* $X_3A_2 = 0$ *and* $X_2 \in A_2\{1\}$;

(b) $X \in A\{r, GD\}$ *if and only if* $X = P \begin{bmatrix} A_1^{-1} & 0 \\ X_4 & X_2 \end{bmatrix} P^{-1}$, *where* $A_2X_4 = 0$ *and* $X_2 \in A_2\{1\}$;

(c) $X \in A\{l, r, GD\}$ *if and only if* $X = P \begin{bmatrix} A_1^{-1} & 0 \\ 0 & X_2 \end{bmatrix} P^{-1}$, *where* $X_2 \in A_2\{1\}$.

Remark 2.10. Notice that if $m = n$ and $W = I_n$, the set of all G-Drazin inverses of A coincides with the set of all left-right G-Drazin inverses of A, that is, $A\{GD\} = A\{l, r, GD\}$.

3. Parametrizing W-weighted left and right G-Drazin inverses

The following two lemmas are due to Penrose.

Lemma 3.1. [1] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{m\times q}$, $A^- \in A\{1\}$, *and* $B^- \in B\{1\}$. *Then, the equation* $AXB = C$ *is consistent (in* X*) if and only if* $AA^-CB^-B = C$, *in which case the general solution is*

$$X = A^-CB^- + Y - A^-AYBB^-,$$

where Y *is an arbitrary matrix.*

Lemma 3.2. [1] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $D \in \mathbb{C}^{m\times p}$, *and* $E \in \mathbb{C}^{n\times q}$. *The matrix equations*

$$AX = B \quad \text{and} \quad XD = E, \tag{3.1}$$

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina — Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

have a common solution if and only if each equation separately has a solution and $AE = BD$. *In this case, the general solution of* (3.1) *is*

$$X = A^-B + ED^- - A^-AED^- + (I_n - A^-A)Y(I_p - DD^-),$$

for arbitrary $A^- \in A\{1\}$, $D^- \in D\{1\}$, *and* $Y \in \mathbb{C}^{n\times p}$.

Now, we provide a parameterized representation of the W-weighted left G-Drazin inverses.

Theorem 3.3. *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, *and* $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. *Then the general solution of the system*

$$AWXWA = A \quad \text{and} \quad XW(AW)^{k+1} = (AW)^k, \tag{3.2}$$

is given by

$$\begin{aligned} X &= (I_m - YWAW)(AW)^dW^- + Y \\ &+ (AW)^-(I_m - AWYW)(I_m - AW(AW)^d)A(WA)^-, \end{aligned}$$

for arbitrary $Y \in \mathbb{C}^{m\times n}$ *and for fixed but arbitrary* $W^- \in W\{1\}$, $(AW)^- \in (AW)\{1\}$, *and* $(WA)^- \in (WA)\{1\}$.

Proof. From Lemma 3.1, the equation $XW(AW)^{k+1} = (AW)^k$ has a solution if and only if

$$(AW)^k(W(AW)^{k+1})^-W(AW)^{k+1} = (AW)^k, \tag{3.3}$$

for some $(W(AW)^{k+1})^- \in W(AW)^{k+1}\{1\}$. Note that (3.3) holds for $(W(AW)^{k+1})^- = [(AW)^d]^{k+1}W^-$. In fact, we know that $W(AW)^d = (WA)^dW$, $(WA)^{k+1}W = W(AW)^{k+1}$ and $BB^d = B^m(B^d)^m$ for any square matrix B with $m \in \mathbb{N}$. Consequently, by taking $B = AW$ and $m = k+1$ we obtain

$$\begin{aligned} W(AW)^{k+1}[(AW)^d]^{k+1}W^-W(AW)^{k+1} &= WAW(AW)^dW^-W(AW)^{k+1} \\ &= WA(WA)^d[WW^-W](AW)^{k+1} \\ &= WA(WA)^dW(AW)^{k+1} \\ &= WA(WA)^d(WA)^{k+1}W \\ &= (WA)^{k+1}W \\ &= W(AW)^{k+1}. \end{aligned}$$

