

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

# ON THE SUM OF THE EIGENVALUES OF THE DISTANCE LAPLACIAN MATRIX OF GRAPHS WITH DIAMETER THREE AND FOUR

UMMER MUSHTAQ, SHARIEFUDDIN PIRZADA, AND SALEEM KHAN

**ABSTRACT.** Let  $G$  be a connected graph with  $n$  vertices. The distance Laplacian matrix of  $G$  is defined by  $D^L(G) = Tr(G) - D(G)$ , where  $Tr(G)$  is the transmission diagonal matrix and  $D(G)$  is the distance matrix of  $G$ . If the eigenvalues of the distance Laplacian matrix are given as  $\partial_1^L \geq \partial_2^L \geq \dots \geq \partial_n^L = 0$ , we take  $U_r(G) = \partial_1^L + \partial_2^L + \dots + \partial_r^L$  to be the sum of the  $r$  largest distance Laplacian eigenvalues of  $G$ . Recently, Zhou et al. [18] proposed a problem to determine connected graphs of order  $n$ , and any  $r = 2, \dots, n$ , which satisfy

$$U_r(G) \leq W(G) + \binom{r+2}{3}, \quad (\text{A})$$

where  $W(G)$  is the Wiener index of  $G$ . Let  $\Gamma_n$  represent the family of graphs with  $n$  vertices and diameter 3 which contain a spanning tree of diameter 3. In this paper, we prove that any graph  $G$  in  $\Gamma_n$  satisfies the inequality (A) for  $n \geq 95$ . This will prove that the inequality (A) is true for any tree with diameter 3. Moreover, we show that for any graph  $G$  with maximum degree of  $n-2$ , the inequality (A) holds for all  $1 \leq r \leq n$ . Finally, we prove that the sun graphs and partial sun-type graphs, with diameter 4, for all  $1 \leq r \leq n$ , also satisfy the inequality (A).

## 1. INTRODUCTION

We consider  $G$  to be a simple connected graph having vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(G) = \{e_1, e_2, \dots, e_m\}$  with  $|V(G)| = n$  and  $|E(G)| = m$ . The adjacency matrix  $A(G) = (a_{ij})$  of  $G$  is a  $(0, 1)$ -matrix of order  $n$ , where the  $(i, j)$ -entry is equal to 1 if  $v_i$  is adjacent to  $v_j$ , and 0 otherwise.

Let  $\mathcal{D}'(G) = \text{diag}(d_1, d_2, \dots, d_n)$  be the diagonal matrix of vertex degrees, where  $d_i = d_G(v_i)$ , for  $i = 1, 2, \dots, n$ . The Laplacian matrix of  $G$  is defined as  $L(G) = \mathcal{D}'(G) - A(G)$ , and the signless Laplacian matrix as  $Q(G) = \mathcal{D}'(G) + A(G)$ . The spectra of these matrices are known as the Laplacian spectrum and the signless Laplacian spectrum of  $G$ , respectively. These matrices are real, symmetric, and positive semi-definite. We write the Laplacian spectrum of  $G$  as  $0 = \mu_n \leq \mu_{n-1} \leq \dots \leq \mu_1$ , and the signless Laplacian spectrum as  $0 \leq q_n \leq q_{n-1} \leq \dots \leq q_1$ .

2020 *Mathematics Subject Classification.* 05C12, 05C50, 15A18.

*Key words and phrases.* distance Laplacian matrix; Wiener index; trees; diameter; sun type graphs.

The research of S. Pirzada is supported by National Board for Higher Mathematics (NBHM) research project number NBHM/02011/20/2022.

Submitted: February 6, 2025

Accepted: July 16, 2025

Published (early view): July 17, 2025

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

In a graph  $G$ , the distance  $d_G(u, v)$  (or  $d_{uv}$  for short) between the vertices  $u$  and  $v$  is the length of a shortest path connecting them. The diameter of  $G$  is the maximum distance between any two vertices in  $G$ . The transmission  $\text{Tr}_G(v)$  of a vertex  $v$  is the sum of the distances from  $v$  to all other vertices in  $G$ , i.e.,  $\text{Tr}_G(v) = \sum_{u \in V(G)} d_{uv}$ . A graph  $G$  is said to be  $r$ -transmission regular if  $\text{Tr}_G(v) = r$  for each  $v \in V(G)$ . The Wiener index (or transmission) of graph  $G$ , denoted by  $W(G)$ , is the sum of the distances between all unordered pairs of vertices in  $G$ , which can be expressed as  $W(G) = \frac{1}{2} \sum_{v \in V(G)} \text{Tr}_G(v)$ . The distance matrix of  $G$ , denoted by  $D(G)$ , is defined as  $D(G) = (d_{uv})_{u, v \in V(G)}$ . The quantity  $\text{Tr}_i = \text{Tr}_G(v_i)$  is referred to as the transmission degree of the vertex  $v_i$ , and the sequence  $\{\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n\}$  is the transmission degree sequence of the graph  $G$ .

