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THE E-RELATIVE m-WEAK GROUP INVERSE

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Abstract. In this paper the notion of the m-weak group inverse of a complex square matrix relative to an invertible Hermitian weight E is introduced. We show several equivalent algebraic characterizations of this new type of generalized inverse. Additionally, different properties, representations, and characterizations involving the column space, null space, rank equalities, and projectors are presented. Finally, we solve a linear equation using this new generalized inverse.

1. Introduction

In 2018, the weak group inverse (or WG inverse) was introduced in [20] as an alternative extension of the group inverse. Several characterizations and representations of the WG inverse and its extension to rectangular matrices were discussed in [6, 7]. This inverse has since been studied in more general contexts such as rings and Hilbert spaces [9, 14, 23]. The latest results related to the WG inverse can be found in [12, 13, 21].

To extend the WG inverse, the m-weak group inverse (or m-WG inverse) was introduced in [22] for an element in a ring with involution. Later, [8, 15] studied the m-WG inverse of square matrices in depth. Various research studies have arisen from the m-WG inverse because it represents a significant extension of other known inverses, depending on the different values of the parameter m. More precisely, if $m = 1$, the m-WG inverse coincides with the WG inverse [20]; if $m = 2$, it reduces to the generalized group inverse (or GG inverse) [5]; and if m is greater than or equal to the index of the matrix, it reduces to the Drazin inverse, and therefore the group inverse. In [10], the authors presented the m-WG inverse for rectangular matrices. Interesting applications of m-WG inverses in optimization problems and matrix partial orders can be found in [9, 11]. Prassad and Bapat [17] introduced the weighted Moore-Penrose inverse of a rectangular matrix with respect to two Hermitian positive definite matrices. The weighted Moore-Penrose

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inverse is particularly useful in various applications, such as, least squares problems, signal processing, control theory, machine learning [3, 19].

Recently, Behera et al. [2] extended the core-EP inverse [18] with respect to an invertible Hermitian weight, introducing the so-called E-weighted core-EP inverse, which is unique whenever it exists. Some recent results in this direction for element in rings can be found in [16, 24].

Motivated by previous research, our main goal is to study the m-WG inverse of a square matrix relative to an invertible Hermitian matrix E which will be used as a weight. The rest of this paper is organized as follows. Section 2 provides the necessary notations and some preliminary results. In Section 3, we introduce the notion of the E-relative m-WG inverse by using an invertible Hermitian matrix E as a weight. Then, assuming the existence of the E-weighted core-EP inverse, it follows that this new inverse exists and is unique. We explore various of its characterizations and properties. In particular, we consider the relation between the E-relative m-WG inverse and the E-relative $(m+1)$-WG inverse. Section 4 deals with expressions of the E-relative m-WG inverse as an outer inverse with prescribed range and null space and we derive some special idempotent matrices determined by this inverse. Section 5 provides additional characterizations involving the column space, null space, rank equalities, and projectors. Finally, in Section 6, we solve a linear equation using the E-relative m-WG inverse.

2. Notation and preliminaries results

For $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $\mathrm{R}(A)$, $\mathrm{N}(A)$, and $\mathrm{rk}(A)$ will be stand for the conjugate transpose, the column space, the null space, and the rank of A, respectively. The index of $A \in \mathbb{C}^{n\times n}$ is the smallest nonnegative integer k such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$ and is denoted by $\mathrm{Ind}(A)$. By convention $A^0 = I_n$ (when $m = n$). We denote by $P_{S,T}$ the projector onto S along T when $\mathbb{C}^n$ is equal to the direct sum of subspaces S and T.

The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger = X \in \mathbb{C}^{n\times m}$ that satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

Denote by $P_A := AA^\dagger$ the orthogonal projector onto $\mathrm{R}(A)$.

The Drazin inverse of a matrix $A \in \mathbb{C}^{n\times n}$ of index k is the unique matrix $A^d = X \in \mathbb{C}^{n\times n}$ that satisfies

$$XA^{k+1} = A^k, \quad XAX = X, \quad AX = XA.$$

When $\mathrm{Ind}(A) = 1$, the Drazin inverse is called the group inverse of A and is denoted by $A^\#$.

The core-EP inverse of a matrix $A \in \mathbb{C}^{n\times n}$ of index k is the unique matrix $A^{\textcircled{\dagger}} = X \in \mathbb{C}^{n\times n}$ satisfying the conditions $XAX = X$ and $\mathrm{R}(X) = \mathrm{R}(X^*) = \mathrm{R}(A^k)$ [18]. When $\mathrm{Ind}(A) = 1$, $A^{\textcircled{\dagger}} := A^{\textcircled{\#}} = A^\# AA^\dagger$ becomes the core inverse of A [1].