Thus, the general solution of the second equation in (3.2) is given by

$$X = (AW)^k(W(AW)^{k+1})^- + Z - ZW(AW)^{k+1}(W(AW)^{k+1})^-,$$

for an arbitrary $Z \in \mathbb{C}^{m\times n}$, which is equivalent to

$$\begin{aligned} X &= (AW)^k[(AW)^d]^{k+1}W^- + Z - ZW(AW)^{k+1}[(AW)^d]^{k+1}W^- \\ &= (AW)^dW^- + Z - ZWAW(AW)^dW^-, \end{aligned} \tag{3.4}$$

for an arbitrary $Z \in \mathbb{C}^{m\times n}$.
Now, substituting (3.4) in the first equation of (3.2), we have

$$AW(AW)^dW^-WA + AWZWA - AWZWAW(AW)^dW^-WA = A,$$

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

or equivalently

$$AWZW(I_m - AW(AW)^d)A = (I_m - AW(AW)^d)A. \tag{3.5}$$

Note that the above equation is consistent (in Z). In fact, as $AWXWA = A$, we have that $\operatorname{rank}(AW) = \operatorname{rank}(A) = \operatorname{rank}(WA)$, whence $AW(AW)^-A = A$ and $A(WA)^-WA = A$. Moreover, as $(WA)^- \in [W(I_m - AW(AW)^d)A]\{1\}$, we obtain

$$\begin{aligned}
& AW(AW)^-(I_m - AW(AW)^d)A(WA)^-W(I_m - AW(AW)^d)A = \\
= \; & [AW(AW)^-A(I_m - W(AW)^dA)][(WA)^-WA(I_m - W(AW)^dA)] \\
= \; & [A(I_m - W(AW)^dA)]\,[(WA)^-WA(I_m - W(AW)^dA)] \\
= \; & [(I_m - AW(AW)^d)][A(WA)^-WA](I_m - W(AW)^dA) \\
= \; & [(I_m - AW(AW)^d)]A(I_m - W(AW)^dA) \\
= \; & (I_m - AW(AW)^d)(I_m - AW(AW)^d)A \\
= \; & (I_m - AW(AW)^d)A.
\end{aligned}$$

Thus, the general solution of (3.5) is given by

$$Z = (AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^-, \tag{3.6}$$

where $P := I_m - AW(AW)^d$.
Now, note that as $A(WA)^-WA = A$ we have

$$\begin{aligned}
PA(WA)^-WAW(AW)^d &= (I_m - AW(AW)^d)[A(WA)^-WA]W(AW)^d \\
&= (I_m - AW(AW)^d)AW(AW)^d \\
&= 0.
\end{aligned}$$

Thus, substituting (3.6) in (3.4), we obtain

$$\begin{aligned}
X \; = \; & (AW)^dW^- + (AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^- - \\
& [(AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^-]WAW(AW)^dW^- \\
= \; & (AW)^dW^- + (AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^- \\
- \; & YWAW(AW)^dW^- \\
= \; & (I_m - YWAW)(AW)^dW^- + Y + (AW)^-(I_m - AWYW)PA(WA)^- \\
= \; & (I_m - YWAW)(AW)^dW^- + Y \\
+ \; & (AW)^-(I_m - AWYW)(I_m - AW(AW)^d)A(WA)^-.
\end{aligned}$$

□

Parameterized expressions for the W-weighted right G-Drazin inverses are also given below.

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\operatorname{Ind}(AW), \operatorname{Ind}(WA)\}$. Then the general solution of the system*

$$AWXWA = A \quad \textit{and} \quad (WA)^{k+1}WX = (WA)^k,$$

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

is given by

$$\begin{aligned} X &= W^-(WA)^d(I_n - WAWY) + Y \\ &+ (AW)^-A(I_n - WA(WA)^d)(I_n - WYWA)(WA)^-, \end{aligned}$$

for arbitrary $Y \in \mathbb{C}^{m\times n}$ *and for fixed but arbitrary* $W^- \in W\{1\}$, $(AW)^- \in (AW)\{1\}$, *and* $(WA)^- \in (WA)\{1\}$.

