Revista de la Unión Matemática Argentina Accepted article · Early view version

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WEIGHTED WEAK GROUP INVERSE IN A RING WITH INVOLUTION

HUANYIN CHEN

Abstract. We introduce the concept of the weighted weak group inverse for elements in a ring with involution. This notion naturally extends the weak group inverse for complex matrices and the weighted weak group inverse for Hilbert operators. We characterize this generalized inverse through a novel decomposition (referred to as the w-group decomposition) that involves weighted group inverses and nilpotent elements. Additionally, the interrelationships among weighted weak group inverses, weighted Drazin inverses, and weighted core-EP core inverses are thoroughly investigated.

1. Introduction

Let R be an associative ring with an identity. An involution of R is an anti-automorphism whose square is the identity map 1. A ring R with involution $*$ is called a *-ring. An element a in a *-ring R has group inverse provided that there exists $x \in R$ such that

$$ax^2 = x, \ ax = xa, \ a = xa^2.$$

Such x is unique if exists, denoted by $a^{\#}$, and called the group inverse of a.

An element $a \in R$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$ax^2 = x, \quad (ax)^* = ax, \quad xa^{n+1} = a^n.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{D}}$. The core-EP inverse has been investigated from many different views, e.g., [4, 9, 10, 15, 16, 17, 23, 27].

Wang and Chen (see [24]) introduced and studied a weak group inverse for square complex matrices. A square complex matrix A has a weak group inverse X if it satisfies the equations

$$AX^2 = X, \quad AX = A^{\textcircled{\dagger}}A.$$

Here, $A^{\textcircled{\dagger}}$ is just the core-EP inverse of A. The involution $*$ is proper if $x^*x = 0 \Longrightarrow x = 0$ for any $x \in R$. In [25], Zou et al. generalize the concept of the weak group inverse from complex matrices to elements within a ring with a proper involution. An element $a \in R$ has a weak group inverse if there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$ax^2 = x, \ (a^*a^2x)^* = a^*a^2x, \ xa^{n+1} = a^n.$$

2020 *Mathematics Subject Classification.* Primary 16U50; Secondary 16W10, 15A09.

Key words and phrases. weighted group inverse, weighted Drazin inverse, weak group inverse, weighted core-EP inverse, ring element.

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

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If such x exists, it is unique, and denote it by $a^{ⓦ}$. The weak group inverse has been extensively studied from many different views, e.g., [3, 5, 6, 7, 8, 12, 18, 19, 20, 21, 24, 25, 26].

Let $a, w \in R$. We recall that

Definition 1.1. An element $a \in R$ has w-Drazin inverse if there exist $x \in R$ such that

$$xwawx = x, \ awx = xwa, \ (aw)^n = xw(aw)^{n+1}$$

for some $n \in \mathbb{N}$. The preceding x is unique if it exists, and we denote it by $a^{D,w}$. The most smallest positive integer n is denoted by $i(a^{D,w})$. The set of all w-Drazin invertible elements in R is denoted by $R^{D,w}$.

We say that a has Drazin inverse if $w = 1$ and denote $a^{D,1}$ by a^D.

Definition 1.2. An element $a \in R$ has w-group inverse if there exist $x \in R$ such that

$$xwawx = x, awx = xwa, awxwa = a.$$

The preceding x is unique if it exists, and we denote it by $a^{\#}_w$. The set of all w-group invertible elements in R is denoted by $R^{\#}_w$.

Evidently, $a^{D,w} = [(aw)^D]^2 a$ and $a^{\#}_w = [(aw)^{\#}]^2 a$. Recently, Ferreyra et al. introduced and studied the weighted group inverse for complex matrices in [6]. In [18], Mosić and Zhang investigated the weighted weak group inverse for Hilbert space operators. The aim of this paper is to introduce and explore a new type of generalized inverse that incorporates a weight, serving as a natural generalization of the generalized inverses mentioned above. In Section 2, we introduce the weighted weak group inverse by way of an innovative weighted group decomposition. This approach enables us to generalize numerous properties of the standard weak group inverse to this broader context.

Let $R^{\text{nil}}_w = \{x \in R \mid xw \in R \text{ is nilpotent}\}$.

Definition 1.3. An element $a \in R$ has a weak w-group decomposition if there exist $a_1, a_2 \in R$ such that

$$a = a_1 + a_2, \quad a_1^* a_2 = a_2 w a_1 = 0, \quad a_1 \in R^{\#}_w, a_2 \in R^{\text{nil}}_w.$$

In Section 2, we will establish the connections between the weak w-group decomposition and a type of generalized inverse, followed by the introduction of the weighted weak group inverse.

In Section 3, we explore the equivalent characterizations of the weighted weak group inverse in a ring. The relationship between weighted weak group inverses and weighted Drazin inverses is investigated.

Following Mosić et al. (see [22]), an element $a \in R$ has w-core-EP inverse if there exists $x \in R$ such that

$$awx^2 = x, \quad x(aw)^{k+1}a = (aw)^k a, \quad (awx)^* = awx$$

for some nonnegative integer k. Such x is unique if it exists and is called the w-core-EP inverse of a. In Section 4, we examine the interplay between weighted

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core-EP inverses and weighted weak group inverses. Additionally, we delineate the circumstances under which an element and its weighted weak group inverse are commutative with respect to the weight.

Throughout the paper, all rings are associative ring with a proper involution $*$. We use $R^{\#}, R^{D}, R^{\textcircled{\tiny D}}$ and $R^{\textcircled{\tiny W}}$ to denote the sets of all group invertible, Drazin invertible, core-EP invertible and weak group invertible elements in R, respectively. $\mathbb{N}$ denotes the set of all natural numbers.

2. Weak w-group inverse

The purpose of this section is to introduce a new weighted weak inverse which is a natural generalization of weak group inverse. We start with the following lemma, which is both simple and crucial. We include the proof to make this presentation self-contained.

Lemma 2.1. *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R^{\#}_{w}$.
(2) $aw, wa \in R^{\#}$.
(3) *There exist $x \in R$ such that $x(wa)^2 = a,\ a(wx)^2 = x,\ awx = xwa$.*

In this case, $a^{\#}_{w} = (aw)^{\#}a(wa)^{\#}$.

Proof. (1) $\Rightarrow$ (3) Set $x = a^{\#}_{w}$. Then $(xwa)wx = x, awx = xwa, (awx)wa = a$. Hence, $x(wa)^2 = a,\ a(wx)^2 = x,\ awx = xwa$.

(3) $\Rightarrow$ (2) By hypothesis, $xw(aw)^2 = aw,\ aw(xw)^2 = xw,\ (aw)(xw) = (xw)(aw)$. Hence, $(aw)^{\#} = xw$. Likewise, we have $(wa)^{\#} = wx$, as required.

(2) $\Rightarrow$ (1) Set $x = (aw)^{\#}a(wa)^{\#}$. By using Cline's formula (see [13, Theorem 2.2]), we have

$$\begin{aligned} awx &= aw(aw)^{\#}a(wa)^{\#} = aw(aw)^{\#}a[w((aw)^{\#})^2a] \\ &= (aw)^2((aw)^{\#})^3a = (aw)^{\#}a, \\ xwa &= (aw)^{\#}a(wa)^{\#}wa = (aw)^{\#}a[w((aw)^{\#})^2a]wa \\ &= [(aw)^{\#}aw((aw)^{\#})^2]awa = [(aw)^{\#}]^2awa = (aw)^{\#}a. \end{aligned}$$

This implies that $awx = xwa$. Analogously, we directly check that $x(wa)^2 = a$ and $a(wx)^2 = x$. Therefore $a^{\#}_{w} = x = (aw)^{\#}a(wa)^{\#}$, as asserted. □

Lemma 2.2. *Let $a \in R^{\#}_{w}$. Then $(aw)^2a^{\#}_{w} = a$ and $a[wa^{\#}_{w}]^2 = a^{\#}_{w}$.*

Proof. Let $x = a^{\#}_{w}$. Then

$$awxwa = a,\ xwawx = x,\ awx = xwa.$$

Hence, $(aw)^2x = aw(awx) = (aw)x(wa) = a$. Moreover, we have $a[wa^{\#}_{w}]^2 = a(wx)^2 = x$ by Lemma 2.1. □

We now proceed to prove the following result.