In [23], Zhou et al. introduced the concept of m-weak group in rings with involution and provided several characterizations and properties of this new generalized

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inverse. For $A \in \mathbb{C}^{n\times n}$ a matrix of index k and $m \in \mathbb{N}$, the m-WG inverse is the unique matrix $A^{Ⓦ_m} = X \in \mathbb{C}^{n\times n}$ satisfying the equations

$$XA^{k+1} = A^k, \quad AX^2 = X, \quad (A^*)^k A^{m+1} X = (A^*)^k A^m, \quad m \in \mathbb{N}.$$

Using the core-EP inverse, the authors of [8] derived the following characterization of the m-WG inverse:

$$AX^2 = X, \quad AX = (A^{\textcircled{\dagger}})^m A^m, \tag{2.1}$$

where the unique solution of the above system is given by $A^{Ⓦ_m} = (A^{\textcircled{\dagger}})^{m+1}A^m$. Note that if $m = 1$, the m-weak group inverse reduces to the WG inverse [20], i.e. $A^{Ⓦ_1} = (A^{\textcircled{\dagger}})^2 A$. When $m = 2$, the m-WG inverse coincides with the GG inverse studied recently by Ferreyra and Malik in [5], that is, $A^{Ⓦ_2} = (A^{\textcircled{\dagger}})^3 A^2$. Moreover, if $m \geq \mathrm{Ind}(A)$, $A^{Ⓦ_m} = A^d$. Therefore, the m-WG inverse extends the notions of WG inverse, GG inverse, Drazin inverse, and therefore of group inverse.

In 1992, Prassad and Bapat [17] introduced the weighted Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ respect to two Hermitian positive definite matrices $E \in \mathbb{C}^{m\times m}$ and $F \in \mathbb{C}^{n\times n}$ as the unique matrix $A^{\dagger}_{E,F} = X \in \mathbb{C}^{n\times m}$ satisfying the fourth conditions

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3^E)\ (EAX)^* = EAX,\ (4^F)\ (FXA)^* = FXA. \tag{2.2}$$

It was proved that the unique solution to (2.2) is given by

$$A^{\dagger}_{E,F} = (A^*EA)^{\dagger}A^*E = F^{-1}A^*(AF^{-1}A^*)^{\dagger}.$$

When $E = I_m$ and $F = I_n$ in (2.2), X represents the classical Moore-Penrose inverse of A.

Inspired by the ideas of previous works, Behera et al. recently introduced the concept of core-EP inverse with respect to an invertible Hermitian weight. More specifically, they presented the following definition:

Definition 2.1 ([2]). Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. A matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$(1^k)\ XA^{k+1} = A^k, \quad (7)\ AX^2 = X, \quad (3^E)\ (EAX)^* = EAX,$$

is called the E-weighted core-EP inverse of A and is denoted by $A^{\textcircled{\dagger},\mathrm{E}}$. In particular, if $k = 1$, it is called E-weighted core inverse of A and is denoted by $A^{\textcircled{\#},\mathrm{E}}$.

Any matrix satisfying equations involved in $\gamma \subseteq \{1, 2, 1^k, 7, 3^E, 4^F\}$, is a γ-inverse of A, and $A\{\gamma\}$ is the set of all γ-inverses of A.

A necessary and sufficient condition for the existence of the E-weighted core-EP inverse in the context of rings was given in [24, Theorem 3.9]. In the matrix setting, we have

$$A^{\textcircled{\dagger},\mathrm{E}} \text{ exists} \iff A^k\{1, 3^E\} \neq \emptyset,$$

In this case, $A^{\textcircled{\dagger},\mathrm{E}} = A^d A^k Y$, for some $Y \in A^k\{1, 3^E\}$. This characterization can also be derived from [2, Lemma 3.7] and [2, Theorem 3.13]. Moreover, from [2, Proposition 3.17] we can deduce

$$A^k\{1, 3^E\} \neq \emptyset \iff A^k = Z(A^*)^k EA^k, \text{ for some } Z \in \mathbb{C}^{n\times n}.$$

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It is also proved in [2] that if the E-weighted core-EP inverse exists, then it is unique and can be characterized by three specific conditions

$$XAX = X, \quad \mathrm{R}(X) = \mathrm{R}(A^k), \quad \mathrm{R}(X^*) = \mathrm{R}(EA^k). \tag{2.3}$$

In order to present some properties of the E-weighted core-EP inverse of a matrix we establish the following results.

Proposition 2.2 ([2, 4]). *Let $A \in \mathbb{C}^{n\times n}$, $\ell \geq \mathrm{Ind}(A)$, and $p \in \mathbb{N}$. Then the following hold:*

(a) *If $X \in A\{7\}$, then $AX = A^p X^p$.*
(b) *If $X \in A\{1^\ell, 7\}$ then $X \in A\{2\}$ and $\mathrm{R}(X) = \mathrm{R}(A^\ell)$.*

Lemma 2.3 ([2]). *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\ell \geq \mathrm{Ind}(A) = k$, and $p \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the following hold:*

(a) $A^d = \left(A^{\textcircled{\dagger},E}\right)^{\ell+1} A^\ell$.
(b) $A^\ell (A^{\textcircled{\dagger},E})^\ell A^\ell = A^\ell$.
(c) $(A^{\textcircled{\dagger},E})^p = (A^p)^{\textcircled{\dagger},E}$.

3. The E-relative m-WG inverse

In this section, we introduce the concept of m-weak group inverse relative to an invertible Hermitian weight. Different characterizations and properties of this new generalized inverse are determined.