Let  $\text{Tr}(G) = \text{diag}(\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n)$  be the diagonal matrix of vertex transmissions of  $G$ . Aouchiche and Hansen [3] defined the distance Laplacian matrix of  $G$  as  $D^L(G) = \text{Tr}(G) - D(G)$ , and the distance signless Laplacian matrix as  $D^Q(G) = \text{Tr}(G) + D(G)$ . If  $G$  is connected, then  $D(G)$  is symmetric, non-negative and irreducible. Some recent work can be seen in [13, 15, 16]. Let  $S_k(G) = \sum_{i=1}^k \mu_i$ , for  $1 \leq k \leq n$ , be the sum of the  $k$ -largest Laplacian eigenvalues of  $G$ . For a graph with  $n$  vertices and degree sequence  $\{d_v | v \in V(G)\}$ , Grone and Merris [12] conjectured that  $S_k(G) = \sum_{i=1}^k \mu_i(G) \leq \sum_{i=1}^k |\{v \in V(G) | d_v \geq i\}|$ , for  $k = 1, 2, \dots, n$ . This was proved by Bai [5]. As a variation of the Grone-Merris theorem, Brouwer [6] conjectured that for a graph  $G$  with  $n$  vertices and  $m$  edges,  $S_k(G) = \sum_{i=1}^k \mu_i \leq m + \binom{k+1}{2}$ , for  $k = 1, 2, \dots, n$ . For the progress on this conjecture, we refer to [7–11, 14].

Analogous to the Brouwer's conjecture, Alhevaz et al. [1] conjectured the following for the sum of the  $k$ -largest distance signless Laplacian eigenvalues  $U_k(G)$ .

**Problem 1.** If  $G$  is a connected graph of order  $n$ , which is not a path, then  $U_k(G) \leq W(G) + \binom{k+2}{3}$ , for any  $k = 2, \dots, n$ .

They proved that this inequality is true for graphs of diameter one and two.

For  $1 \leq r \leq n - 1$ , let  $U_r$  be the sum of the  $r$  largest distance Laplacian eigenvalues of a graph  $G$ , i.e.,  $U_r = \sum_{i=1}^r \partial_i^L(G)$ , where  $\partial_i^L(G)$ ,  $i = 1, 2, \dots, n$  are the distance Laplacian eigenvalues of  $G$ . Similar to Alhevaz et al. [1], the following problem was proposed by Zhou et al. [18].

**Problem 2.** Determine which connected graphs of order  $n$ , and any  $r = 2, \dots, n$ , satisfy

$$U_r(G) \leq W(G) + \binom{r+2}{3}. \quad (A)$$

Zhou et al. [18] confirmed the validity of Inequality (A) for graphs with diameter one and graphs of diameter two with given maximum degree for all  $r$ .

**Definition.** If  $d(v) = 1$ , then  $v$  is called a pendant vertex. A tree containing exactly two non-pendant vertices is called a double-star. A double-star with degree sequence  $(k_1 + 1, k_2 + 1, 1, 1, \dots, 1)$  is denoted by  $S_{k_1, k_2}$ .

The following observations will be used in what follows.

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

**Lemma 1.1.** [4] *Let  $G$  be a graph on  $n$  vertices and  $N(v)$  represent the set of all vertices adjacent to vertex  $v$ . If  $S = \{v_1, v_2, \dots, v_p\}$  is an independent set of  $G$  such that  $N(v_i) = N(v_j)$  for all  $i, j \in \{1, 2, \dots, p\}$ , then  $\partial = \text{Tr}(v_i) = \text{Tr}(v_j)$  for all  $i, j \in \{1, 2, \dots, p\}$ , and  $\partial + 2$  is an eigenvalue of  $D^L(G)$  with multiplicity at least  $p - 1$ .*

**Lemma 1.2.** [3] *Let  $G$  be a connected graph on  $n$  vertices and  $m \geq n$  edges. Consider the connected graph  $G'$  obtained from  $G$  by the deletion of an edge. Let  $\partial_1^L(G) \geq \partial_2^L(G) \geq \dots \geq \partial_n^L(G)$  and  $\partial_1^L(G') \geq \partial_2^L(G') \geq \dots \geq \partial_n^L(G')$  be respectively the distance Laplacian eigenvalues of  $G$  and  $G'$ . Then  $\partial_i^L(G') \geq \partial_i^L(G)$  holds for all  $1 \leq i \leq n$ .*

The paper is organized as follows. In Section 2, we prove that Inequality (A) holds for all graphs in the family  $\Gamma_n$  with  $n \geq 95$  and for any graph with a maximum degree  $n - 2$  and  $n \geq 95$ . In Section 3, we establish that Inequality (A) is true for all sun-type and partial sun-type graphs.