Theorem 3.5. *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, *and* $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. *Then the general solution of the system*

$$AWXWA = A, \quad XW(AW)^{k+1} = (AW)^k \quad and \quad (WA)^{k+1}WX = (WA)^k, \tag{3.7}$$

is given by

$$\begin{aligned} X &= (AW)^dW^- \qquad (3.8)\\ &- (AW)^-A(I_n - WA(WA)^d)WYW(I_m - AW(AW)^d)A(WA)^- \\ &+ [W^-(WA)^d + (AW)^-A(I_n - WA(WA)^d) \\ &+ (I_m - W^-(WA)^dWAW)Y](I_n - WAW(AW)^dW^-), \end{aligned}$$

for arbitrary $Y \in \mathbb{C}^{m\times n}$ *and for fixed but arbitrary* $W^- \in W\{1\}$, $(AW)^- \in (AW)\{1\}$, *and* $(WA)^- \in (WA)\{1\}$.

Proof. As in the proofs of Theorem 3.3 and Theorem 3.4, each equation $XW(AW)^{k+1} = (AW)^k$ and $(WA)^{k+1}WX = (WA)^k$ separately has a solution. Also, we know that $[(AW)^d]^{k+1}W^- \in W(AW)^{k+1}\{1\}$ and $W^-[(WA)^d]^{k+1} \in (WA)^{k+1}W\{1\}$. Thus, from Lemma 3.2, the general solution of the two last equations in (3.7) is given by

$$\begin{aligned} X &= W^-(WA)^d + (AW)^dW^- - W^-(WA)^dWAW(AW)^dW^- \\ &+(I_m - W^-(WA)^dWAW)Z(I_n - WAW(AW)^dW^-), \end{aligned} \tag{3.9}$$

for arbitrary Z.

Substituting (3.9) in $AWXWA = A$ we get

$$A(I_n - WA(WA)^d)WZW(I_m - AW(AW)^d)A = A(I_n - WA(WA)^d). \tag{3.10}$$

Since $(AW)^-$ and $(WA)^-$ are inner inverses of $A(I_n - WA(WA)^d)W$ and $W(I_m - AW(AW)^d)A$, respectively, the equation (3.10) is consistent (in Z), and therefore from Lemma 3.1 we obtain

$$\begin{aligned} Z &= (AW)^-A(I_n - WA(WA)^d)(WA)^- + Y \\ &- (AW)^-A(I_n - WA(WA)^d)WYW(I_m - AW(AW)^d)A(WA)^-. \end{aligned}$$

Now, replacing this expression of Z in (3.9) and using the facts that $(I_m - AW(AW)^d)A(WA)^-WAW(AW)^d = 0$ and $(WA)^dWAW(AW)^-A(I_n -$

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

$WA(WA)^d) = 0$ we obtain

$$\begin{aligned} X &= (AW)^dW^- + W^-(WA)^d(I_n - WAW(AW)^dW^-) \\ &\quad +(AW)^-A(I_n - WA(WA)^d)(I_n - WAW(AW)^dW^-) \\ &\quad +(I_m - W^-(WA)^dWAW)Y(I_n - WAW(AW)^dW^-) \\ &\quad -(AW)^-A(I_n - WA(WA)^d)WYW(I_m - AW(AW)^d)A(WA)^-, \end{aligned}$$

which implies (3.8). □

When $m = n$ and $W = I_n$, we get some useful parameterized representations of the different lateral G-Drazin inverses.

Corollary 3.6. *Let $A \in \mathbb{C}^{n\times n}$ and $A^- \in A\{1\}$. Then*

$$\begin{aligned} A\{l, GD\} &= \{Y + (I_n - YA)A^d + A^-(I_n - AY)(I_n - AA^d)AA^- : Y \in \mathbb{C}^{n\times n}\}; \\ A\{r, GD\} &= \{Y + A^d(I_n - AY) + A^-A(I_n - AA^d)(I_n - YA)A^- : Y \in \mathbb{C}^{n\times n}\}; \\ A\{l, r, GD\} &= \{A^d + A^-A(I_n - AA^d)[I_n - Y(I_n - AA^d)AA^-] \\ &\quad + (I_n - AA^d)Y(I_n - AA^d) : Y \in \mathbb{C}^{n\times n}\}. \end{aligned}$$

4. Weighted left and right G-Drazin partial orders

It is well known that the binary relation $\preceq^d$ (called Drazin pre-order) is only a pre-order on $\mathbb{C}^{n\times n}$, because the nilpotent part of matrices is not considered. To extend the Drazin pre-order to the rectangular case, the order $\preceq^{d,W}$ [7] and more recently the order $\preceq_W^{GD}$ [4] were introduced and characterized. However, none of these weighted matrix orders have the antisymmetric property, and therefore only result pre-orders on $\mathbb{C}^{m\times n}$. The aim of this section is to give new binary relations on the set $\mathbb{C}^{m\times n}$ by using W-weighted left (resp. right) G-Drazin inverses resulting matrix partial orders.