Definition 3.1. Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. The matrix $A^{\textcircled{w}_m^E} \in \mathbb{C}^{n\times n}$ satisfying

$$A^{\textcircled{w}_m^E} = (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^m, \tag{3.1}$$

is called the m-weak group inverse of A relative to E (shortly, E-relative m-WG inverse).

In the above definition, we have assumed that $A^k\{1, 3^E\} \neq \emptyset$, which ensures the existence of the E-weighted core-EP inverse of $A \in \mathbb{C}^{n\times n}$. Consequently, $A^{\textcircled{w}_m^E}$ also exists and is unique.

Remark 3.2. (a) If $m \geq \mathrm{Ind}(A)$, note that from Lemma 2.3 (a) we have $A^{\textcircled{w}_m^E} = A^d$.
(b) According to item (a), it is evident that $A^{\textcircled{w}_m^E} = A^\#$ when $\mathrm{Ind}(A) = 1$.
(c) If $E = I_n$, the E-relative m-WG inverse coincides with the m-WG inverse, that is, $A^{\textcircled{w}_m^E} = A^{\textcircled{w}_m}$.

In the following example, we show that when $1 \leq m < k$, the E-relative m-WG inverse is different from the E-weighted core-EP inverse and the m-WG inverse (whenever $E \neq I_n$).

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Example 3.3. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As $\mathrm{Ind}(A) = 3$, we consider $1 \le m < 3$. Thus,

$$A^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^{\textcircled{\dagger},\mathrm{E}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{\textcircled{w}_1} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^{\textcircled{w}_1^E} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{\textcircled{w}_2} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^{\textcircled{w}_2^E} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In order to establish characterizations and properties of the E-relative m-WG inverse, we first show an auxiliary lemma.

Lemma 3.4. *Let $A \in \mathbb{C}^{n\times n}$, $\ell \ge \mathrm{Ind}(A) = k$, and $p, q \in \mathbb{N}$. Then the following hold:*

(a) *If $X \in A\{7\}$, then $A^q X^{p+q} = X^p$.*
(b) *If $X \in A\{1^k\}$, then $X^p A^{\ell+p} = A^\ell$.*

Proof. (a) In order to prove the assertion, we will apply mathematical induction on p and q. In fact, as $X \in A\{7\}$, the affirmation is true for $p = q = 1$.
Now, we assume that $A^q X^{p+q} = X^p$ is true for some $p, q \in \mathbb{N}$. Then, $A^q X^{q+p+1} = (A^q X^{p+q})X = X^{p+1}$. Moreover, Proposition 2.2 (a) yields to

$$A^{q+1} X^{(q+1)+p} = (A^{q+1} X^{q+1}) X^p = A^q X^{q+p} = X^p.$$

Thus, the statement is true for all $p, q \in \mathbb{N}$.
(b) As $X \in A\{1^k\}$, it is evident that $XA^{\ell+1} = A^\ell$ for each integer $\ell \ge k$. Therefore, assertion (b) holds for $p = 1$. Now, suppose that $X^p A^{\ell+p} = A^\ell$ is true for some $p \in \mathbb{N}$. Then $X^{p+1} A^{\ell+p+1} = X(X^p A^{\ell+p})A = XA^{\ell+1} = A^\ell$, which implies that the affirmation is true for all $p \in \mathbb{N}$. ☐

From Definition 2.1, we know that $A^{\textcircled{\dagger},\mathrm{E}} \in A\{1^k, 7\}$. Therefore, the following corollary can be deduced from the above lemma.

Corollary 3.5. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\ell \ge \mathrm{Ind}(A) = k$, and $p, q \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \ne \emptyset$. Then the following hold:*

(a) $A^q (A^{\textcircled{\dagger},E})^{p+q} = (A^{\textcircled{\dagger},E})^p$.

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(b) $(A^{\textcircled{\dagger},E})^p A^{\ell+p} = A^\ell$.

Now, we give the first algebraic characterization of the E-relative m-WG inverse.

Theorem 3.6. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the system of equations*

$$AX = (A^{\textcircled{\dagger},E})^m A^m \quad \text{and} \quad \mathrm{R}(X) \subseteq \mathrm{R}(A^k), \tag{3.2}$$

is consistent and its unique solution is the matrix $X = A^{\textcircled{w}_m^E}$.

Proof. Existence. Let $X := A^{\textcircled{w}_m^E} = (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^m$. By applying Corollary 3.5 (a) with $p = m$ and $q = 1$, we have that X satisfies the first condition in (3.2). Also, from (2.3) it follows that $\mathrm{R}(X) \subseteq \mathrm{R}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{R}(A^k)$, which implies that X satisfies the second condition in (3.2).
Uniqueness. Let X_1 and X_2 be two solutions of the system (3.2), that is,

$$AX_1 = AX_2 = (A^{\textcircled{\dagger},\mathrm{E}})^m A^m, \quad \mathrm{R}(X_1) \subseteq \mathrm{R}(A^k), \quad \mathrm{R}(X_2) \subseteq \mathrm{R}(A^k).$$

Thus, $\mathrm{R}(X_1 - X_2) \subseteq \mathrm{N}(A) \subseteq \mathrm{N}(A^k)$ and $\mathrm{R}(X_1 - X_2) \subseteq \mathrm{R}(A^k)$. Therefore, $\mathrm{R}(X_1 - X_2) \subseteq \mathrm{R}(A^k) \cap \mathrm{N}(A^k) = \{0\}$. Thus, $X_1 = X_2$. □

Note that we always have

$$AX^2 = X \quad \Longrightarrow \quad \mathrm{R}(X) \subseteq \mathrm{R}(A^k), \tag{3.3}$$

where $k = \mathrm{Ind}(A)$ [4, Lemma 4.1]. So, we derive the following characterization.