## 2. SUM OF THE DISTANCE LAPLACIAN EIGENVALUES OF GRAPHS WITH DIAMETER THREE

Graphs with diameter three can be classified into two distinct categories, one that includes graphs that have spanning tree with diameter not greater than three, while the other consists of graphs with spanning tree whose diameter can exceed three. We use the notation  $\Gamma_n$  for the first category. It is important to note that every graph in  $\Gamma_n$  contains the double star  $S_{k_1, k_2}$ , where  $k_1 + k_2 + 2 = n$ , as a spanning subgraph. The graph  $H$  given in Figure 1 is the largest possible graph in  $\Gamma_n$  in terms of edge count. The graph  $H$  is formed by connecting an isolated vertex  $u$  to a complete graph  $K_{k_1}$ , and another isolated vertex  $v$  to a complete graph  $K_{k_2}$ . Finally, the two complete graphs  $K_{k_1}$  and  $K_{k_2}$  are joined together. This construction yields the largest graph by edge count in the family  $\Gamma_n$ . In this section, we prove that Inequality (A) is true for all graphs in  $\Gamma_n$  for which  $n \geq 95$ . As a consequence of this, we finally show that Inequality (A) is true for any graph  $G$  with maximum degree  $\Delta = n - 2$ .

In the following lemma, we first obtain the distance Laplacian spectrum of  $S_{k_1, k_2}$ .

**Lemma 2.1.** *The distance Laplacian spectrum of  $S_{k, n-k-2}$  is  $\{(3n-k-3)^{k-1}, (2n+k-1)^{n-k-3}\}$  together with the roots of the polynomial*

$$p(\lambda) = \lambda(\lambda^3 - (6n-4)\lambda^2 + (11n^2 - 13n - k^2 + kn - 2k + 3)\lambda + (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)).$$

*Proof.* Suppose that  $V_1 = \{u, u_1, u_2, \dots, u_k\}$  and  $V_2 = \{v, v_1, v_2, \dots, v_{n-k-2}\}$  are the set of vertices of  $K_{1, k}$  and  $K_{1, n-k-2}$ , respectively. We assume that the double star graph  $S_{k, n-k-2}$  is obtained by joining the central vertices  $u$  and  $v$  of the star graphs  $K_{1, k}$  and  $K_{1, n-k-2}$ . Then the vertex set of  $S_{k, n-k-2}$  is  $V(S_{k, n-k-2}) = V_1 \cup V_2$ . The sets of vertices  $\{u_1, u_2, \dots, u_k\}$  and  $\{v_1, v_2, \dots, v_{n-k-2}\}$  are clearly independent, with the transmission values  $\text{Tr}(u_i) = 3n - k - 5$  and  $\text{Tr}(v_i) =$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

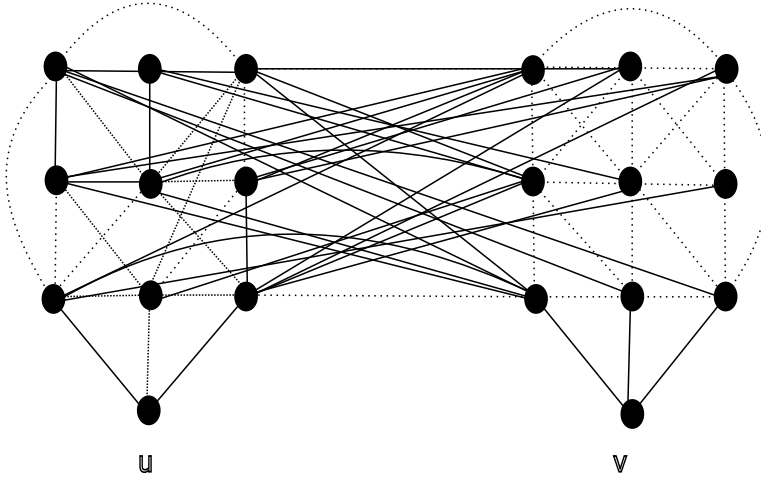


FIGURE 1. Graph H

$2n+k-3$ . By Lemma 1.1, the distance Laplacian eigenvalues  $3n-k-3$  and  $2n-k-1$  of  $S_{k,n-k-2}$  have multiplicities of at least  $k-1$  and  $n-k-3$ , respectively.