Theorem 2.3. *Let $a, w \in R$. Then the following are equivalent:*

(1) *$a \in R$ has a weak w-group decomposition.*

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(2) *There exist* $x \in R$ *and* $n \in \mathbb{N}$ *such that*

$$x = a(wx)^2, \quad \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, \quad (aw)^n = xw(aw)^{n+1}.$$

Proof. (1) $\Rightarrow$ (2) Let $a = a_1 + a_2$ be the weak w-group decomposition of a. Let $x = (a_1)^{\#}_w$. By virtue of Lemma 2.2, we have

$$\begin{aligned} awx &= (a_1w + a_2w)(a_1)^{\#}_w = a_1w(a_1)^{\#}_w, \\ a(wx)^2 &= a_1[w(a_1)^{\#}_w]^2 = [a_1w(a_1)^{\#}_w][w(a_1)^{\#}_w] = [(a_1)^{\#}_w wa_1][w(a_1)^{\#}_w] \\ &= (a_1)^{\#}_w wa_1w(a_1)^{\#}_w = (a_1)^{\#}_w = x. \end{aligned}$$

Since $a_2 \in R^{\rm nil}_w$, $(a_2w)^n = 0$ for some $n \in \mathbb{N}$. As $a_2wa_1 = 0$, we see that

$$\begin{aligned} aw - xw(aw)^2 &= (a_1w + a_2w) - [(a_1)^{\#}_w wa_1w + (a_1)^{\#}_w wa_2w](a_1w + a_2w) \\ &= [1 - (a_1)^{\#}_w wa_1w - (a_1)^{\#}_w wa_2w]a_2w. \end{aligned}$$

Hence,

$$\begin{aligned} (aw)^n - xw(aw)^{n+1} &= [aw - xw(aw)^2](aw)^{n-1} \\ &= [1 - (a_1)^{\#}_w wa_1w - (a_1)^{\#}_w wa_2w]a_2w(aw)^{n-1} \\ &= [1 - (a_1)^{\#}_w wa_1w - (a_1)^{\#}_w wa_2w](a_2w)^n = 0. \end{aligned}$$

Thus, $(aw)^n = xw(aw)^{n+1}$.

Since $aw = a_1w + a_2w, (a_2w)(a_1w) = 0$ and $(a_2w)^n = 0$, we have

$$(aw)^n = \sum_{i=0}^{n}(a_1w)^i(a_2w)^{n-i} = (a_1w)^n + \sum_{i=1}^{n-1}(a_1w)^i(a_2w)^{n-i}.$$

As $(a_1)^*a_2 = 0$, we deduce that

$$\big((aw)^n\big)^*a_2 = \big((a_1w)^n\big)^*a_2 + \sum_{n=1}^{n-1}[(a_1w)^i(a_2w)^{n-i}]^*a_2 = 0;$$

and then $\big((aw)^n\big)^*a_1 = \big((aw)^n\big)^*a$. Accordingly,

$$\begin{aligned} \big((aw)^n\big)^*(aw)^2x &= \big((aw)^n\big)^*(a_1w + a_2w)(a_1w + a_2w)a_1^{\#} \\ &= \big((aw)^n\big)^*(a_1w)^2a_1^{\#} = \big((aw)^n\big)^*a_1 \\ &= \big((aw)^n\big)^*a. \end{aligned}$$

Therefore $x = a(wx)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1}$.

(2) $\Rightarrow$ (1) By hypothesis, there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$x = a(wx)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1}.$$

Without loss of generality, we may assume that $n \geq 2$. Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$.

Claim 1. $a_2wa_1 = 0$. Clearly, we have

$$x = (aw)x(wx) = (aw)^2x(wx)^2 = (aw)^{n-1}x(wx)^{n-1}.$$

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Then

$$\begin{aligned}
a_2wa_1 &= [a-(aw)^2x]w(aw)^2x\\
&= (aw)^3x-(aw)^2xw(aw)^2x\\
&= (aw)^3x-(aw)^2xw(aw)^2x\\
&= (aw)^3x-(aw)^2xw(aw)^2(aw)^{n-1}x(wx)^{n-1}\\
&= (aw)^3x-(aw)^2[xw(aw)^{n+1}]x(wx)^{n-1}\\
&= (aw)^3x-(aw)^{n+2}x(wx)^{n-1}\\
&= (aw)^3x-(aw)^3[(aw)^{n-1}x(wx)^{n-1}]=0.
\end{aligned}$$

Claim 2. $a_1^*a_2=0$. Obviously, we have

$$(aw)^2x=(aw)^2[(aw)^{n-2}x(wx)^{n-2}]=(aw)^nx(wx)^{n-2}.$$

Then

$$\begin{aligned}
a_1^*a_2 &= [(aw)^2x]^*[a-(aw)^2x]\\
&= [(aw)^nx(wx)^{n-2}]^*[a-(aw)^2x]\\
&= [x(wx)^{n-2}]^*[(aw)^n]^*[a-(aw)^2x]=0.
\end{aligned}$$

Claim 3. $a_1\in R_w^{\#}$. Evidently, we verify that

$$\begin{aligned}
a_1wx &= (aw)^2xwx=awa(wx)^2=awx,\\
xwa_1 &= xw(aw)^2x=xw(aw)^2[(aw)^{n-1}x(wx)^{n-1}]\\
&= [xw(aw)^{n+1}]x(wx)^{n-1}=(aw)^nx(wx)^{n-1}=awx,
\end{aligned}$$

and then $a_1wx=xwa_1$. Moreover, we have

$$\begin{aligned}
a_1wxwa_1 &= (a_1wx)wa_1=awxw(aw)^2x\\
&= awxw(aw)^2[(aw)^{n-1}x(wx)^{n-1}]\\
&= aw[xw(aw)^{n+1}]x(wx)^{n-1}\\
&= (aw)^{n+1}x(wx)^{n-1}=(aw)^2x=a_1,\\
xwa_1wx &= (xwa_1)wx=a(wx)^2=x.
\end{aligned}$$

Hence, $a_1\in R_w^{\#}$ and $(a_1)_w^{\#}=x$.

Claim 4. $a_2\in R_w^{\mathrm{nil}}$. It is easy to verify that

$$\begin{aligned}
[aw-xw(aw)^2]x &= [aw-xw(aw)^2][(aw)^{n-1}x(wx)^{n-2}]\\
&= [(aw)^n-xw(aw)^{n+1}]x(wx)^{n-2}=0.
\end{aligned}$$

Hence,

$$\begin{aligned}
[aw-xw(aw)^2]^2 &= [aw-xw(aw)^2]aw-[aw-xw(aw)^2]xw(aw)^2\\
&= [aw-xw(aw)^2](aw).
\end{aligned}$$

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This implies that

$$\begin{aligned}
[aw - xw(aw)^2]^n &= [aw - xw(aw)^2]^{n-2}[aw - xw(aw)^2]^2 \\
&= [aw - xw(aw)^2]^{n-3}[aw - xw(aw)^2]^2(aw) \\
&= [aw - xw(aw)^2]^{n-3}[aw - xw(aw)^2](aw)^2 \\
&\vdots \\
&= [aw - xw(aw)^2](aw)^{n-1} \\
&= (aw)^n - xw(aw)^{n+1} = 0.
\end{aligned}$$

Thus, $[1 - xw(aw)]aw \in R^{\mathrm{nil}}$, and then $aw[1 - xw(aw)] = aw - aw(xw)aw \in R^{\mathrm{nil}}$. Hence, $[1 - aw(xw)]aw \in R^{\mathrm{nil}}$. This implies that $aw[1 - aw(xw)] \in R^{\mathrm{nil}}$. That is, $aw - (aw)^2xw \in R^{\mathrm{nil}}$; hence, $a_2 = a - (aw)^2x \in R^{\mathrm{nil}}_w$. Therefore $a = a_1 + a_2$ is weak w-group decomposition of a, as required. □

Corollary 2.4. *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R$ has a weak w-group decomposition.
(2) There exists $x \in R$ such that

$$x = a(wx)^2, \quad \big((aw)^m\big)^*(aw)^2x = \big((aw)^m\big)^*a, \quad (aw)^n = xw(aw)^{n+1}$$

for some $m, n \in \mathbb{N}$.

Proof. (1) ⇒ (2) This is trivial by Theorem 2.3.