Corollary 3.7. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the system of equations*

$$AX = (A^{\textcircled{\dagger},E})^m A^m \quad \text{and} \quad AX^2 = X, \tag{3.4}$$

is consistent and its unique solution is the matrix $X = A^{\textcircled{w}_m^E}$.

Proof. Let $X := A^{\textcircled{w}_m^E} = (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^m$. As in the proof of Theorem 3.6, it is clear that X satisfies the first condition of system (3.4). In consequence, from Corollary 3.5 (a) with $q = m$ and $p = 1$, we obtain

$$AX^2 = (A^{\textcircled{\dagger},\mathrm{E}})^m (A^m (A^{\textcircled{\dagger},\mathrm{E}})^{m+1}) A^m = (A^{\textcircled{\dagger},\mathrm{E}})^m A^{\textcircled{\dagger},\mathrm{E}} A^m = X.$$

Finally, the uniqueness immediately follows from (3.3) and Theorem 3.6. □

Now, some properties and recursive expressions (depending on $m \in \mathbb{N}$) of the E-relative m-WG inverse are established.

Theorem 3.8. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\ell \geq \mathrm{Ind}(A) = k$, and $m, p, q \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the following hold:*

(a) $A^{\textcircled{w}_m^E} \in A\{1^\ell, 2, 7\}$.
(b) $(A^{\textcircled{w}_m^E})^\ell \in (A^\ell)\{1\}$.
(c) $A^q (A^{\textcircled{w}_m^E})^{p+q} = (A^{\textcircled{w}_m^E})^p$.
(d) $(A^{\textcircled{w}_m^E})^p A^{\ell+p} = A^\ell$.
(e) $A^{\textcircled{w}_{m+1}^E} = A^{\textcircled{\dagger},E} A^{\textcircled{w}_m^E} A$.

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(f) $A^{\textcircled{w}_{m}^{E}} A^{\textcircled{w}_{m+1}^{E}} A = A^{\textcircled{w}_{m}^{E}} A A^{\textcircled{w}_{m+2}^{E}}$.

Proof. (a) By Corollary 3.7 we get $A^{\textcircled{w}_{m}^{E}} \in A\{7\}$. Also, as $A^{\textcircled{w}_{m}^{E}} = (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^m$, from Corollary 3.5 (a) with $q = m$ and $p = 1$, we obtain that $A^{\textcircled{w}_{m}^{E}}$ is an outer inverse of A. Moreover, from Corollary 3.5 (b) with $p = m + 1$, we deduce $A^{\textcircled{w}_{m}^{E}} A^{\ell+1} = (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^{\ell+m+1} = A^{\ell}$, that is, $A^{\textcircled{w}_{m}^{E}} \in A\{1^{\ell}\}$.
(b) From (a) we know that $A^{\textcircled{w}_{m}^{E}} \in A\{7\}$. Thus, from Proposition 2.2 (a) and Theorem 3.6 we obtain

$$A^{\ell}(A^{\textcircled{w}_{m}^{E}})^{\ell} A^{\ell} = A A^{\textcircled{w}_{m}^{E}} A^{\ell} = (A^{\textcircled{\dagger},\mathrm{E}})^{m} A^{m+\ell} = A^{\ell},$$

where the last equality is due to Corollary 3.5 (b) with $p = m$.
(c) Follows from statement (a) and Lemma 3.4.
(d) It is due to (a) and Lemma 3.4 (b).
(e) As $A^{\textcircled{w}_{m}^{E}} = (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^m$ we get $A^{\textcircled{\dagger},\mathrm{E}} A^{\textcircled{w}_{m}^{E}} A = (A^{\textcircled{\dagger},\mathrm{E}})^{m+2} A^{m+1} = A^{\textcircled{w}_{m+1}^{E}}$.
(f) By using (e) and the first condition of Theorem 3.6 with $m + 2$ instead of m, we deduce

$$A^{\textcircled{w}_{m}^{E}} A^{\textcircled{w}_{m+1}^{E}} A = A^{\textcircled{w}_{m}^{E}} A^{\textcircled{\dagger},\mathrm{E}} A^{\textcircled{w}_{m}^{E}} A^2 = A^{\textcircled{w}_{m}^{E}} (A^{\textcircled{\dagger},\mathrm{E}})^{m+2} A^{m+2} = A^{\textcircled{w}_{m}^{E}} A A^{\textcircled{w}_{m+2}^{E}}.$$

□

Corollary 3.9. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$ and define $A_1^E = A A^{\textcircled{\dagger},E} A$. Then the following properties hold:*
(a) $A^{\textcircled{w}_{m}^{E}} \in A_1^E\{2\}$.
(b) *If $k = 1$, then* $A^{\textcircled{w}_{m}^{E}} \in A_1^E\{1, 2\}$.