Definition 4.1. Let $W \in \mathbb{C}^{n\times m}$ be a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$. Then we say that

(i) A is below B under the W-weighted left G-Drazin order (denoted by $A \leq_W^{l,GD} B$) if there exist $X_1, X_2 \in A\{l, W - GD\}$ such that
$$AWX_1 = BWX_1, \quad X_2WA = X_2WB;$$
(ii) A is below B under the W-weighted right G-Drazin order (denoted by $A \leq_W^{r,GD} B$) if there exist $X_1, X_2 \in A\{r, W - GD\}$ such that
$$AWX_1 = BWX_1, \quad X_2WA = X_2WB;$$
(iii) A is below B under the W-weighted left-right G-Drazin order (denoted by $A \leq_W^{l,r,GD} B$) if there exist $X_1, X_2 \in A\{l, r, W - GD\}$ such that
$$AWX_1 = BWX_1, \quad X_2WA = X_2WB.$$

Remark 4.2. When $m = n$ and $W_n = I_n$ in the above definition we recover [16, Definition 7.1] and [18, Definition 3.1].

Our first result gives some useful characterizations of the binary relation $\leq_W^{l,GD}$.

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

Theorem 4.3. *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (2.1), *then the following conditions are equivalent:*

(a) $A \leq_W^{l,GD} B$;
(b) *there exists* $X \in A\{l, W-GD\}$ *such that* $AWX = BWX$ *and* $XWA = XWB$;
(c) *there exists* $X \in A\{l, W - GD\}$ *such that*

$$X = P\begin{bmatrix} (W_1A_1W_1)^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} Q^{-1} \quad and \quad B = P\begin{bmatrix} A_1 & -A_1W_1X_3W_2B_2 \\ 0 & B_2 \end{bmatrix} Q^{-1}, \tag{4.1}$$

where $X_3W_2A_2 = 0$, $W_2X_2W_2 \in A_2\{1\}$, $A_2W_2X_2 = B_2W_2X_2$ *and* $X_2W_2A_2 = X_2W_2B_2$;
(d) *there exists* $X \in A\{l, W - GD\}$ *such that* $AWXWB = BWXWA = A$.

Proof. $(a) \Rightarrow (b)$ If $A \leq_W^{l,GD} B$, there exist $X_1, X_2 \in A\{l, W - GD\}$ such that $AWX_1 = BWX_1$ and $X_2WA = X_2WB$. Now, we consider the matrix $X := X_1WAWX_2$. Firstly, we note that $X \in A\{l, W - GD\}$. In fact, $AWXWA = (AWX_1WA)WX_2WA = AWX_2WA = A$ and $XW(AW)^{k+1} = X_1WAW(X_2W(AW)^{k+1}) = X_1WAW(AW)^k = X_1W(AW)^{k+1} = (AW)^k$. Moreover, $AWX = (AWX_1)WAWX_2 = BWX_1WAWX_2 = BWX$ and $XWA = X_1WAW(X_2WA) = X_1WAWX_2WB = XWB$.
$(b) \Rightarrow (c)$ Let $X \in A\{l, W-GD\}$ be such that $AWX = BWX$ and $XWA = XWB$. Thus, Theorem 2.6 implies

$$X = P\begin{bmatrix} (W_1A_1W_1)^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} Q^{-1}, \text{ where } X_3W_2A_2 = 0 \text{ and } W_2X_2W_2 \in A_2\{1\}. \tag{4.2}$$