(2) ⇒ (1) By hypothesis, there exists $x \in R$ such that

$$x = a(wx)^2, \quad \big((aw)^m\big)^*(aw)^2x = \big((aw)^m\big)^*a, \quad (aw)^n = xw(aw)^{n+1}$$

for some $m, n \in \mathbb{N}$. Then

$$\begin{aligned}
&\big((aw)^n\big)^*(aw)^2x = \big(xw(aw)^{n+1}\big)^*(aw)^2x = \big(a(wx)^2w(aw)^{n+1}\big)^*(aw)^2x \\
&= \big((aw)(xw)^2(aw)^{n+1}\big)^*(aw)^2x = \big((aw)^m(xw)^{m+1}(aw)^{n+1}\big)^*(aw)^2x \\
&= \big((xw)^{m+1}(aw)^{n+1}\big)^*[((aw)^m)^*(aw)^2x] = \big((xw)^{m+1}(aw)^{n+1}\big)^*[\big((aw)^m\big)^*a] \\
&= \big((aw)^m(xw)^{m+1}(aw)^{n+1}\big)^*a = \big((aw)(xw)^2(aw)^{n+1}\big)^*a \\
&= \big(a(wx)^2w(aw)^{n+1}\big)^*a = \big(xw(aw)^{n+1}\big)^*a = \big((aw)^n\big)^*a.
\end{aligned}$$

In light of Theorem 2.3, we complete the proof. □

Theorem 2.5. *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R$ has a weak w-group decomposition.
(2) There exists unique $x \in R$ such that

$$x = a(wx)^2, \quad \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, \quad (aw)^n = xw(aw)^{n+1}$$

for some $n \in \mathbb{N}$.

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Proof. (2) ⇒ (1) Suppose that there exist $x, y \in R$ and $m, n \in \mathbb{N}$ such that

$$x = a(wx)^2, \quad \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, \quad (aw)^n = xw(aw)^{n+1};$$
$$y = a(wy)^2, \quad \big((aw)^m\big)^*(aw)^2y = \big((aw)^m\big)^*a, \quad (aw)^m = yw(aw)^{m+1}.$$

Choose $k = \max(m, n)$. Then

$$x = a(wx)^2, \quad \big((aw)^k\big)^*(aw)^2x = \big((aw)^k\big)^*a, \quad (aw)^k = xw(aw)^{k+1};$$
$$y = a(wy)^2, \quad \big((aw)^k\big)^*(aw)^2y = \big((aw)^k\big)^*a, \quad (aw)^k = yw(aw)^{k+1}.$$

Claim 1. $xw = yw$. By hypothesis, we see that xw and yw are both the weak group inverse of aw. In view of [25, Theorem 3.5], we have $xw = yw$.

Claim 2. $(aw)^2x = (aw)^2y$. Since $x = a(wx)^2$, we have $x = (aw)^{n-2}x(wx)^{n-2}$, and so $(aw)^2x = (aw)^nx(wx)^{n-2}$. Then

$$((aw)^2x)^*(aw)^2x - ((aw)^2x)^*a = (x(wx)^{n-2})^*[((aw)^n)^*(aw)^2x - ((aw)^n)^*a] = 0.$$

Therefore $((aw)^2x)^*(aw)^2x = ((aw)^2x)^*a$. Similarly, we have

$$\begin{aligned} ((aw)^2x)^*(aw)^2y &= ((aw)^2x)^*a, \\ ((aw)^2y)^*(aw)^2x &= ((aw)^2y)^*a, \\ ((aw)^2y)^*(aw)^2y &= ((aw)^2y)^*a. \end{aligned}$$

Let $z = (aw)^2x - (aw)^2y$. Then we check that

$$\begin{aligned} z^*z &= ((aw)^2x - (aw)^2y)^*((aw)^2x - (aw)^2y) \\ &= ((aw)^2x)^*(aw)^2x - ((aw)^2x)^*(aw)^2y \\ &\quad - ((aw)^2y)^*(aw)^2x + ((aw)^2y)^*(aw)^2y \\ &= ((aw)^2x)^*a - ((aw)^2x)^*a - ((aw)^2y)^*a + ((aw)^2y)^*a = 0. \end{aligned}$$

Since R is a ring with proper involution, we have $z = 0$; hence, $(aw)^2x = (aw)^2y$.

Claim 3. $x = y$. We see that

$$\begin{aligned} (xw)(awx) &= xwaw[(aw)^kx(wx)^k]w = [xw(aw)^{k+1}]x(wx)^k \\ &= (aw)^kx(wx)^k = x. \end{aligned}$$

Moreover, we have

$$\begin{aligned} (xw)^2(aw)^2x &= xw[xw(aw)^2](aw)^{k-1}x(wx)^{k-1} \\ &= xw[xw(aw)^{k+1}]x(wx)^{k-1} \\ &= xwaw[(aw)^{k-1}x(wx)^{k-1}] = (xw)(awx). \end{aligned}$$

Therefore $x = (xw)^2(aw)^2x$. Likewise, $y = (yw)^2(aw)^2y$. By using Claim 1 and Claim 2, we have

$$x = (xw)^2[(aw)^2x] = (yw)^2[(aw)^2y] = y,$$

as desired. □

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We denote x in Theorem 2.5 by $a_w^{\textcircled{w}}$, and call it the weak w-group inverse of a. We use $R_w^{\textcircled{w}}$ to stand for the set of all weak w-group invertible elements in R. We now derive

Corollary 2.6. *Let $a \in R_w^{\textcircled{w}}$. Then the following hold.*

(1) $a_w^{\textcircled{w}} = a_w^{\textcircled{w}} wawa_w^{\textcircled{w}}$.
(2) $(aw)(a_w^{\textcircled{w}} w) = (aw)^m (a_w^{\textcircled{w}} w)^m$ for any $m \in \mathbb{N}$.

Proof. (1) Let $x = a_w^{\textcircled{w}}$. In view of Theorem 2.5, we have $x = a(wx)^2$ and $(aw)^n = xw(aw)^{n+1}$ for some $n \in \mathbb{N}$. Then $x = (aw)^n x(wx)^n$. Hence,

$$\begin{aligned} a_w^{\textcircled{w}} wawa_w^{\textcircled{w}} &= (xwaw)[(aw)^n x(wx)^n] \\ &= [xw(aw)^{n+1}]x(wx)^n = (aw)^n x(wx)^n = x, \end{aligned}$$

as required.

(2) We easily see that

$$(aw)^2(a_w^{\textcircled{w}} w)^2 = aw[a(w(a_w^{\textcircled{w}})^2]w = (aw)(a_w^{\textcircled{w}} w).$$

By induction, we complete the proof. □

Theorem 2.7. *Let $a, x \in R$ and $w \in R^{-1}$. Then the following are equivalent:*

(1) $x = a_w^{\textcircled{w}}$.
(2) *There exists some $n \in \mathbb{N}$ such that*

$$x = a(wx)^2, \quad [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, \quad (aw)^n = xw(aw)^{n+1}.$$

In this case, $x = (aw)^{\textcircled{w}} w^{-1}$.

Proof. (1) $\Rightarrow$ (2) By hypothesis, a has the weak w-group decomposition $a = a_1 + a_2$. Let $x = (a_1)_w^{\#}$. In virtue of Theorem 2.3, there exists $n \in \mathbb{N}$ such that

$$x = a(wx)^2, \quad \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, \quad (aw)^n = xw(aw)^{n+1}.$$

Moreover, we have

$$\begin{aligned} (aw)^*(aw)^2xw &= (a_1w + a_2w)^*(a_1w + a_2w)^2(a_1)_w^{\#}w \\ &= (a_1w + a_2w)^*(a_1w + a_2w)a_1w(a_1)_w^{\#}w \\ &= (a_1w + a_2w)^*(a_1w)^2(a_1)_w^{\#}w \\ &= (a_1w + a_2w)^*a_1w = (a_1w)^*(a_1w). \end{aligned}$$

Therefore $[(aw)^*(aw)^2xw]^* = [(a_1w)^*(a_1w)]^* = (a_1w)^*(a_1w) = (aw)^*(aw)^2xw$, as desired.

(2) $\Rightarrow$ (1) By hypotheses, there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$x = a(wx)^2, \quad [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, \quad (aw)^n = xw(aw)^{n+1}.$$

Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$. As in the proof of Theorem 2.3, we prove that

$$a_2wa_1 = 0, a_1 \in R_w^{\#} \quad \text{and} \quad a_2 \in R_w^{\text{nil}}.$$

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5074.