Proof. (a) From (2.3) and Theorem 3.8 (a) we have

$$A^{\textcircled{w}_{m}^{E}} A_1^E A^{\textcircled{w}_{m}^{E}} = A^{\textcircled{w}_{m}^{E}} A A^{\textcircled{\dagger},\mathrm{E}} A (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^m = A^{\textcircled{w}_{m}^{E}} A A^{\textcircled{w}_{m}^{E}} = A^{\textcircled{w}_{m}^{E}}.$$

(b) From Theorem 3.6, Corollary 3.5 (a) with $p = m$ and $q = 1$, and Corollary 3.5 (b) with $p = m$ and $\ell = 1$, we have

$$\begin{aligned} A_1^E A^{\textcircled{w}_{m}^{E}} A_1^E &= A A^{\textcircled{\dagger},\mathrm{E}} (A A^{\textcircled{w}_{m}^{E}}) A A^{\textcircled{\dagger},\mathrm{E}} A = (A (A^{\textcircled{\dagger},\mathrm{E}})^{m+1}) A^{m+1} A^{\textcircled{\dagger},\mathrm{E}} A \\ &= ((A^{\textcircled{\dagger},\mathrm{E}})^{m} A^{m+1}) A^{\textcircled{\dagger},\mathrm{E}} A = A A^{\textcircled{\dagger},\mathrm{E}} A = A_1^E. \end{aligned}$$

□

4. Some geometric properties of the E-relative m-WG inverse

In this section, we show that the E-relative m-WG inverse can be written as a generalized inverse with prescribed range and null space. Also, we consider some special idempotent matrices determined by this inverse.

Recall that the outer inverse of a matrix $A \in \mathbb{C}^{m\times n}$ which is uniquely determined by the null space S and the column space T is denoted by $A^{(2)}_{T,S} = X \in \mathbb{C}^{n\times m}$ and satisfies $XAX = X$, $\mathrm{N}(X) = S$, and $\mathrm{R}(X) = T$, where $s \leq r = \mathrm{rk}(A)$ is dimension of the subspace $T \subseteq \mathbb{C}^n$ and $n - s$ is dimension of the subspace $S \subseteq \mathbb{C}^n$.

From (2.3), for each $\ell \geq \mathrm{Ind}(A)$, we have $\mathrm{R}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{R}(A^{\ell})$. Next, we will derive information about the null space.

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Proposition 4.1. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\ell \geq \operatorname{Ind}(A) = k$. Suppose $A^k\{1,3^E\} \neq \emptyset$. Then* $\mathrm{N}(A^{\textcircled{\dagger},E}) = \mathrm{N}((A^\ell)^* E)$.

Proof. From (2.3), we have $\mathrm{N}(A^{\textcircled{\dagger},\mathrm{E}})^\perp = \mathrm{R}((A^{\textcircled{\dagger},\mathrm{E}})^*) = \mathrm{R}(EA^k) = E\,\mathrm{R}(A^k) = \mathrm{R}(EA^\ell)$. Thus, as E is Hermitian, $\mathrm{N}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{R}(EA^\ell)^\perp = \mathrm{N}((A^\ell)^*E)$. □

Remark 4.2. Let A, B, and C be complex rectangular matrices of adequate size such that $\mathrm{N}(A) = \mathrm{N}(B)$. Then $\mathrm{N}(AC) = \mathrm{N}(BC)$.

Theorem 4.3. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$ and* $\operatorname{Ind}(A) = k$*. Suppose $A^k\{1,3^E\} \neq \emptyset$. Then the following hold:*

(a) $A^{\textcircled{\dagger},E} = A^{(2)}_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*E)} = A^k((A^k)^*EA^{k+1})^\dagger(A^k)^*E.$
(b) $AA^{\textcircled{\dagger},E} = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*E)}.$
(c) $A^{\textcircled{\dagger},E}A = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA)}.$

Proof. (a) The first equality follows from (2.3) and Proposition 4.1. By Urquhart formula [3], we obtain the second equality.
(b) From (2.3), $AA^{\textcircled{\dagger},\mathrm{E}}$ is idempotent and $\mathrm{R}(AA^{\textcircled{\dagger},\mathrm{E}}) = A\,\mathrm{R}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{R}(A^{k+1}) = \mathrm{R}(A^k)$. Moreover, Proposition 4.1 yields to $\mathrm{N}(AA^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{N}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{N}((A^k)^*E)$.
(c) Since $A^{\textcircled{\dagger},\mathrm{E}}$ is an outer inverse, it is evident that $A^{\textcircled{\dagger},\mathrm{E}}A$ is idempotent and $\mathrm{R}(A^{\textcircled{\dagger},\mathrm{E}}A) = \mathrm{R}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{R}(A^k)$. Also, from Proposition 4.1 it follows that $\mathrm{N}(A^{\textcircled{\dagger},\mathrm{E}}A) = \mathrm{N}((A^k)^*EA)$ due to Remark 4.2. □

Theorem 4.4. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\ell \geq \operatorname{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1,3^E\} \neq \emptyset$. Then the following hold:*

(a) $\mathrm{R}(A^{\textcircled{w}^E_m}) = \mathrm{R}(A^\ell)$.
(b) $\mathrm{rk}(A^{\textcircled{w}^E_m}) = \mathrm{rk}(A^\ell) = \mathrm{rk}((A^\ell)^*EA^m)$.
(c) $\mathrm{N}(A^{\textcircled{w}^E_m}) = \mathrm{N}((A^\ell)^*EA^m)$.