The remaining are the eigenvalues of matrix  $B$ , which is given below

$$B = \begin{pmatrix} 2n-k-3 & -k & -1 & -2n+2k+4 \\ -1 & 3n-3k-3 & -2 & -3n+3k+6 \\ -1 & -2k & n+k-1 & -n+2+k \\ -2 & -3k & -1 & 3k+3 \end{pmatrix}.$$

The characteristic polynomial of the matrix  $B$  is given by

$$p(\lambda) = \lambda(\lambda^3 - (6n-4)\lambda^2 + (11n^2 - 13n - k^2 + kn - 2k + 3)\lambda + (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)).$$

Clearly, 0 is one of the zeros of the polynomial  $p(\lambda)$  and the sum of the remaining three zeros is  $6n-4$ . This completes the proof.  $\square$

**Theorem 2.2.** *Let  $G$  be a graph in the class  $\Gamma_n$  with  $n \geq 95$ , containing  $S_{k,n-k-2}$  as a spanning subgraph. Then, for all  $1 \leq r \leq k-1$  and  $k \geq 2$ , Inequality (A) holds for  $G$ .*

*Proof.* The graph  $H$  has the smallest Wiener index among all possible connected graphs in  $\Gamma_n$  and its Wiener index is  $\frac{n^2+n}{2}$ . Moreover, for any graph  $G'$  of diameter 3,  $\partial_i^L(S_{k,n-k-2}) \geq \partial_i^L(G')$ , for all  $i = 1, 2, \dots, n$ . Therefore, in view of Lemma 1.2 and the fact that the Wiener index of any graph in  $\Gamma_n$  is at least  $\frac{n^2+n}{2}$ , the following

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

inequality will establish the result.

$$U_r(S_{k,n-k-2}) \leq \frac{n^2+n}{2} + \binom{r+2}{3}. \quad (2.1)$$

From Lemma 2.1, the distance Laplacian spectrum of  $S_{k,n-k-2}$  is  $\{(3n-k-3)^{k-2}, (2n+k-1)^{n-k-2}\}$  together with the roots of the polynomial

$$p(\lambda) = \lambda(\lambda^3 - (6n-4)\lambda^2 + (11n^2 - 13n - k^2 + kn - 2k + 3)\lambda + (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)).$$

Since  $P(3n-k) < 0$  and  $P(6n-4) > 0$ , it follows that the spectral radius is one of the zeros of the polynomial  $P(\lambda)$  and lies in the interval  $(3n-k, 6n-4)$ . To establish the result, we consider the following two cases based on the multiplicity of the second largest distance Laplacian eigenvalue  $\partial_2^L(S_{k,n-k-2})$ .

**Case (i).**  $n \geq 2k+2$  and  $\partial_2^L = 3n-k-3$  with multiplicity at least  $k-1$ .

For  $n \geq 2k+2$ , we have  $3n-k-3 \geq 2n+k-1$  and  $3n-k-3$  is a decreasing function of  $k$ , so we can write  $(3n-k-3)(r-1) \leq (3n-4)(r-1)$ . As  $\partial_1^L(S_{k,n-k-2}) < 6n-4$ , we prove that, for any  $r$  satisfying  $1 \leq r \leq k-1$ , the following inequality

$$(6n-4) + (3n-4)(r-1) \leq \frac{n^2+n}{2} + \binom{r+2}{3}.$$

holds, which can be further simplified to  $r^3 + 3r^2 + 26r - 18nr + 3n^2 - 15n \geq 0$ . Since  $r \geq 1$ , in order to prove  $r^3 + 3r^2 + 26r - 18nr + 3n^2 - 15n \geq 0$ , we need to prove that  $r^3 + 3r^2 - 18nr + 3n^2 - 15n \geq 0$ . Let  $f(r) = r^3 + 3r^2 - 18nr + 3n^2 - 15n$ ,  $r \in [1, k-2]$ . The function  $f(r)$  is decreasing in  $[1, -1 + \sqrt{1+6n}]$  and increasing in  $[-1 + \sqrt{1+6n}, k-2]$ . Therefore, to conclude the proof it is necessary to prove that  $f(-1 + \sqrt{1+6n}) \geq 0$ , that is,

$$\begin{aligned} f(-1 + \sqrt{1+6n}) &= (-1 + \sqrt{1+6n})^3 + 3(-1 + \sqrt{1+6n})^2 - 18n(-1 + \sqrt{1+6n}) + 3n^2 - 15n \\ &= (-1 + (1+6n)\sqrt{1+6n} + 3\sqrt{1+6n} - 3(1+6n)) \\ &\quad + 6 + 18n - 6\sqrt{1+6n} + 18n - 18n(\sqrt{1+6n}) + 3n^2 - 15n \\ &\geq 0, \end{aligned}$$

which is true for  $n \geq 95$ .