Moreover, we verify that

$$\begin{aligned}
a_1^*a_2w &= [(aw)^2x]^*[a-(aw)^2x]w\\
&= [(aw)^2x]^*aw - [awx]^*[(aw)^*(aw)^2xw]\\
&= [(aw)^2x]^*aw - [awx]^*[(aw)^*(aw)^2xw]^*\\
&= [(aw)^2x]^*aw - [(aw)^*(aw)^2xwawx]^*\\
&= [(aw)^2x]^*aw - [(aw)^2xwawx]^*aw\\
&= [(aw)^2x]^*aw - [(aw)^2xwaw((aw)^nx(xw)^n)]^*aw\\
&= [(aw)^2x]^*aw - [(aw)^2(xw(aw)^{n+1})x(xw)^n)]^*aw\\
&= [(aw)^2x]^*aw - [(aw)^2((aw)^nx(xw)^n)]^*aw\\
&= [(aw)^2x]^*aw - [(aw)^2x]^*aw = 0.
\end{aligned}$$

As $w \in R^{-1}$, we deduce that $a_1^*a_2 = 0$. Therefore $a = a_1 + a_2$ is weak w-group decomposition of a.

Obviously, we have

$$xw = aw(xw)^2, \quad [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, \quad (aw)^n = xw(aw)^{n+1}.$$

Hence, $xw = (aw)^{Ⓦ}$. As $w \in R^{-1}$, we have $x = (aw)^{Ⓦ}w^{-1}$, as required. □

3. Equivalent characterizations

In this section we establish some equivalent characterizations of weak weighted group inverses. We now derive

Theorem 3.1. *Let $a \in R$. Then $a \in R_w^{Ⓦ}$ if and only if*

(1) *$a \in R^{D,w}$;*

(2) *There exists $x \in R$ such that*

$$xR = a^{D,w}R, \quad ((aw)^n)^*(aw)^2x = ((aw)^n)^*a, \quad (aw)^n = xw(aw)^{n+1}$$

for some $n \in \mathbb{N}$.

In this case, $a_w^{Ⓦ} = x$.

Proof. $\Longrightarrow$ In view of Theorem 2.3, there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$x = a(wx)^2, ((aw)^n)^*(aw)^2x = ((aw)^n)^*a, (aw)^n = xw(aw)^{n+1}.$$

Here, $x = a_w^{Ⓦ}$. Hence, $xw = (aw)(xw)^2$ and $(aw)^n = (xw)(aw)^{n+1}$. By virtue of [27, Lemma 2.2], $aw \in R^D$. This implies that $a \in R^{D,w}$.

We claim that $xR = a^{D,w}R$. By virtue of Theorem 2.3, there exist $z, y \in R$ such that

$$a = z + y, z^*y = ywz = 0, z \in R_w^{\#}, y \in R_w^{\text{nil}}.$$

Moreover, we have $x = z_w^{\#}$. In view of Lemma 2.1, $zw \in R^{\#}$. Write $(yw)^{k+1} = 0$ for some $k \in \mathbb{N}$. Since $aw = zw + yw$, $yw \in R^{\text{nil}}$ and $(yw)(zw) = 0$, it follows by

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version
This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.5074.

[1, Corollary 3.5] that $aw \in R^D$ and

$$(aw)^D = (zw)^{\#} + \sum_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n.$$

Since $x = a(wx)^2$, we see that $x = awxwx = (aw)^m(xw)^m x$; hence,

$$\begin{aligned} xwawx &= xwaw(aw)^m(xw)^m x = [xw(aw)^{m+1}](xw)^m x \\ &= (aw)^m(xw)^m x = (aw)(xw)x = a(wx)^2 = x. \end{aligned}$$

Obviously, we have $(zw)^{\#}(zw)z_w^{\#} = z_w^{\#}(wz)(wz)^{\#}$. Then we directly verify that

$$\begin{aligned} (aw)(aw)^D x &= (aw)^D(zw + yw)z_w^{\#} \\ &= (aw)^D zwz_w^{\#} \\ &= [(zw)^{\#} + \sum_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n](zw)z_w^{\#} \\ &= (zw)^{\#}(zw)z_w^{\#} = x. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} xw(aw)(aw)^D &= z_w^{\#}w(zw + yw)[(zw)^{\#} + \sum_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n] \\ &= z_w^{\#}(wzw)[(zw)^{\#} + \sum_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n] \\ &= (zw)^{\#}wzw(zw)^{\#} + z_w^{\#}(wzw)\sum_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n \\ &= (zw)^{\#} + \sum_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n = (aw)^D. \end{aligned}$$

Accordingly, $xR = xwR = a^{D,w}R$, as asserted.

$\Longleftarrow$ We directly check that

$$\begin{aligned} awxw(aw)^D &= awxw(aw)^{n+1}[(aw)^D]^{n+2} \\ &= aw[xw(aw)^{n+1}][(aw)^D]^{n+2} \\ &= aw(aw)^n[(aw)^D]^{n+2} = (aw)^D. \end{aligned}$$

Since $a^{D,w} = (aw)^D a(wa)^D$, we have $awxwa^{D,w} = a^{D,w}$, and so $(1 - awxw)a^{D,w} = 0$. As $xR = a^{D,w}R$, we derive that $(1 - awxw)x = 0$. Therefore $x = awxwx = a(wx)^2$. By virtue of Theorem 2.3, $a \in R_w^{Ⓦ}$. In this case, $a_w^{Ⓦ} = x$, required. □

Corollary 3.2. *Let $a \in R$. Then $a \in R_w^{Ⓦ}$ if and only if*

(1) $a \in R^{D,w}$;

Submitted: December 16, 2024
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Revista de la Unión Matemática Argentina Accepted article · Early view version
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(2) There exists $x \in R$ such that

$$xwawx = x, \quad xR = (aw)^m R = (aw)^{m+1} R, \quad a^*(aw)^m R \subseteq x^* R$$

for some $m \in \mathbb{N}$.

In this case, $a_w^{Ⓦ} = x$.

Proof. $\Longrightarrow$ In view of Theorem 3.1, $a \in R^{D,w}$ and there exist $x \in R$ and $m \in \mathbb{N}$ such that

$$x = a(wx)^2, \quad \big((aw)^m\big)^*(aw)^2 x = \big((aw)^m\big)^* a, \quad (aw)^m = xw(aw)^{m+1}.$$

Then $x = awxwx = (aw)^m(xw)^m x$; hence,

$$\begin{aligned} xwawx &= xwaw(aw)^m(xw)^m x = [xw(aw)^{m+1}](xw)^m x \\ &= (aw)^m(xw)^m x = (aw)(xw)x = a(wx)^2 = x. \end{aligned}$$

Thus, $x = awxwx = (aw)^m x(wx)^m$; whence $xR \subseteq (aw)^m R$. On the other hand, $(aw)mR \subseteq xR$. Thus, $xR = (aw)^m R$. Obviously, $(aw)^{m+1}R \subseteq (aw)^m R$. On the other hand, $(aw)^m = xw(aw)^{m+1} = (aw)^{m+1}x(wx)^{m+1}w(aw)^{m+1}$; hence, $(aw)^m R \subseteq (aw)^{m+1} R$. This implies that $(aw)^m R = (aw)^{m+1} R$. Moreover, $\big((aw)^m\big)^*(aw)^2 x = \big((aw)^m\big)^* a$; hence, $a^*(aw)^m = x^*[\big((aw)^m\big)^*(aw)^2]^*$. Accordingly, $a^*(aw)^m R \subseteq x^* R$, as required.

$\Longleftarrow$ By hypothesis, $a \in R^{D,w}$ and there exists $x \in R$ such that

$$xwawx = x, xR = (aw)^m R = (aw)^{m+1} R, a^*(aw)^m R \subseteq x^* R$$

for some $m \in \mathbb{N}$.

Claim 1. $xR = a^{D,w}R$. Let $k = i(aw)$. Then $xR = (aw)^{m+k}R$. Since $(aw)^k = (aw)^D(aw)^{k+1}$, we have $xR = (aw)^D(aw)^{m+k+1}R = (aw)^D R = a^{D,w}wR = a^{D,w}R$, as desired.

Claim 2. $((aw)^m)^*(aw)^2 x = ((aw)^m)^* a$. Since $x = xwawx$, we have $x(1 - wawx) = 0$. Hence $(1 - wawx)^* x^* = 0$, and then $(1 - wawx)^* a^*(aw)^m = 0$. Therefore $[(aw)^m]^*(aw)^2 x = [(aw)^m]^* a$, as required.