Proof. (a) From Theorem 3.8, (a) we have that $A^{\textcircled{w}^E_m} \in A\{1^\ell,7\}$. Now, the assertion follows from Proposition 2.2 (b).
(b) From (a), it is clear that $\mathrm{rk}(A^{\textcircled{w}^E_m}) = \mathrm{rk}(A^\ell)$. Moreover, using the fact that $A^{\textcircled{w}^E_m} = (A^{\textcircled{\dagger},\mathrm{E}})^{m+1}A^m$ and (2.3) we obtain

$$\begin{aligned}\mathrm{rk}(A^{\textcircled{w}^E_m}) &\leq \mathrm{rk}(A^{\textcircled{\dagger},\mathrm{E}}A^m) = \mathrm{rk}((A^m)^*(A^{\textcircled{\dagger},\mathrm{E}})^*) = \mathrm{rk}((A^m)^*EA^\ell)\\ &= \mathrm{rk}((A^\ell)^*EA^m) \leq \mathrm{rk}(A^\ell).\end{aligned}$$

(c) Theorem 3.8 (a) implies $A^{\textcircled{w}^E_m} \in A\{2\}$, whence $\mathrm{N}(A^{\textcircled{w}^E_m}) = \mathrm{N}(AA^{\textcircled{w}^E_m})$. In consequence, from Theorem 3.6, Corollary 3.5 (a) with $p = 1$ and $q = m$, Proposition 4.1, and Remark 4.2, we obtain

$$\mathrm{N}(A^{\textcircled{w}^E_m}) = \mathrm{N}((A^{\textcircled{\dagger},\mathrm{E}})^mA^m) \subseteq \mathrm{N}(A^{\textcircled{\dagger},\mathrm{E}}A^m) = \mathrm{N}((A^\ell)^*EA^m).$$

Now, the affirmation follows from (b). □

According to Theorem 4.4, by a procedure similar to the proof of Theorem 4.3, one can show the following result.

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Theorem 4.5. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\ell \geq \operatorname{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the following hold:*

(a) $A^{\circledR{w}^E_m} = A^{(2)}_{\mathrm{R}(A^\ell),\mathrm{N}((A^\ell)^* E A^m)} = A^\ell((A^\ell)^* E A^{m+\ell+1})^\dagger (A^\ell)^* E A^m$.

(b) $AA^{\circledR{w}^E_m} = P_{\mathrm{R}(A^\ell),\mathrm{N}((A^\ell)^* E A^m)}$.

(c) $A^{\circledR{w}^E_m} A = P_{\mathrm{R}(A^\ell),\mathrm{N}((A^\ell)^* E A^{m+1})}$.

5. Further characterizations of the E-relative m-WG inverse

In this section, we present more characterizations of the E-relative m-WG inverse involving the column space, null space, rank equalities, and projectors.

We begin with an auxiliary lemma.

Lemma 5.1. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{Ind}(A) = k$. Then the following are equivalent:*

(a) $\mathrm{R}(X) \subseteq \mathrm{R}(A^k)$ *and* $XA^{k+1} = A^k$;

(b) $\mathrm{R}(A^k) = \mathrm{R}(X)$ *and* $XAX = X$;

(c) $AX^2 = X$ *and* $XA^{k+1} = A^k$.

Proof. (a)⇔(b) It was proved in [4, Lemma 4.2].

(b)⇔(c) By [5, Lemma 4.3]. □

Theorem 5.2. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\operatorname{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the following are equivalent:*

(a) $X = A^{\circledR{w}^E_m}$;

(b) $XAX = X$, $\mathrm{R}(X) = \mathrm{R}(A^k)$, $((A^m)^* E A^{m+1} X)^* = (A^m)^* E A^{m+1} X$;

(c) $XAX = X$, $\mathrm{R}(X) = \mathrm{R}(A^k)$, $\mathrm{N}(X) = \mathrm{N}((A^k)^* E A^m)$;

(d) $XAX = X$, $XA^{k+1} = A^k$, $AX = (A^{\textcircled{\dagger},E})^m A^m$.

Proof. (a)⇒(b) Let $X := A^{\circledR{w}^E_m}$. The first two conditions follow from Theorem 4.5. Also, by Theorem 3.6 and Proposition 2.2 (a), we obtain

$$(A^m)^* E A^{m+1} X = (A^m)^* E A^{m+1} (A^{\textcircled{\dagger},\mathrm{E}})^{m+1} A^m = (A^m)^* E A A^{\textcircled{\dagger},\mathrm{E}} A^m,$$

which is an Hermitian matrix because $EAA^{\textcircled{\dagger},\mathrm{E}}$ is Hermitian too.