**Case (ii).**  $n < 2k+2$  and  $\partial_2^L = 2n+k-1$  with multiplicity at least  $n-k-2$ . For  $n < 2k+1$ , we have,  $3n-k-3 < 2n+k-1$  and  $2n+k-1$  is increasing function of  $k$ , so we can write  $(2n+k-1)(r-1) \leq (3n-4)(r-1)$ . Now proceeding in the same way as in Case (i), we can establish the result.  $\square$

**Theorem 2.3.** *Inequality (A) is true for all connected graphs in  $\Gamma_n$ , for which  $n \geq 95$ .*

*Proof.* Assume that  $G$  is any graph in  $\Gamma_n$  and  $S_{k,n-k-2}$  is the spanning tree of  $G$ . If  $r \leq k-2$ , then the result is true by Theorem 2.2. We need to prove this for  $r > k-2$ . The graph  $H$  has the smallest Wiener index among all possible connected graphs in  $\Gamma_n$  and its Wiener index is  $\frac{n^2+n}{2}$ . Moreover, for any graph  $G'$  in  $\Gamma_n$ ,  $\partial_i^L(S_{k,n-k-2}) \geq \partial_i^L(G')$ , for all  $i = 1, 2, \dots, n$ . Therefore, in view of

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

Lemma 1.2 and the fact that the Wiener index of any graph in  $\Gamma_n$  at least  $\frac{n^2+n}{2}$ , the following inequality will establish the result.

$$U_r(S_{k,n-k-2}) \leq \frac{n^2+n}{2} + \binom{r+2}{3}. \quad (2.2)$$

From Lemma 2.1, the distance Laplacian spectrum of  $S_{k,n-k-2}$  is  $\{(3n-k-3)^{k-2}, (2n+k-1)^{n-k-2}\}$  together with the roots of the polynomial

$$p(\lambda) = \lambda(\lambda^3 - (6n-4)\lambda^2 + (11n^2 - 13n - k^2 + kn - 2k + 3)\lambda + (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)).$$

We consider the following cases.

**Case (i).**  $n \geq 2k+2$  and assume that the multiplicity of  $\partial_{n-4}^L = 2n+k-1$  is  $s$ .

Now, for  $n \geq 2k+2$ , we have  $3n-k-3 \geq 2n+k-1$  and  $3n-k-3$  is a decreasing function of  $k$  and  $2n+k-1$  is an increasing function of  $k$ , so we can write

$$(3n-k-3)(r-t) + (2n+k-1)t \leq (3n-4)(r-t) + \left(\frac{5n}{2} - 2\right)t,$$

where  $1 \leq t \leq s$ .

For any  $r$  satisfying  $1 \leq r \leq n-4$ , we prove that the inequality

$$(3n-4)(r-t) + \left(\frac{5n}{2} - 2\right)t \leq \frac{n^2+n}{2} + \binom{r+2}{3}$$

holds. Since  $(3n-4)(r-t) + \left(\frac{5n}{2} - 2\right)t \leq (3n-4)(r)$ , so we show that  $(3n-4)(r) \leq \frac{n^2+n}{2} + \binom{r+2}{3}$ , which can be further simplified as

$$r^3 + 3r^2 + 6r - 18nr + 3n^2 + 3n \geq 0.$$

Since  $r \geq 1$ , in order to prove that  $r^3 + 3r^2 + 6r - 18nr + 3n^2 + 3n \geq 0$ , we need to prove that  $r^3 - 18nr + 3n^2 + 3n + 9 \geq 0$ . Let  $f(r) = r^3 - 18nr + 3n^2 + 3n + 9$ ,  $r \in [1, n-4]$ . The function  $f(r)$  is decreasing in  $[1, \sqrt{6n}]$  and increasing in  $[\sqrt{6n}, n-4]$ . Hence, to conclude the proof, it is necessary to prove that  $f(\sqrt{6n}) \geq 0$ , that is,

$$\begin{aligned} f(\sqrt{6n}) &= (\sqrt{6n})^3 - 18n(\sqrt{6n}) + 3n^2 + 3n + 9 \\ &= (-12n)\sqrt{6n} + 3n^2 + 3n + 9 \\ &\geq 0, \end{aligned}$$

which is true for  $n \geq 95$ .

**Case (ii).**  $n < 2k+1$  and  $\partial_2^L = 2n+k-2$  with multiplicity at least  $n-k-2$ . For  $n < 2k+1$ , we have  $3n-k-3 < 2n+k-2$  and  $2n+k-2$  is an increasing function of  $k$ , so we can write  $(2n+k-2)r \leq (3n-3)r$ . Now, proceeding similarly as in Case (i), we can establish the result.  $\square$

Zhou et al. [18] proved the following result for graphs with diameter 2 and maximum degree  $n-2$ .