Claim 3. $(aw)^m = xw(aw)^{m+1}$. Since $(1 - xwaw)x = 0$, we see that $(1 - xwaw)(aw)^m = 0$. Thus $(aw)^m = xw(aw)^{m+1}$.

Therefore $a \in R_w^{Ⓦ}$ by Theorem 3.1. □

Theorem 3.3. *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R_w^{Ⓦ}$.

(2) $a \in R^{D,w}$ *and there exist* $n \in \mathbb{N}, x \in R$ *such that*

$$x = a(wx)^2, \quad ((aw)^n)^*(aw)^2 x = ((aw)^n)^* a.$$

(3) $a \in R^{D,w}$ *and there exists some* $y \in R$ *such that*

$$(a^{D,w}w)^* a^{D,w} wy = (a^{D,w}w)^* a.$$

In this case, $a_w^{Ⓦ} = (aw)(aw)^D x = (a^{D,w}w)^3 y$.

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

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Proof. (1) ⇒ (2) By virtue of Theorem 3.1, $a \in R^{D,w}$ and there exists $x \in R$ such that $\big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a$, as desired.

(2) ⇒ (3) By hypothesis, $a \in R^{D,w}$ and there exist $n \in \mathbb{N}, x \in R$ such that $x = a(wx)^2, ((aw)^n)^*(aw)^2x = ((aw)^n)^*a$. Then we see that $((aw)^D)^*(aw)^2x = ((aw)^D)^*a$. Clearly, $a^{D,w}w = (aw)^D$, and so $(a^{D,w}w)^*(aw)^2x = (a^{D,w}w)^*a$.

Since $x = a(wx)^2 = (aw)x(wx) = (aw)^nx(wx)^n$ for any $n \in \mathbb{N}$, we observe that

$$\begin{aligned}&(a^{D,w}w)^*(aw)^2x - (a^{D,w}w)^*(aw)^D(aw)^3x\\ &\qquad = (a^{D,w}w)^*(aw)^{n+2}x(wx)^n - (a^{D,w}w)^*(aw)^D(aw)^{n+3}x(wx)^n\\ &\qquad = (a^{D,w}w)^*[(aw)^n - (aw)^D(aw)^{n+1}](aw)^2x(wx)^n.\end{aligned}$$

Since $(aw)^n = (aw)^D(aw)^{n+1}$, we get

$$(a^{D,w}w)^*(aw)^2x - (a^{D,w}w)^*(aw)^D(aw)^3x = 0;$$

hence, $(a^{D,w}w)^*(aw)^D(aw)^3x = (a^{D,w}w)^*a$. Set $y = (aw)^3x$. Then we verify that

$$\begin{aligned}(a^{D,w}w)^*a^{D,w}wy &= (a^{D,w}w)^*[a^{D,w}w]awx\\ &= (a^{D,w}w)^*(aw)^Dawx = (a^{D,w}w)^*a,\end{aligned}$$

as desired.

(3) ⇒ (1) By hypothesis, $(a^{D,w}w)^*a^{D,w}wy = (a^{D,w}w)^*a$ for a $y \in R$. Then $((aw)^D)^*(aw)^Dy = ((aw)^D)^*a$. It is easy to verify that

$$[aw(aw)^D]^*aw(aw)^D = [aw(aw)^D]^*aw[(aw)^D]^2yw(aw)^D.$$

Since the involution $*$ is proper, we get $aw(aw)^D = (aw)^Dyw(aw)^D$. Let $z = ((aw)^D)^3y$. Then we verify that

$$\begin{aligned}a(wz)^2 &= aw((aw)^D)^3yw((aw)^D)^3y\\ &= ((aw)^D)^2yw((aw)^D)^3y\\ &= (aw)^D[(aw)^Dyw(aw)^D]((aw)^D)^2y\\ &= (aw)^Daw(aw)^D((aw)^D)^2y\\ &= ((aw)^D)^3y = z;\\ \big((aw)^D\big)^*(aw)^2z &= \big((aw)^D\big)^*(aw)^2((aw)^D)^3y\\ &= \big((aw)^D\big)^*(aw)^Dy\\ &= \big((aw)^D\big)^*a.\end{aligned}$$

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

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Set $n=i(aw)$. Then $(aw)^n=(aw)^D(aw)^{n+1}$. Thus,

$$\begin{aligned}(aw)^n-zw(aw)^{n+1}&=[(aw)^n-(aw)^D(aw)^{n+1}]+[(aw)^D(aw)^{n+1}-zw(aw)^{n+1}]\\&=(aw)^D(aw)^{n+1}-((aw)^D)^3yw(aw)^{n+1}\\&=-((aw)^D)^3yw[1-(aw)^D(aw)](aw)^{n+1}\\&\quad+[(aw)^D(aw)^{n+1}-((aw)^D)^3yw(aw)^D(aw)^{n+2}]\\&=(aw)^D(aw)^{n+1}-((aw)^D)^3yw(aw)^D(aw)^{n+2}\\&=[(aw)^D]^2[(aw)(aw)^D](aw)^{n+2}-((aw)^D)^3yw(aw)^D(aw)^{n+2}\\&=[(aw)^D]^2[(aw)^Dyw(aw)^D](aw)^{n+2}\\&\quad-((aw)^D)^3yw(aw)^D(aw)^{n+2}\\&=0.\end{aligned}$$

That is, $(aw)^n=zw(aw)^{n+1}$. Accordingly, $a\in R_w^{Ⓦ}$. In this case,

$$a_w^{Ⓦ}=((aw)^D)^3y=(a^{D,w}w)^3y=(a^{D,w}w)^3(aw)^3x=(aw)(aw)^Dx,$$

as asserted. □

Corollary 3.4. *Let $a\in R$. Then $a\in R_w^{Ⓦ}$ if and only if*

(1) $a\in R^{D,w}$;
(2) There exists an idempotent $q\in R$ such that
$$a^{D,w}R=qR\quad\text{and}\quad(a^{D,w})^*awq=(a^{D,w})^*a.$$

Proof. $\Longrightarrow$ By using Theorem 3.3, $a\in R^{D,w}$ and there exists some $y\in R$ such that

$$(a^{D,w}w)^*a^{D,w}wy=(a^{D,w}w)^*a.$$

Then

$$(a^{D,w}w)^*(aw)[(aw)^D]^2y=(a^{D,w}w)^*a.$$

Set $q=[(aw)^D]^2y$. Then $(a^{D,w})^*awq=(a^{D,w})^*a$. Obviously, $qR\subseteq a^{D,w}wR$. Moreover, $a_w^{Ⓦ}=(a^{D,w}w)^3y=[(aw)^D]^3y$. Then $a_w^{Ⓦ}=(aw)^Dq$, and then $q=(aw)[(aw)^Dq]=awa_w^{Ⓦ}$. Thus $qw=(aw)(a_w^{Ⓦ}w)$, and so $qw(aw)^D=(aw)(a_w^{Ⓦ}w)(aw)^D$. We observe that

$$\begin{aligned}(aw)^D-qw(aw)^D&=(aw)^D-(aw)(a_w^{Ⓦ}w)(aw)^D\\&=(aw)^{n+1}[(aw)^D]^{n+2}-(aw)(a_w^{Ⓦ}w)(aw)^{n+1}[(aw)^D]^{n+2}\\&=aw[(aw)^n-(a_w^{Ⓦ}w)(aw)^{n+1}][(aw)^D]^{n+2}.\end{aligned}$$

Since $(aw)^n=(a_w^{Ⓦ}w)(aw)^{n+1}$, we get $(aw)^D-qw(aw)^D=0$; hence, $a^{D,w}w=(aw)^D=qw(aw)^D$. Thus $a^{D,w}wR\subseteq qR$. Accordingly, $a^{D,w}R=a^{D,w}wR=qR$, as desired.