(b)⇒(c) The second and third conditions imply

$$\mathrm{R}((A^m)^* E A^k) = \mathrm{R}((A^m)^* E A^{m+k+1}) = \mathrm{R}((A^m)^* E A^{m+1} X) \subseteq \mathrm{R}(X^*).$$

Moreover, from Theorem 4.4 (b), we get $\operatorname{rk}(X^*) = \operatorname{rk}(A^k) = \operatorname{rk}((A^m)^* E A^k)$. So, $\mathrm{R}(X^*) = \mathrm{R}((A^m)^* E A^k)$, or equivalently $\mathrm{N}(X) = \mathrm{N}((A^k)^* E A^m)$.

(c)⇒(d) By Lemma 5.1, we have $XA^{k+1} = A^k$. Also, as $X = XAX$, we have that AX is idempotent. In consequence, Proposition 4.1 and Remark 4.2 imply

$$\mathrm{N}(AX) = \mathrm{N}(X) = \mathrm{N}((A^k)^* E A^m) = \mathrm{N}((A^{\textcircled{\dagger},\mathrm{E}})^m A^m).$$

Also, Corollary 3.5 (b) with $p = m$ and $\ell = k$, yields

$$\mathrm{R}(AX) = \mathrm{R}(A^k) = \mathrm{R}((A^{\textcircled{\dagger},\mathrm{E}})^m A^{m+k}) \subseteq \mathrm{R}((A^{\textcircled{\dagger},\mathrm{E}})^m A^m) \subseteq \mathrm{R}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{R}(A^k).$$

As $(A^{\textcircled{\dagger},\mathrm{E}})^m = (A^m)^{\textcircled{\dagger},\mathrm{E}}$ by Lemma 2.3 (c), it is evident that $(A^{\textcircled{\dagger},\mathrm{E}})^m A^m$ is idempotent. Now, by uniqueness of projectors, we deduce that $AX = (A^{\textcircled{\dagger},\mathrm{E}})^m A^m$.

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(d)⇒(a) From $XA^{k+1} = A^k$, we get $\mathrm{R}(A^k) \subseteq \mathrm{R}(X)$. Also, $AX = (A^{\textcircled{\dagger},\mathrm{E}})^m A^m$ implies $\mathrm{rk}(X) = \mathrm{rk}(AX) = \mathrm{rk}((A^{\textcircled{\dagger},\mathrm{E}})^m A^m) \leq \mathrm{rk}(A^{\textcircled{\dagger},\mathrm{E}}) = \mathrm{rk}(A^k)$. Thus, $\mathrm{R}(X) = \mathrm{R}(A^k)$. Finally, Theorem 3.6 completes the proof. □

Theorem 5.3. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the following are equivalent:*

(a) $X = A^{\textcircled{w}_m^E}$;
(b) $XA^{k+1} = A^k$, $AX^2 = X$, $A^{m+1}X = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*E)}A^m$;
(c) $XA^{k+1} = A^k$, $AX^2 = X$, $((A^m)^*EA^{m+1}X)^* = (A^m)^*EA^{m+1}X$;
(d) $XA^{k+1} = A^k$, $\mathrm{rk}(X) = \mathrm{rk}(A^k)$, $AX = (A^{\textcircled{\dagger},E})^m A^m$.

Proof. (a)⇒(b) Let $X := A^{\textcircled{w}_m^E}$. Clearly, X satisfy the first two conditions due to Theorem 3.8 (a). In order to prove the third condition, note that, by Theorem 3.6, we have $AX = (A^{\textcircled{\dagger},\mathrm{E}})^m A^m$. Multiplying this expression by A^m on the left side and using Proposition 2.2 (a), we have $A^{m+1}X = A^m(A^{\textcircled{\dagger},\mathrm{E}})^m A^m = AA^{\textcircled{\dagger},\mathrm{E}}A^m = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*E)}A^m$, where the last equality is due to Theorem 4.3 (b).
(b)⇒(c) Pre-multiplying the third condition by $(A^m)^*E$ and applying Theorem 4.3 (b), we have that

$$(A^m)^*EA^{m+1}X = (A^m)^*EP_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*E)}A^m = (A^m)^*EAA^{\textcircled{\dagger},\mathrm{E}}A^m,$$

which is Hermitian because $A^{\textcircled{\dagger},\mathrm{E}} \in A\{3^E\}$.
(c)⇒(d) It is a consequence from Lemma 5.1 and Theorem 5.2.
(d)⇒(a) The first two conditions imply $\mathrm{R}(X) = \mathrm{R}(A^k)$. Now, Theorem 3.6 concludes the proof. □

Theorem 5.4. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the following are equivalent:*

(a) $X = A^{\textcircled{w}_m^E}$;
(b) $AX = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^m)}$, $XA = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^{m+1})}$, $XAX = X$;
(c) $AX = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^m)}$, $XA = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^{m+1})}$, $\mathrm{rk}(X) = \mathrm{rk}(A^k)$.