Let us consider the partition of $B = P\begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} Q^{-1}$. Hence,

$$AWX = P\begin{bmatrix} W_1^{-1} & A_1W_1X_3 \\ 0 & A_2W_2X_2 \end{bmatrix} Q^{-1},$$

$$BWX = P\begin{bmatrix} B_1(W_1A_1)^{-1} & B_1W_1X_3 + B_3W_2X_2 \\ B_4(W_1A_1)^{-1} & B_4W_1X_3 + B_2W_2X_2 \end{bmatrix} Q^{-1},$$

$$XWA = P\begin{bmatrix} W_1^{-1} & 0 \\ 0 & X_2W_2A_2 \end{bmatrix} Q^{-1},$$

$$XWB = P\begin{bmatrix} (A_1W_1)^{-1}B_1 & (A_1W_1)^{-1}B_3 + X_3W_2B_2 \\ 0 & X_2W_2B_2 \end{bmatrix} Q^{-1}.$$

Therefore,

$$AWX = BWX \Leftrightarrow A_1 = B_1,\ B_4 = 0,\ B_3W_2X_2 = 0,\ A_2W_2X_2 = B_2W_2X_2. \tag{4.3}$$

In consequence,

$$XWA = XWB \Leftrightarrow B_3 + A_1W_1X_3W_2B_2 = 0,\ X_2W_2A_2 = X_2W_2B_2. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we obtain the desired implication.
$(c) \Rightarrow (d)$ This implication follows from direct calculations.
$(d) \Rightarrow (a)$ Let $X \in A\{l, W - GD\}$ such that $AWXWB = BWXWA = A$. Taking

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

$Y = XWAWX$ we have $AWYWA = AW(XWAWX)WA$ $= (AWXWA)WXWA = AWXWA = A$ and $YW(AW)^{k+1}$ $= XWAW[XW(AW)^{k+1}] = XW(AW)^{k+1} = (AW)^k$. Thus, $Y \in A\{l, W-GD\}$. Moreover, as $AWXWA = A$ and $BWXWA = A$, we obtain $AWY = (AWXWA)WX = AWX = BW(XWAWX) = BWY$. Similarly, from $AWXWB = A$, we get $YWA = YWB$. Thus, $A \leq_W^{l,GD} B$. $\square$

Remark 4.4. By the proof of Theorem 4.3, we deduce that

$$A\{l, W-GD\} \cdot WAW \cdot A\{l, W-GD\} \subseteq A\{l, W-GD\}.$$

Corollary 4.5. *Let $X, B \in \mathbb{C}^{m\times n}$ be of the form (4.1). Then Z is a W-weighted left G-Drazin of B if and only if*

$$Z = P\begin{bmatrix} (W_1A_1W_1)^{-1} & Z_3 \\ 0 & Z_2 \end{bmatrix} Q^{-1}, \tag{4.5}$$

where $Z_3W_2B_2 = X_3W_2B_2$ and $Z_2 \in B_2\{l, W_2 - GD\}$.

Proof. Let $Z \in B\{W - GD\}$ and consider

$$Z = P\begin{bmatrix} Z_1 & Z_3 \\ Z_4 & Z_2 \end{bmatrix} Q^{-1}, \tag{4.6}$$

according to the size of blocks in B. From (4.1) and (4.6), we have

$$BW = P\begin{bmatrix} A_1W_1 & -A_1W_1X_3W_2B_2W_2 \\ 0 & B_2W_2 \end{bmatrix} P^{-1}, \tag{4.7}$$

$$ZWBW = P\begin{bmatrix} Z_1W_1A_1W_1 & M \\ Z_4W_1A_1W_1 & N \end{bmatrix} P^{-1}, \tag{4.8}$$

where $S = -A_1W_1X_3W_2B_2W_2$, $M = Z_1W_1S + Z_3W_2B_2W_2$, and $N = Z_4W_1S + Z_2W_2B_2W_2$.
As BW is an upper-triangular block matrix with its block A_1W_1 nonsingular, from [2, Theorem 7.7.2], we get

$$(BW)^d = P\begin{bmatrix} (A_1W_1)^{-1} & R \\ 0 & (B_2W_2)^d \end{bmatrix} P^{-1}, \quad \text{for some matrix } R. \tag{4.9}$$

Now, from (4.8) and (4.9), we obtain

$$ZWBW(BW)^d = P\begin{bmatrix} Z_1W_1 & Z_1W_1A_1W_1R + M(B_2W_2)^d \\ Z_4W_1 & Z_4W_1A_1W_1R + N(B_2W_2)^d \end{bmatrix} P^{-1}. \tag{4.10}$$