$\Longleftarrow$ By hypothesis, $a\in R^{D,w}$ and there exists an idempotent $q\in R$ such that

$$a^{D,w}R=qR\quad\text{and}\quad(a^{D,w})^*awq=(a^{D,w})^*a.$$

Submitted: December 16, 2024
Accepted: July 18, 2025
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Revista de la Unión Matemática Argentina | Accepted article · Early view version

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Since $a^{D,w}R = a^{D,w}wR = (aw)^D R$, we write $q = (aw)^D z$ with $z \in R$. Choose $y = awz$. Then

$$\begin{aligned}(a^{D,w}w)^* a^{D,w} wy &= (a^{D,w}w)^* a^{D,w} w(aw)z \\ &= (a^{D,w}w)^* (aw)^D (aw)z \\ &= (a^{D,w}w)^* aw[(aw)^D z] \\ &= (a^{D,w}w)^* awq = (a^{D,w}w)^* a,\end{aligned}$$

the result follows by Theorem 3.3. □

4. Relations with weighted core-EP inverses

In this section we investigate relations between weighted weak group and weighted core-EP inverses. We use $a_w^{\textcircled{D}}$ to denote the w-core-EP inverse of a. Our starting point is the following.

Theorem 4.1. *Let $a \in R_w^{\textcircled{D}}$. Then $a \in R_w^{\textcircled{W}}$ and*

$$a_w^{\textcircled{W}} = (a_w^{\textcircled{D}})^2 a = \big(awa_w^{\textcircled{D}}a\big)_w^{\#}.$$

Proof. Let $x = a_w^{\textcircled{D}}$. In view of [22, Theorem 2.4], we have some nonnegative integer k such that $awx^2 = x$, $x(aw)^{k+1}a = (aw)^k a$, $xawx = x$, $awx(aw)^k a = (aw)^k a$, $(awx)^* = awx$. Set $n = k+1$. Then $awx(aw)^n = [awx(aw)^k a]w = [(aw)^k a]w = (aw)^n = x(aw)^{n+1}$. Obviously, we have $x = (aw)^{\textcircled{D}}$. Let $z = x^2 a$. Then we check that

$$\begin{aligned}a(wz)^2 &= aw[x^2a]w[x^2a] \\ &= [awx^2][awx^2]a = x^2a = z, \\ \big((aw)^n\big)^*(aw)^2 z &= \big((aw)^n\big)^*(aw)^2[x^2a] \\ &= \big((aw)^n\big)^*[(aw)^2x^2]a = \big((aw)^n\big)^*(awx)a \\ &= \big((aw)^n\big)^*(awx)^*a = [awx(aw)^n]^*a = [(aw)^n]^*a.\end{aligned}$$

Moreover, we see that

$$\begin{aligned}(aw)^n - zw(aw)^{n+1} &= (aw)^n - [x^2a]w(aw)^{n+1} \\ &= (aw)^n - x(aw)^{n+1} + x(aw)^{n+1} - x^2(aw)^{n+2} \\ &= [(aw)^n - x(aw)^{n+1}] + x[(aw)^n - x(aw)^{n+1})]aw = 0,\end{aligned}$$

and so $(aw)^n = zw(aw)^{n+1}$. Therefore $a_w^{\textcircled{W}} = [a_w^{\textcircled{D}}]^2 a$.

One directly checks that

$$\begin{aligned}awa_w^{\textcircled{D}}a &= aw[aw(a_w^{\textcircled{D}})^2]a \\ &= (aw)^2(a_w^{\textcircled{D}})^2]a \\ &= (aw)^2 a_w^{\textcircled{W}}.\end{aligned}$$

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version
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As in the proof of Theorem 2.3, we have $(aw)^2a_w^{Ⓦ} \in R_w^{\#}$ and

$$a_w^{Ⓦ} = \big((aw)^2a_w^{Ⓦ}\big)_w^{\#} = \big(awa_w^{Ⓓ}a\big)_w^{\#},$$

as asserted. □

Let $A, W \in \mathbb{C}^{n\times n}$. By using Theorem 4.1 and [22, Theorem 2.14], we have

$$A_W^{Ⓦ} = (A_W^{Ⓓ})^2A = [(AW)^{Ⓓ}]^2A = \big(AWA_W^{Ⓓ}A\big)_W^{\#}.$$

Corollary 4.2. *Let $a \in R_w^{Ⓓ}$. Then $a_w^{Ⓦ} = x$ if and only if $a(wx)^2 = x$, $awx = a_w^{Ⓓ}a$.*

Proof. $\Longrightarrow$ In view of Theorem 4.1, $a \in R_w^{Ⓦ}$ and $x := a_w^{Ⓦ} = (a_w^{Ⓓ})^2a$. $a(wx)^2 = x$. Moreover, we have

$$awx = aw(a_w^{Ⓓ})^2a = [aw(a_w^{Ⓓ})^2]a = a_w^{Ⓓ}a,$$

as required.

$\Longleftarrow$ By hypotheses, $a(wx)^2 = x$, $awx = a_w^{Ⓓ}a$. Then we have

$$\begin{aligned} x = a(wx)^2 &= (awx)wx = [a_w^{Ⓓ}a](wx) \\ &= a_w^{Ⓓ}(awx) = a_w^{Ⓓ}(a_w^{Ⓓ}a) = (a_w^{Ⓓ})^2a. \end{aligned}$$

In light of Theorem 4.1, $x = a_w^{Ⓦ}$, as desired. □

Corollary 4.3. *Let $A, W \in \mathbb{C}^{n\times n}$. Then X is the weak W-group inverse of A if and only if X satisfies*

$$A(WX)^2 = X, AWX = A_W^{Ⓓ}A.$$

Proof. Let $\mathbb{C}^{n\times n}$ be the ring of all $n\times n$ complex matrices, with conjugate transpose $*$ as the involution. Then the involution $*$ is proper. Therefore we obtain the result by Corollary 4.2. □

Theorem 4.4. *Let $a \in R_w^{Ⓓ}$. Then the following are equivalent:*

(1) $a_w^{Ⓓ} = a_w^{Ⓦ}w$.
(2) $awa_w^{Ⓓ} = a_w^{Ⓓ}aw$.

Proof. (1) ⇒ (2) In view of Theorem 4.1, $a_w^{Ⓦ} = (a_w^{Ⓓ})^2a$. Hence,

$$\begin{aligned} awa_w^{Ⓓ} = awa_w^{Ⓦ}w &= aw[(a_w^{Ⓓ})^2a]w \\ &= [aw(a_w^{Ⓓ})^2]aw = a_w^{Ⓓ}aw, \end{aligned}$$

as desired.

(2) ⇒ (1) Since $awa_w^{Ⓓ} = a_w^{Ⓓ}aw$, it follows by Theorem 4.1 that $a \in R_w^{Ⓦ}$ and

$$\begin{aligned} a_w^{Ⓦ}w = (a_w^{Ⓓ})^2aw &= a_w^{Ⓓ}[a_w^{Ⓓ}aw] \\ &= a_w^{Ⓓ}[awa_w^{Ⓓ}] = a_w^{Ⓓ}, \end{aligned}$$

as required. □

Corollary 4.5. *Let $a \in R^{Ⓓ}$. Then the following are equivalent:*

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Revista de la Unión Matemática Argentina Accepted article · Early view version
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(1) $a^{Ⓓ} = a^{Ⓦ}$.
(2) $aa^{Ⓓ} = a^{Ⓓ}a$.

Proof. This is obvious by choosing $w = 1$ in Theorem 4.4. □

We turn to investigate when the weighted group inverse possesses the commuting property.