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

**Theorem 2.4.** [18] *Let  $G$  be a connected graph of order  $n$  and size  $m$  having maximum degree  $\Delta$ .*

(1) *If  $\Delta = n - 1$ , then Problem 1 is true for all  $n \geq 21$ .*

(2) *If  $\Delta = n - 2$ , then Problem 1 is true for all  $n \geq 26$ .*

Now, we generalize this to all graphs with maximum degree  $n - 2$ , without any constraints on the diameter.

**Theorem 2.5.** *Let  $G$  be a connected graph of order  $n$  and size  $m$  having maximum degree  $\Delta = n - 2$ . Then Inequality (A) is true for all  $n \geq 95$ .*

*Proof.* Any graph  $G$  with maximum degree  $n - 2$  has diameter either 2 or 3. If  $G$  has diameter 2, the result holds by Theorem 2.4. Therefore, we only need to prove the case where  $G$  has a maximum degree  $n - 2$  and diameter 3.

Now, if  $G$  has diameter 3 and maximum degree  $n - 2$ , then  $G$  belongs to the family  $\Gamma_n$ . Consequently, in this case, the result follows from Theorem 2.3.  $\square$

### 3. SUM OF DISTANCE LAPLACIAN EIGENVALUES OF SUN TYPE GRAPHS

**Definition.** The sun graph  $S_n(a, a)$  is a tree of order  $n = 2a + 1$ , containing pendent vertices, each attached to a vertex of degree 2, and a vertex of degree  $a$ . The sun graph of order  $n$  can be obtained from star graph  $K_{1, n-1}$  by deleting  $\frac{n-1}{2}$  pendent vertices and inserting a new vertex on remaining  $\frac{n-1}{2}$  edges of the star graph  $K_{1, n-1}$ . We define a partial sun graph  $S_n(a, k)$  a tree of order  $n = a + k + 1$ , containing  $k$ ,  $1 \leq k \leq a - 1$  pendent vertices each attached to a vertex of degree 2,  $a - k$  pendent vertices each attached to a vertex of degree  $a$ .

**Theorem 3.1.** [17] *For  $a \geq 2$ , let  $S_n(a, a)$  be the sun graph of order  $n = 2a + 1$ . Then the distance Laplacian spectrum of  $S_n(a, a)$  is*

$$\left\{ 0, \frac{1}{2}(12a - 1 \pm \sqrt{4a^2 + 4a + 17})^{a-1}, \frac{1}{2}(9a - 1 \pm \sqrt{9a^2 - 22a + 17}) \right\}.$$

Now, we prove that Inequality (A) is true for both sun graph  $S_n(a, a)$  and partial sun graph  $S_n(a, k)$ , which are of diameter 4 each.

**Theorem 3.2.** *If  $a \geq 12$ , then Inequality (A) is true for  $S_n(a, a)$ .*

*Proof.* We first find the Wiener index of  $S_n(a, a)$ . Since  $S_n(a, a)$  has ' $a$ ' vertices of transmission  $7a - 4$ , ' $a$ ' of transmission  $5a - 3$  and one vertex of transmission  $3a$ , therefore, the Wiener index of  $S_n(a, a)$  is  $\frac{12a^2 - 4a}{2}$ . Thus, we need to prove the following inequality.

$$U_r(S_n(a, a)) \leq \frac{12a^2 - 4a}{2} + \binom{r+2}{3}. \quad (3.3)$$

Since  $\partial_1^L(S_n(a, a)) = \frac{1}{2}(12a - 1 + \sqrt{4a^2 + 4a + 17})$  is of multiplicity  $a - 1$ , so in order to establish (3.3), we show that the following inequality is true for all  $r$  satisfying  $1 \leq r \leq n$ .

$$\frac{1}{2}(12a - 1 + \sqrt{4a^2 + 4a + 17})r \leq \frac{12a^2 - 4a}{2} + \binom{r+2}{3}.$$

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

As  $\sqrt{4a^2 + 4a + 17} \leq 3a$ , so we show that  $\frac{1}{2}(15a)r \leq \frac{12a^2 - 4a}{2} + \binom{r+2}{3}$ , which can be further simplified to

$$r^3 + 3r^2 - 45ar + 2r + 36a^2 - 12a \geq 0.$$

Again, since  $r \geq 1$ , in order to prove that  $r^3 + 3r^2 - 45ar + 2r + 36a^2 - 12a \geq 0$ , we need to prove that  $r^3 - 45ar + 5 + 36a^2 - 12a \geq 0$ .