Thus, from (4.9) and (4.10) direct calculations imply that $ZWBW(BW)^d = (BW)^d$ holds if and only if

$$\begin{aligned} Z_1 &= (W_1A_1W_1)^{-1}, \\ Z_4 &= 0, \\ Z_3W_2B_2W_2 &= X_3W_2B_2W_2, \\ Z_2W_2B_2W_2(B_2W_2)^d &= (B_2W_2)^d. \end{aligned} \tag{4.11}$$

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

From (4.11), we obtain

$$BWZWB = P\begin{bmatrix} A_1 & A_1W_1Z_3W_2B_2 - A_1W_1X_3W_2B_2W_2Z_2W_2B_2 - A_1W_1X_3W_2B_2 \\ 0 & B_2W_2Z_2W_2B_2 \end{bmatrix} Q^{-1},$$

whence

$$BWZWB = B \;\Leftrightarrow\; B_2W_2Z_2W_2B_2 = B_2, \quad Z_3W_2B_2 = X_3W_2B_2. \tag{4.12}$$

Thus, (4.11) and (4.12) imply (4.5). □

In a manner similar to Theorem 4.3, we derive characterizations of the W-weighted right G-Drazin order.

Theorem 4.6. *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (2.1)*, then the following conditions are equivalent:*

(a) $A \leq_W^{r,GD} B$*;*
(b) there exists $X \in A\{r, W-GD\}$ such that $AWX = BWX$ and $XWA = XWB$;
(c) there exists $X \in \mathbb{C}^{m\times n}$ such that

$$X = P\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ X_4 & X_2 \end{bmatrix} Q^{-1} \quad \textit{and} \quad B = P\begin{bmatrix} A_1 & 0 \\ -B_2W_2X_4A_1W_1 & B_2 \end{bmatrix} Q^{-1}, \tag{4.13}$$

where $A_2W_2X_4 = 0$, $W_2X_2W_2 \in A_2\{1\}$, $A_2W_2X_2 = B_2W_2X_2$, and $X_2W_2A_2 = X_2W_2B_2$;
(d) there exists $X \in A\{r, W-GD\}$ such that $AWXWB = BWXWA = A$.

Corollary 4.7. *Let $X, B \in \mathbb{C}^{m\times n}$ be of the form* (4.13)*. Then Z is a W-weighted right G-Drazin of B if and only if*

$$Z = P\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ Z_4 & Z_2 \end{bmatrix} Q^{-1},$$

where $B_2W_2Z_4 = B_2W_2X_4$ and $Z_2 \in B_2\{r, W_2 - GD\}$.

Theorem 4.8. *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (2.1)*, then the following conditions are equivalent:*

(a) $A \leq_W^{l,r,GD} B$*;*
(b) there exists $X \in A\{l, r, W-GD\}$ such that $AWX = BWX$ and $XWA = XWB$;
(c) there exists $X \in \mathbb{C}^{m\times n}$ such that

$$X = P\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & X_2 \end{bmatrix} Q^{-1} \quad \textit{and} \quad B = P\begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix} Q^{-1}, \tag{4.14}$$

where $W_2X_2W_2 \in A_2\{1\}$, $A_2W_2X_2 = B_2W_2X_2$, and $X_2W_2A_2 = X_2W_2B_2$;
(d) there exists $X \in A\{l, r, W-GD\}$ such that $AWXWB = BWXWA = A$.

Corollary 4.9. *Let $X, B \in \mathbb{C}^{m\times n}$ be of the form* (4.14)*. Then Z is a W-weighted left-right G-Drazin of B if and only if*

$$Z = P\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & Z_2 \end{bmatrix} Q^{-1},$$

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

where $Z_2 \in B_2\{l, r, W_2 - GD\}$.

In order to prove that the binary relations $\leq_W^{l,GD}$, $\leq_W^{r,GD}$ and $\leq_W^{l,r,GD}$ are partial orders on the set of complex rectangular matrices, we need an auxiliary lemma.