Lemma 4.6. *Let $a \in \mathcal{A}_w^{Ⓦ}$. Then $awa \in \mathcal{A}_w^{Ⓦ}$. In this case,*

$$(awa)_w^{Ⓦ} = (a_w^{Ⓦ}w)^3 a.$$

Proof. Let $x = (a_w^{Ⓦ}w)^3 a$ and $c = awa$. Then we can find $n \in \mathbb{N}$ such that

$$((aw)^n)^*(aw)^2 a_w^{Ⓦ} = [(aw)^n]^* a \quad \text{and} \quad (aw)^n = a_w^{Ⓦ}w(aw)^{n+1}.$$

Accordingly, we have

$$\begin{aligned} c(wx)^2 &= awaw(a_w^{Ⓦ}w)^3 aw(a_w^{Ⓦ}w)^3 a \\ &= aw[a(wa_w^{Ⓦ})^2 w](a_w^{Ⓦ}w)aw(a_w^{Ⓦ}w)^3 a \\ &= a(wa_w^{Ⓦ})^2 waw(a_w^{Ⓦ}w)^3 a \\ &= [a_w^{Ⓦ}wawa_w^{Ⓦ}]w(a_w^{Ⓦ}w)^2 a \\ &= a_w^{Ⓦ}wa_w^{Ⓦ}wa_w^{Ⓦ}wa \\ &= (a_w^{Ⓦ}w)^3 a = x, \\ \big((cw)^n\big)^*(cw)^2 x &= ((aw)^{2n})^*(aw)^4(a_w^{Ⓦ}w)^3 a \\ &= ((aw)^{2n})^*(aw)^2[(aw)^2(a_w^{Ⓦ}w)^2](a_w^{Ⓦ}w)a \\ &= ((aw)^{2n})^*(aw)^2(aw)(a_w^{Ⓦ}w)(a_w^{Ⓦ}w)a \\ &= ((aw)^{2n})^*(aw)^2[a(wa_w^{Ⓦ})^2]wa \\ &= [(aw)^n]^*[((aw)^n)^*(aw)^2 a_w^{Ⓦ}]wa \\ &= [(aw)^n]^*[((aw)^n)^* a]wa \\ &= [(aw)^{2n}]^*(awa) \\ &= \big((cw)^n\big)^* c. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} (cw)^n - xw(cw)^{n+1} &= (aw)^{2n} - (a_w^{Ⓦ}w)^3 aw(aw)^{2n+2} \\ &= (aw)^{2n} - (a_w^{Ⓦ}w)^2 a_w^{Ⓦ}w(aw)^{2n+3} \\ &= (aw)^{2n} - a_w^{Ⓦ}w(aw)^{2n+1} \\ &= [(aw)^n - a_w^{Ⓦ}w(aw)^{n+1}](aw)^n = 0. \end{aligned}$$

Hence, $(cw)^n = xw(cw)^{n+1}$. Therefore we complete the proof. □

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version
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Theorem 4.7. *Let $a \in R_w^{\textcircled{w}}$. Then the following are equivalent:*

(1) $awa_w^{\textcircled{w}} = a_w^{\textcircled{w}}wa$.
(2) $(awa)_w^{\textcircled{w}} = a_w^{\textcircled{w}}wa_w^{\textcircled{w}}$.
(3) *a has a weak w-group decomposition $a = a_1 + a_2$ with $a_1wa_2 = 0$.*

Proof. Let $a = a_1 + a_2$ be the weak w-group decomposition of a.

(1) ⇔ (3) We check that

$$awa_w^{\textcircled{w}} = aw(a_1)_w^{\#} = (a_1w + a_2w)(a_1)_w^{\#} = a_1w(a_1)_w^{\#},$$
$$a_w^{\textcircled{w}}wa = (a_1)_w^{\#}wa = (a_1)_w^{\#}(wa_1 + wa_2) = (a_1)_w^{\#}wa_1 + (a_1)_w^{\#}wa_2.$$

Thus, $(a_1)_w^{\#}wa_2 = 0$ if and only if $awa_w^{\textcircled{w}} = a_w^{\textcircled{w}}wa$.

If $a_1wa_2 = 0$, then

$$(a_1)_w^{\#}wa_2 = [(a_1)_w^{\#}wa_1w(a_1)_w^{\#}]wa_2 = [(a_1)_w^{\#}w]^2[a_1wa_2] = 0.$$

If $(a_1)_w^{\#}wa_2 = 0$, then

$$a_1wa_2 = [a_1w(a_1)_w^{\#}wa_1]wa_2 = [a_1w]^2[(a_1)_w^{\#}wa_2] = 0.$$

Thus, $a_1wa_2 = 0$ if and only if $(a_1)_w^{\#}wa_2 = 0$. Accordingly, $awa_w^{\textcircled{w}} = a_w^{\textcircled{w}}wa$ if and only if $a_1wa_2 = 0$.

(2) ⇔ (3) In light of Lemma 4.6, we verify that

$$\begin{aligned}
(awa)_w^{\textcircled{w}} &= (a_w^{\textcircled{w}}w)^3a \\
&= ((a_1)_w^{\#}w)^3(a_1 + a_2) \\
&= ((a_1)_w^{\#}w)^3a_1 + ((a_1)_w^{\#}w)^3a_2 \\
&= ((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_1] + ((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] \\
&= ((a_1)_w^{\#}w)^2[a_1w(a_1)_w^{\#}] + ((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] \\
&= ((a_1)_w^{\#}w)[(a_1)_w^{\#}wa_1w(a_1)_w^{\#}] + ((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] \\
&= (a_1)_w^{\#}w(a_1)_w^{\#} + ((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] \\
&= (a_1)_w^{\#}w(a_1)_w^{\#} + ((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] \\
&= a_w^{\textcircled{w}}wa_w^{\textcircled{w}} + ((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2].
\end{aligned}$$

If $a_1wa_2 = 0$, as in the argument above, we have $(a_1)_w^{\#}wa_2 = 0$. Hence $((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] = 0$.

If $((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] = 0$, then $[a_1w(a_1)_w^{\#}w(a_1)_w^{\#}]w[(a_1)_w^{\#}wa_2] = 0$; hence,

$$[(a_1)_w^{\#}wa_1w(a_1)_w^{\#}]w[(a_1)_w^{\#}wa_2] = 0.$$

This implies that $(a_1)_w^{\#}w[(a_1)_w^{\#}wa_2] = 0$. Moreover, $[a_1(w(a_1)_w^{\#})^2]wa_2 = 0$. We infer that $(a_1)_w^{\#}wa_2 = 0$. Similarly to the preceding argument, we have $a_1wa_2 = 0$. Then $a_1wa_2 = 0$ if and only if $((a_1)_w^{\#}w)^2[(a_1)_w^{\#}wa_2] = 0$. Therefore $a_1wa_2 = 0$ if and only if $(awa)_w^{\textcircled{w}} = a_w^{\textcircled{w}}wa_w^{\textcircled{w}}$, as asserted. □

Corollary 4.8. *Let $a \in R_w^{\textcircled{w}}$ and $m \in \mathbb{N}$. Then the following are equivalent:*

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

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(1) $awa^{\textcircled{w}}_w = a^{\textcircled{w}}_w wa$.
(2) $(aw)^m a^{\textcircled{w}}_w = a^{\textcircled{w}}_w (wa)^m$.

Proof. (1) ⇒ (2) Since $awa^{\textcircled{w}}_w = a^{\textcircled{w}}_w wa$, we verify that

$$\begin{aligned}(aw)^2 a^{\textcircled{w}}_w &= aw[awa^{\textcircled{w}}_w] = aw[a^{\textcircled{w}}_w wa] = [awa^{\textcircled{w}}_w](wa)\\ &= [(a^{\textcircled{w}}_w wa](wa) = a^{\textcircled{w}}_w (wa)^2.\end{aligned}$$

By iteration of this process, we prove that $(aw)^m a^{\textcircled{w}}_w = a^{\textcircled{w}}_w (wa)^m$.

(2) ⇒ (1) Let $a = a_1 + a_2$ be the weak w-group decomposition of a. Let $m = 2$. Since $a_2wa_1 = 0$, we derive

$$\begin{aligned}(aw)^2 &= (a_1w + a_2w)^2 = (a_1w)^2 + (a_1w)(a_2w) + (a_2w)^2;\\ (wa)^2 &= (wa_1 + wa_2)^2 = (wa_1)^2 + (wa_1)(wa_2) + (wa_2)^2.\end{aligned}$$

Clearly, $a_2wa^{\textcircled{w}}_w = a_2w(a_1)^{\#}_w = (a_2wa_1)w[(a_1)^{\#}_w]^2 = 0$. Then

$$\begin{aligned}(aw)^2 a^{\textcircled{w}}_w &= (a_1w)^2 a^{\textcircled{w}}_w + (a_1w)(a_2w)a^{\textcircled{w}}_w + (a_2w)^2 a^{\textcircled{w}}_w\\ &= (a_1w)^2 a^{\textcircled{w}}_w;\\ a^{\textcircled{w}}_w (wa)^2 &= a^{\textcircled{w}}_w (wa_1)^2 + a^{\textcircled{w}}_w (wa_1)(wa_2) + a^{\textcircled{w}}_w (wa_2)^2.\end{aligned}$$

As $(a_1w)^2 a^{\textcircled{w}}_w = (a_1w)^2(a_1)^{\#}_w = (a_1)^{\#}_w (wa_1)^2 = a^{\textcircled{w}}_w (wa_1)^2$, we deduce that

$$[a_1w(a_1)^{\#}_w](wa_2) + (a_1)^{\#}_w (wa_2)^2 = (a_1)^{\#}_w (wa_1)(wa_2) + (a_1)^{\#}_w (wa_2)^2 = 0.$$