Let  $f(r) = r^3 - 45ar + 5 + 36a^2 - 12a$ ,  $r \in [1, n]$ . The function  $f(r)$  is decreasing in  $[1, \sqrt{15a}]$  and increasing in  $[\sqrt{15a}, n]$ . Thus, to conclude the proof, it is necessary to prove that  $f(\sqrt{15a}) \geq 0$ , that is,

$$\begin{aligned} f(\sqrt{15a}) &= (\sqrt{15a})^3 - 45a(\sqrt{15a}) + 5 + 36a^2 - 12a \\ &= -30a\sqrt{15a} + 36a^2 - 12a \\ &\geq 0, \end{aligned}$$

if  $a \geq 12$ . □

**Theorem 3.3.** [17] For  $a \geq 2$  and  $1 \leq k \leq a - 1$ , let  $S_n(a, k)$  be the partial sun graph of order  $n = a + k + 1$ . Then the distance Laplacian spectrum of  $S_n(a, k)$  consists of the simple eigenvalue 0, the eigenvalues  $\frac{1}{2}(5a + 7k - 1 \pm \sqrt{(a + k)^2 + 2(a + k) + 17})$  each with multiplicities  $k - 1$ , the eigenvalue  $2a + 3k + 1$  with multiplicity  $a - k - 1$  and the zeros of the polynomial

$$\begin{aligned} x^3 - (6a + 8k)x^2 + (-5 + a + 11a^2 + 4k + 31ak + 21k^2)x + 4 + 7a - 3a^2 - 6a^3 + 12k \\ - 10ak - 27a^2k - 10k^2 - 39ak^2 - 18k^3. \end{aligned}$$

**Theorem 3.4.** For  $a \geq 450$ , Inequality (A) is true for partial sun graph  $S_n(a, k)$ .

*Proof.* The Wiener index of  $S_n(a, k)$  is  $\frac{6ak + 4k^2 + 2a^2 - 4k}{2}$ . Therefore, we need to prove the following inequality

$$U_r(S_n(a, k)) \leq \frac{6ak + 4k^2 + 2a^2 - 4k}{2} + \binom{r+2}{3}.$$

Since,  $\frac{1}{2}(5a + 7k - 1 + \sqrt{(a + k)^2 + 2(a + k) + 17})$  is the spectral radius of  $S_n(a, k)$  with multiplicity  $k - 1$ , we prove that

$$\left(\frac{1}{2}(5a + 7k - 1 + \sqrt{(a + k)^2 + 2(a + k) + 17})\right)r \leq \frac{6ak + 4k^2 + 2a^2 - 4k}{2} + \binom{r+2}{3}.$$

for any  $r$ ,  $1 \leq r \leq n - 1$ . Since,  $k \leq a - 1$  and  $a + k \geq 4$ , therefore  $\frac{1}{2}(5a + 7k - 1 + \sqrt{(a + k)^2 + 2(a + k) + 17}) \leq 8a$ . So to establish the above inequality, we prove that

$$(8a)r \leq \frac{6ak + 4k^2 + 2a^2 - 4k}{2} + \binom{r+2}{3}.$$

This can be further simplified as

$$r^3 + 3r^2 + 2r - 48ar + 18ak + 12k^2 + 6a^2 - 12k \geq 0.$$

Since  $r \geq 1$ , to prove that



This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

$$r^3 + 3r^2 + 2r - 48ar + 18ak + 12k^2 + 6a^2 - 12k \geq 0,$$

we can instead show that

$$r^3 - 48ar + 18ak + 12k^2 + 6a^2 - 12k \geq 0.$$

Let  $f(r) = r^3 - 48ar + 18ak + 12k^2 + 6a^2 - 12k$ . The function  $f(r)$  is decreasing in  $[1, 4\sqrt{a}]$  and increasing in  $[4\sqrt{a}, n]$ . Thus, to conclude the proof, it is necessary to prove that  $f(4\sqrt{a}) \geq 0$ , that is,

$$\begin{aligned} f(4\sqrt{a}) &= (4\sqrt{a})^3 - 48a(4\sqrt{a}) + 18ak + 12k^2 + 6a^2 - 12k \\ &= -128a\sqrt{a} + 18ak + 12k^2 + 6a^2 - 12k \\ &\geq 0, \end{aligned}$$

if  $a \geq 450$ . □

**Conclusion.** We have extended the study of Zhou et al. [18] by determining the class of connected graphs of diameter 3 and 4 that satisfy the inequality  $U_r(G) \leq W(G) + \binom{r+2}{3}$  for any  $r = 2, \dots, n$ , where  $U_r(G)$  represents the sum of the  $r$  largest distance Laplacian eigenvalues of  $G$ . Our findings demonstrate that this upper bound holds for certain classes of graphs with diameters beyond 2, expanding the understanding of the spectral properties of graphs with larger diameters. Further research may continue to explore the validity of this bound for all graphs of diameter 3, 4, and beyond, potentially providing deeper insights into the relationships between graph structure and spectral bounds for distance Laplacian eigenvalues.