Proof. (a)⇔(b) By Theorem 4.5 and [3, Theorem 14, p. 72].
(b)⇒(c) As $XAX = X$, we have $\mathrm{rk}(X) = \mathrm{rk}(AX) = \mathrm{rk}(P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^m)}) = \mathrm{rk}(A^k)$.
(c)⇒(b) The second and third conditions imply $\mathrm{R}(X) = \mathrm{R}(A^k) = \mathrm{R}(XA) = \mathrm{N}(I_n - XA)$ because XA is idempotent. In consequence, $XAX = X$. □

Corollary 5.5. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the following are equivalent:*

(a) $X = A^{\textcircled{w}_m^E}$;
(b) $AX = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^m)}$ *and* $\mathrm{R}(X) \subseteq \mathrm{R}(A^k)$;
(c) $XA = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^{m+1})}$ *and* $\mathrm{N}((A^k)^*EA^m) \subseteq \mathrm{N}(X)$.

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Proof. (a)⇔(b) It is sufficient to note that $P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^m)} = (A^{\textcircled{\dagger},\mathrm{E}})^m A^m$. Now, the equivalence follows from Theorem 3.6.
(a)⇔(c) Let $X := A^{\textcircled{w}_m^E}$. From Theorem 5.2 and Theorem 5.4 we can see that X satisfy $XA = P_{\mathrm{R}(A^k),\mathrm{N}((A^k)^*EA^{m+1})}$ and $\mathrm{N}((A^k)^*EA^m) \subseteq \mathrm{N}(X)$. The uniqueness can be proved in a similar manner to the proof of Theorem 3.6. □

6. Applications of the E-relative m-WG inverse

In order to solve a linear equation in this section, we apply the E-relative m-WG inverse.

Firstly, we present new representation of the E-relative m-WG inverse, which will be useful.

Lemma 6.1. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m, p \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then, for $(A^k)^{(1,3^E)} \in (A^k)\{1, 3^E\}$, the following hold:*
(a) $(A^{\textcircled{\dagger},E})^p = (A^d)^p A^k (A^k)^{(1,3^E)}$.
(b) $A^{\textcircled{w}_m^E} = (A^d)^{m+1} A^k (A^k)^{(1,3^E)} A^m$.

Proof. By [2], $A^{\textcircled{\dagger},\mathrm{E}} = A^d A^k (A^k)^{(1,3^E)}$. Then item (a) follows by induction on p and item (b) is a direct consequence of item (a). □

Theorem 6.2. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$, and $m \in \mathbb{N}$. Suppose $A^k\{1, 3^E\} \neq \emptyset$. Then the equation*

$$(A^k)^* EA^{m+1} X = (A^k)^* EA^m B, \tag{6.1}$$

has the general solution expressed by

$$X = A^{\textcircled{w}_m^E} B + (I_n - A^{\textcircled{w}_m^E} A)Y, \tag{6.2}$$

for arbitrary $Y \in \mathbb{C}^{n\times q}$.

Proof. Using Lemma 6.1, $A^{\textcircled{w}_m^E} = (A^d)^{m+1} A^k (A^k)^{(1,3^E)} A^m$, where $(A^k)^{(1,3^E)} \in (A^k)\{1, 3^E\}$. Now,

$$\begin{aligned}(A^k)^* EA^{m+1} A^{\textcircled{w}_m^E} &= (A^k)^* EA^{m+1} (A^d)^{m+1} A^k (A^k)^{(1,3^E)} A^m \\ &= (A^k)^* EAA^d A^k (A^k)^{(1,3^E)} A^m \\ &= (A^k)^* EA^k (A^k)^{(1,3^E)} A^m = (EA^k (A^k)^{(1,3^E)} A^k)^* A^m \\ &= (A^k)^* EA^m.\end{aligned}$$

For X given by (6.2), it follows that (6.1) holds. Assume that X is a solution to (6.1). Since E is invertible and

$$\begin{aligned}EA^k (A^k)^{(1,3^E)} A^{m+1} X &= ((A^k)^{(1,3^E)})^* ((A^k)^* EA^{m+1} X) \\ &= ((A^k)^{(1,3^E)})^* (A^k)^* EA^m B \\ &= EA^k (A^k)^{(1,3^E)} A^m B,\end{aligned}$$

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we deduce that $A^k(A^k)^{(1,3^E)}A^{m+1}X = A^k(A^k)^{(1,3^E)}A^mB$. Therefore,

$$\begin{aligned} A^{ⓦ_m^E}B &= (A^d)^{m+1}A^k(A^k)^{(1,3^E)}A^mB = (A^d)^{m+1}A^k(A^k)^{(1,3^E)}A^{m+1}X \\ &= A^{ⓦ_m^E}AX, \end{aligned}$$

which gives $X = A^{ⓦ_m^E}B + (I_n - A^{ⓦ_m^E}A)X$ has the form (6.2). □

Notice that, for $E = I_n$ in Theorem 6.2, we recover [15, Theorem 4.1].

We also can give the next result related to the E-weighted core-EP inverse.

Corollary 6.3. *Let $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix and let $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$, and $\operatorname{Ind}(A) = k$. Suppose $A^k\{1,3^E\} \neq \emptyset$. Then the equation*

$$(A^k)^*EAX = (A^k)^*EB,$$

has the general solution expressed by

$$X = A^{Ⓣ,E}B + (I_n - A^{Ⓣ,E}A)Y,$$

for arbitrary $Y \in \mathbb{C}^{n\times q}$.

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