As $wa_2 \in R^{\text{nil}}$, we write $(wa_2)^k = 0$ for some $k \in \mathbb{N}$. Then $[wa_1w(a_1)^{\#}_w](wa_2)^{k-1} = 0$. By iteration of this process, we have $[wa_1w(a_1)^{\#}_w]^{k-1}(wa_2) = 0$. As $a_1 \in R^{\#}_w$, we see that $a_1 = (a_1w)^2(a_1)^{\#}_w = a_1[wa_1w(a_1)^{\#}_w] = a_1[wa_1w(a_1)^{\#}_w]^{k-1}$, we deuce that $a_1wa_2 = 0$. By using Theorem 4.7, $awa^{\textcircled{w}}_w = a^{\textcircled{w}}_w wa$. The general case can be proved in a similar way. □

Corollary 4.9. *Let $a \in R^{\textcircled{w}}_w$ and $i(a^{D,w}) = k$. Then the following hold:*

(1) $awa^{\textcircled{w}}_w = a^{\textcircled{w}}_w wa$.
(2) $(aw)^k a^{\textcircled{w}}_w = a^{D,w}(wa)^k$.

Proof. In view of Theorem 3.1, $a \in R^{D,w}$. Let $x = a^{\textcircled{w}}_w$. Then $a(wx)^2 = x, (aw)^k = (xw)(aw)^{k+1}$. Hence $(aw)(xw)^2 = xw, (aw)^k = (xw)(aw)^{k+1}$. In view of [27, Lemma 2.2], $aw \in R^D$ and $(aw)^D = (xw)^{k+1}(aw)^k$. Also we have $wa(wx)^2 = wx, (wa)^{k+1} = (wx)(wa)^{k+2}$. By [27, Lemma 2.2] again, $(wa)^D = (wx)^{k+2}(wa)^{k+1}$.

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version
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Accordingly,

$$\begin{aligned} a^{D,w} &= (aw)^D a(wa)^D = (xw)^{k+1}(aw)^k a(wx)^{k+2}(wa)^{k+1} \\ &= (xw)^{k+1}(aw)^{k+1}(xw)^{k+1}x(wa)^{k+1} \\ &= (xw)^{k+1}(aw)(xw)x(wa)^{k+1} = (xw)^{k+1}[a(wx)^2](wa)^{k+1} \\ &= (xw)^{k+1}x(wa)^{k+1} = x(wx)^{k+1}(wa)^{k+1}. \end{aligned}$$

One directly verifies that

$$\begin{aligned} a^{D,w}(wa)^k &= x(wx)^{k+1}(wa)^{k+1}(wa)^k \\ &= x(wx)^{k+1}(wa)^{2k+1} = x(wx)^k[(wx)(wa)^{k+1}](wa)^k \\ &= x(wx)^k(wa)^{2k} = x(wx)^{k-1}[(wx)(wa)^{k+1}](wa)^{k-1} \\ &= x(wx)^{k-1}(wa)^{2k-1} = \cdots = x(wx)(wa)^{k+1} \\ &= x(wa)^k = a_w^{\textcircled{w}}(wa)^k. \end{aligned}$$

This completes the proof by Corollary 4.8. □

Theorem 4.10. *Let $a \in R_w^{\textcircled{w}}$. Then the following are equivalent:*

(1) $awa_w^{\textcircled{w}} = a_w^{\textcircled{w}}wa$.
(2) $(aw)^D a_w^{\textcircled{w}} = a_w^{\textcircled{w}}(wa)^D$.

Proof. (1) ⇒ (2) In view of Theorem 3.1, $aw \in R^D$. By using Cline's formula (see [13, Theorem 2.2]), $wa \in R^D$. Therefore $(aw)^D a_w^{\textcircled{w}} = a_w^{\textcircled{w}}(wa)^D$ by [2, Theorem 2.3].

(2) ⇒ (1) By virtue of Theorem 2.3, there exist $z, y \in R$ such that

$$a = z + y, z^* y = ywz = 0, z \in R_w^{\#}, y \in R_w^{\text{nil}}.$$

Explicitly, $a_w^{\textcircled{w}} = z_w^{\#}$ and $(yw)^k = (wy)^k = 0$ for some $k \in \mathbb{N}$. Since $aw = zw + yw, wa = wz + wy$ and $(yw)(zw) = (wy)(wz) = 0$, it follows by [1, Corollary 3.5] that

$$(aw)^D = (zw)^D + \sum_{i=1}^{k-1}((zw)^D)^{i+1}(yw)^i,$$
$$(wa)^D = (wa)^D + \sum_{i=1}^{k-1}((wz)^D)^{i+1}(wy)^i.$$

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Accepted: July 18, 2025
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It is easy to verify that

$$\begin{aligned}(aw)^D a_w^{Ⓦ} &= [(zw)^D + \sum_{i=1}^{k-1}((zw)^D)^{i+1}(yw)^i]z_w^{\#} = (zw)^D z_w^{\#},\\ a_w^{Ⓦ}(wa)^D &= z_w^{\#}[(wz)^D + \sum_{i=1}^{k-1}((wz)^D)^{i+1}(wy)^i]\\ &= z_w^{\#}(wz)^D + z_w^{\#}\sum_{i=1}^{k-1}((wz)^{\#})^{i+1}(wy)^i.\end{aligned}$$

Sine $(zw)z_w^{\#} = z_w^{\#}(wz)$, it follows by [2, Theorem 2.3] that $(zw)^D z_w^{\#} = z_w^{\#}(wz)^D$, we have

$$z_w^{\#}\sum_{i=1}^{k-1}((wz)^D)^{i+1}(wy)^i = 0.$$

By Cline's formula (see [13, Theorem 2.2]), we have $wzwz_w^{\#} = wzw\big([(zw)^D]^2 z\big) = wz\big(w[(zw)^D]^2 z\big) = wz(wz)^D$. Then

$$\sum_{i=1}^{k-1}((wz)^D)^{i+1}(wy)^i = wzw\big[z_w^{\#}\sum_{i=1}^{k-1}((wz)^D)^{i+1}(wy)^i\big] = 0.$$

That is,

$$[(wz)^D]^2(wy) + [(wz)^D]^3(wy)^2 + \cdots + [(wz)^D]^k(wy)^{k-1} = 0.$$

Since $(wy)^k = 0$, we have $[(wz)^D]^2(wy)^{k-1} = 0$; hence, $[(wz)^D]^2(wy)^{k-2} = 0$. By iteration of this process, we see that $[(wz)^D]^2 wy = 0$. Hence, $z[(wz)^D]^2 wy = 0$. By Cline's formula again, we get $(zw)^D y = 0$. As $z \in R_w^{\#}$, we see that $zw \in R^{\#}$. This implies that $zwy = (zw)^2[(zw)^D y] = 0$. Therefor we have

$$\begin{aligned}awa_w^{Ⓦ} &= (zw + yw)z_w^{\#}\\ &= zwz_w^{\#} + (ywz)[wz_w^{\#}]^2 = zwz_w^{\#},\\ a_w^{Ⓦ}wa &= z_w^{\#}w(z+y) = z_w^{\#}wz + z_w^{\#}wy\\ &= z_w^{\#}wz + (z_w^{\#}w)^2(zwy) = z_w^{\#}wz.\end{aligned}$$

Hence, $awa_w^{Ⓦ} = a_w^{Ⓦ}wa$, as asserted. □

Following directly from Theorem 4.10, we present a property of the weak group inverse as follows.

Corollary 4.11. *Let $a \in R^{Ⓦ}$. Then the following are equivalent:*

(1) $aa^{Ⓦ} = a^{Ⓦ}a$.
(2) $a^D a^{Ⓦ} = a^{Ⓦ}a^D$.

Submitted: December 16, 2024
Accepted: July 18, 2025
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Acknowledgement

The author would like to express the sincere gratitude to the referees for their thorough review and valuable suggestions for improving this paper.

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Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025

Revista de la Unión Matemática Argentina Accepted article · Early view version

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School of Big Data, Fuzhou University of International Studies and Trade, Fuzhou 350202, China

Email address: huanyinchenfz@163.com

Submitted: December 16, 2024
Accepted: July 18, 2025
Published (early view): August 12, 2025