Lemma 4.10. *Let* $W \in \mathbb{C}^{n\times m}$ *and* $A, B \in \mathbb{C}^{m\times n}$.
(a) If $A \leq_W^{l,GD} B$ *then* $B\{l, W - GD\} \subseteq A\{l, W - GD\}$.
(b) If $A \leq_W^{r,GD} B$ *then* $B\{r, W - GD\} \subseteq A\{r, W - GD\}$.
(c) If $A \leq_W^{l,r,GD} B$ *then* $B\{l, r, W - GD\} \subseteq A\{l, r, W - GD\}$.

Proof. (a) Since $A \leq_W^{l,GD} B$, by Theorem 4.3 (b), there exists $X \in A\{l, W - GD\}$ such that $AWX = BWX$ and $XWA = XWB$. Suppose that $Y \in B\{l, W - GD\}$, i.e. $BWYWB = B$ and $YWBW(BW)^d = (BW)^d$. We will prove that $Y \in A\{l, W - GD\}$. In fact, as $A = AWXWA$ due to $X \in A\{l, W - GD\}$, we have

$$\begin{aligned} AWYWA &= (AWXWA)WYW(AWXWA) \\ &= AW(XWA)WYW(AWX)WA \\ &= AW(XWB)WYW(BWX)WA \\ &= AWXW(BWYWB)WXWA \\ &= AW(XWB)WXWA \\ &= AW(XWA)WXWA \\ &= (AWXWA)WXWA \\ &= AWXWA \\ &= A. \end{aligned}$$

From (4.1) and [2, Theorem 7.7.1], we have

$$(AW)^d = P\begin{bmatrix} (A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \quad (BW)^d = P\begin{bmatrix} (A_1W_1)^{-1} & R \\ 0 & (B_2W_2)^d \end{bmatrix}P^{-1}, \tag{4.15}$$

for some matrix R.
From (4.15), it easy to see that $(AW)^d = (BW)^d BW(AW)^d$. In consequence, using that $AWX = BWX$, $XWAW(AW)^d = (AW)^d$, and $YWBW(BW)^d = (BW)^d$, we obtain

$$\begin{aligned} YWAW(AW)^d &= YW(AWX)WAW(AW)^d = YWBW(XWAW(AW)^d) \\ &= YWBW(AW)^d = YWBW(BW)^d BW(AW)^d \\ &= (BW)^d BW(AW)^d = (AW)^d. \end{aligned}$$

A similar proof can be applied to statements (b) and (c). □

Theorem 4.11. *Let* $W \in \mathbb{C}^{n\times m}$. *Then the binary relation* $\leq_W^{l,GD}$ *is a partial order on* $\mathbb{C}^{m\times n}$.

Proof. Clearly, the binary relation $\leq_W^{l,GD}$ is reflexive. Suppose $A, B, C \in \mathbb{C}^{m\times n}$ such that $A \leq_W^{l,GD} B$ and $B \leq_W^{l,GD} C$. By Theorem 4.3 (b) we know that there exist $X \in A\{l, W - GD\}$ and $Y \in B\{l, W - GD\}$ such that $AWX = BWX$,

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

$XWA = XWB$, $BWY = CWY$, and $YWB = YWC$. Now, by applying Lemma 4.10, we deduce that $Y \in A\{l, W-GD\}$. Thus,

$$\begin{aligned} AWYWC &= AWYWB = AW(XWA)WYWB = AWXW(BWYWB) \\ &= AW(XWB) = AWXWA = A \end{aligned}$$

Similarly, we obtain $CWYWA = A$. Therefore, by Theorem 4.3 (d) we get $A \leq_W^{l,GD} C$.

In order to prove that the binary relation is antisymmetric, we have to show that the conjunction $A \leq_W^{l,GD} B$ and $B \leq_W^{l,GD} A$ implies $A = B$. By Theorem 4.3 (b), the second of these conditions ensure that there exists $Y \in B\{l, W-GD\}$ such that $BWY = AWY$ and $YWB = YWA$. By Lemma 4.10, we know that $Y \in A\{l, W-GD\}$. In consequence, $A = (AWY)WA = BW(YWA) = BWYWB = B$. $\square$

By a similar procedure, from Theorem 4.6, Theorem 4.8, and Lemma 4.10 we can identifies two new matrix partial orders on the set of complex rectangular matrices.