## REFERENCES

- [1] A. Alhevaz, M. Baghipur, H. Ganie, S. Pirzada, Brouwer type conjecture for the eigenvalues of distance signless Laplacian matrix of a graph, *Linear Multilinear Algebra*. **69** (2021) 2423-2440.
- [2] M. Aouchiche, P. Hansen, Distance spectra of graphs: a survey, *Linear Algebra Appl.* **458** (2014) 301-386.
- [3] M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.* **439** (2013) 21-33.
- [4] M. Aouchiche, P. Hansen, Some properties of the distance Laplacian eigenvalues of a graph, *Czechoslov. Math. J.* **64** (2014) 751-761.
- [5] H. Bai, The Grone-Merris conjecture, *Trans. Amer. Math. Soc.* **363** (2011) 4463-4474.
- [6] A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*, Springer, New York, 2012.
- [7] H. A. Ganie, A. Alghamdi, S. Pirzada, On the sum of the Laplacian eigenvalues of a graph and Brouwer's conjecture, *Linear Algebra Appl.* **501** (2016) 376-389.
- [8] H. A. Ganie, S. Pirzada, R. Shaban, X. Li, Upper bounds for the sum of Laplacian eigenvalues of a graph and Brouwer's conjecture, *Discrete Math. Algorithms Appl.* **11(2)** (2019) 105008 (15 pages).
- [9] H. A. Ganie, S. Pirzada, V. Trevisan, On the sum of  $k$  largest Laplacian eigenvalues of a graph and clique number, *Mediterranean J. Math.* **18** (2021) Article No. 15.
- [10] H. A. Ganie, S. Pirzada, B. A. Rather, R. Shaban, On Laplacian eigenvalues of graphs and Brouwer's conjecture, *J. Ramanujan Math. Soc.* **36(1)** (2021) 13-21.

*Submitted: February 6, 2025*

*Accepted: July 16, 2025*

*Published (early view): July 17, 2025*

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: <https://doi.org/10.33044/revuma.5147>.

- [11] H. A. Ganie, S. Pirzada, B. A. Rather, V. Trevisan, Further developments on Brouwer's conjecture for the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.* **588** (2020) 1-18.
- [12] R. Grone, R. Merris, The Laplacian spectrum of a graph II, *SIAM J. Discrete Math.* **7** (1994) 221-229.
- [13] Saleem Khan, S. Pirzada, A. Somasundaram, On graphs with distance Laplacian eigenvalues of multiplicity  $n - 4$ , *AKCE International J. Graphs Comb.* **20(3)** (2023) 282-286.
- [14] P. Kumar, Merajuddin, S. Pirzada, Computing the sum of  $k$  largest Laplacian eigenvalues of tricyclic graphs, *Discrete Math. Letters* **11** (2023) 14-18.
- [15] S. Pirzada, Saleem Khan, On the sum of distance Laplacian eigenvalues of graphs, *Tamkang J. Math.* **54(1)** (2023) 83-91.
- [16] S. Pirzada, Saleem Khan, Distance Laplacian eigenvalues of graphs and chromatic and independence number, *Revista de la Union Matematica Argentina* **67(1)** (2024) 145-159.
- [17] B. A. Rather, H. A. Ganie, Y. Shang, Distance Laplacian spectral ordering of sun type graph, *Appl. Math Comput.* **445** (2023) 127847.
- [18] Y. Zhou, L. Wang, Y. Chai, Brouwer type conjecture for the eigenvalues of distance Laplacian matrix of a graph, *Comput. Appl. Math.* **44(1)** (2025) 138.

(Ummer Mushtaq) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KASHMIR, SRINAGAR, INDIA

*Email address:* [mmermushtaq1995@gmail.com](mailto:mmermushtaq1995@gmail.com)

(Shariefuddin Pirzada) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KASHMIR, SRINAGAR, INDIA

*Email address:* [pirzadasd@kashmiruniversity.ac.in](mailto:pirzadasd@kashmiruniversity.ac.in)

(Saleem Khan) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KASHMIR, SRINAGAR, INDIA

*Email address:* [khansaleem1727@gmail.com](mailto:khansaleem1727@gmail.com)

*Submitted: February 6, 2025*

*Accepted: July 16, 2025*

*Published (early view): July 17, 2025*