Theorem 4.12. *Let $W \in \mathbb{C}^{n\times m}$. Then the binary relations $\leq_W^{r,GD}$ and $\leq_W^{l,r,GD}$ are partial orders on the set $\mathbb{C}^{m\times n}$.*

Acknowledgment

We are deeply indebted to the anonymous referees whose insightful comments and invaluable suggestions have greatly helped us to improve the quality of the paper.

References

[1] A. Ben-Israel, T.N.E Greville, *Generalized Inverses: Theory and Applications*, Springer-Verlag, 2003.

[2] S.L. Campbell, C.D. Meyer, *Generalized Inverses of Linear transformations*, SIAM, Philadelphia, 2009.

[3] R.E. Cline, T.N.E. Greville, *A Drazin inverse for rectangular matrices*, Linear Algebra Appl. **29** (1980) 53-62.

[4] C. Coll, M. Lattanzi, N. Thome, *Weighted G-Drazin inverses and a new pre-order on rectangular matrices*, Appl. Math. Comput. **317** (2018) 12-24.

[5] D.E. Ferreyra, M. Lattanzi, F.E. Levis, N. Thome, *Parametrized solutions X of the system $AXA = AEA$ and $A^kEAX = XAEA^k$ for a matrix A having index k*, Electronic J. Linear Algebra **35** (2019) 503-510.

[6] D.E. Ferreyra, M. Lattanzi, F.E. Levis, N. Thome, *Solving an open problem about the G-Drazin partial order*, Electronic J. Linear Algebra **36** (2020) 55-66.

[7] A. Hernández, M. Lattanzi, N. Thome, *Weighted binary relation involving the Drazin inverse*, Appl. Math. Comput. **282** (2015) 215-223.

[8] A. Hernández, M. Lattanzi, N. Thome, *On some new pre-orders defined by weighted Drazin inverses*, Appl. Math. Comput. **282** (2016) 108-116.

[9] S.K. Mitra, P. Bhimasankaram, S. Malik, *Matrix partial orders, shorted operators and applications*, World Scientific Publishing Company, 2010.

[10] D. Mosić, *G-outer inverse of Banach spaces operators*, J. Math. Anal. Appl. **481** (2) (2020) 123-501.

[11] D. Mosić, *Solvability to some systems of matrix equations using G-outer inverses*, Electron. J. Linear Algebra **36** (2020) 265-276.

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5023.

[12] D. Mosić, *Weighted binary relations for operators on Banach spaces*, Aequat. Math. **90** (4) (2016) 787-798.
[13] D. Mosić, *Weighted G-Drazin inverse for operators on Banach spaces*, Carpathian J. Math. **35** (2)(2019) 171-184.
[14] D. Mosić, D.S. Djordjević, *Weighted pre-orders involving the generalized Drazin inverse*, Appl. Math. Comput. **270** (2015) 496-504.
[15] D. Mosić, P.S. Stanimirović, *Strong weighted GDMP inverse for operators*, Bull. Iran. Math. Soc. **50** (2024) 43.
[16] D. Mosić, L. Wang, *Left and right G-outer inverses*, Linear Multilinear Algebra **70** (17) (2022) 3319-3334.
[17] Y. Wei, *Integral representation of the W-weighted Drazin inverse*, Appl. Math. Comput., **144** (2003) 3-10.
[18] H. Wang , X. Liu, *Partial orders based on core-nilpotent decomposition*, Linear Algebra Appl. **488** (2016) 235-248.

(D. E. Ferreyra) CONICET, Departamento de Matemática, FCEFQyN, Universidad Nacional de Río Cuarto, Km 601, 5800 Río Cuarto, Argentina
Email address: `deferreyra@exa.unrc.edu.ar`

(D. Mosić) Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia
Email address: `dijana@pmf.ni.ac.rs`

(A. N. Priori) Departamento de Matemática, FCEFQyN, Universidad Nacional de Río Cuarto, Km 601, 5800 Río Cuarto, Argentina
Email address: `apriori@exa.unrc.edu.ar`

(M. L. Llanes) Departamento de Matemática, FCEFQyN, Universidad Nacional de Río Cuarto, Km 601, 5800 Río Cuarto, Argentina
Email address: `mllanes@exa.unrc.edu.ar`

Submitted: October 30, 2024
Accepted: March 3, 2025
Published (early view): March 16, 2025