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BOTT–DUFFIN (e, f) INVERSES IN e-SYMMETRIC RINGS

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Abstract. We first investigate several properties of Bott–Duffin (e, f) inverses in e-symmetric rings. In particular, we present new characterizations of Bott–Duffin (e, f) inverses in terms of units in such rings. We also study the EP property of an element a in an ab-symmetric ring, where b is a generalized inverse of a. Finally, we consider the Bott–Duffin decomposition associated with idempotents.

1. Introduction

Throughout this paper we assume that R is a ring with the unity 1. Drazin [2] introduced a class of outer generalized inverses relative to a pair of idempotents e and f, which intermediates between the Bott–Duffin inverse [1] and the (b, c)-inverse [2] in a ring. This particular type of generalized inverse is denominated as the Bott–Duffin (e, f) inverse. The B-D (e, f) inverse of a is defined as the unique element $y \in R$, when it exists, which satisfies the equations $y = ey = yf$, $yae = e$ and $fay = f$. Whenever it is more convenient, we shall use the widely accepted shorthand "B-D" for "Bott–Duffin" in the text.

In [2, Theorem 2.2], it is proven that a is B-D (e, f) invertible if and only if $e \in Rfae$ and $f \in faeR$. In particular, a has a B-D inverse if and only if $e \in Reae \cap eaeR$. Note that $e \in Reae \cap eaeR$ holds if and only if $eae + 1 - e$ is invertible in R. Consequently, it is natural to determine whether the existence of B-D (e, f) inverses can be characterized by means of the invertibility of specific elements. To our knowledge, the investigation of this topic can be found in [5, 16]. In [5], it has been demonstrated that a is B-D (e, f) invertible if and only if there exist $s, t \in R$ such that both $u = sfae+1-e$ and $v = faet+1-f$ are invertible in R. In [16, Theorem 2.2], under the premise that e and f are self-adjoint idempotents, the criteria for the existence of B-D (e, f) inverses are presented in terms of units. Herein, we recall that $*$ is an involution in R if it is a map $* : R \to R$ such that for all $a, b \in R : (a^*)^* = a, (a + b)^* = a^* + b^*, (ab)^* = b^*a^*$. In this case, we need to assume that R is a ring with an involution (That is to say, R is a $*$-ring). Moreover, an idempotent e is said to be self-adjoint if $e = e^*$. It has been established that

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the element a is B-D (e, f) invertible if and only if both $u = (fae)^*fae + 1 - e$ and $v = fae(fae)^* + 1 - f$ are invertible [16].

In the reference [9], the authors introduced a class of weakly symmetric rings, namely e-symmetric rings, where e is an idempotent in R. Recall that R is e-symmetric if the condition $abc = 0$ necessarily implies $acbe = 0$. An idempotent e of a ring R is called left (right) semicentral if $re = ere$ ($er = ere$) for each $r \in R$. It is proven that R is e-symmetric if and only if e is left semicentral and eRe is symmetric. The studies of symmetric rings and their generalized forms can also be traced in a series of works, including [7, 4, 8, 9, 10, 11, 15, 12, 19]. Inspired by the above discussion, in this article, with a focus on the characterizations of B-D (e, f) inverses, we endeavor to further explore how the existence of B-D (e, f) inverses can be characterized by the invertibility of specific elements within the context of e-symmetric rings.

In what follows, some standard terminology and notations in rings are given. We use $E(R)$ to denote the set of all idempotents, and use R^{-1} denote the set of all invertible elements in R. An element $a \in R$ is said to be (von Neumann) regular if there exists an element $x \in R$ such that $axa = a$. Such x is referred to as a regular (or inner) inverse of a and is denoted by a^-. The group inverse of a is the unique element $a^\sharp$, whenever it exists, satisfying $aa^\sharp a = a$, $a^\sharp aa^\sharp = a^\sharp$ and $aa^\sharp = a^\sharp a$. We use $R^\sharp$ to denote the set of all group invertible elements. The set consisting of all self-adjoint idempotent elements of R (i.e., $e = e^2 = e^*$) will be denoted by $P(R)$. An element $a \in R$ is said to be Moore-Penrose invertible with respect to the involution $*$ if the four equations

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa$$

have a unique common solution. When such a solution exists, it is denoted by $a^\dagger$, and the set of all Moore-Penrose invertible elements in R is denoted by $R^\dagger$. If an element $x \in R$ satisfies both Eqs. (1) and (3), then x is called an $\{1, 3\}$-inverse of a, and is denoted by $a^{(1,3)}$. The set of all $\{1, 3\}$-invertible elements of R is denoted by $R\{1, 3\}$. Analogously, if $x \in R$ satisfies both Eqs. (1) and (4), then x is referred to as an $\{1, 4\}$-inverse of a, and is denoted by $a^{(1,4)}$, and the set of all $\{1, 4\}$-invertible elements in R is denoted by $R\{1, 4\}$.

2. Bott–Duffin (e, f) inverses in e-symmetric rings

The inspiration for this section stems from the following results in [9]. In this reference, the authors consider some properties of B-D (e, f) inverses in e-symmetric rings.

Lemma 2.1 ([9, Proposition 2.6]). *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a has a B-D (e, f) inverse y, then R is f-symmetric and $eR = fR$.*

In point of fact, a straightforward verification reveals that the converse also holds true.

Lemma 2.2. *Let $e, f \in E(R)$ such that $fR \subseteq eR$. If R is e-symmetric, then R is f-symmetric.*

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Proof. For arbitrary elements $a, b, c \in R$, whenever $abc = 0$ holds, it can be deduced that $acbe = 0$. As $fR \subseteq eR$, then it follows that $f = ef$. As a result, we have $acbf = acbef = 0$. This shows that R is also f-symmetric. $\square$

Proposition 2.3. *Let $a \in R$ and $e, f \in E(R)$. If a is B-D (e, f) invertible, then R is e-symmetric if and only if R is f-symmetric.*

Proof. Necessity: As demonstrated in [9, Proposition 2.6], the necessary condition holds. Sufficiency: Suppose that the ring R is f-symmetric. Then, it can be deduced that the idempotent f is left semicentral. Given that y represents the B-D (e, f) inverse of a, we have the following sequence of equalities:

$$e = yae = yfae = fyfae = fyae = fe.$$

This implies that, $eR \subseteq fR$, and then by Lemma 2.2 we obtain that R is also e-symmetric. $\square$

In the aforementioned proposition, the condition that a is B-D (e, f) invertible must be presupposed. Subsequently, we shall explore the equivalent conditions for the existence of B-D (e, f) inverses.

Lemma 2.4. *Let $a \in R$ and $e \in E(R)$. Then the following conditions are equivalent:*

(i) $e \in eaeR \cap Reae$.
(ii) $eae + 1 - e$ *is invertible.*
(ii) $ae + 1 - e$ *is invertible.*
(ii) $ea + 1 - e$ *is invertible.*

Lemma 2.5 (Jacobson's Lemma)**.** *Let $a, b \in R$. Then $1 - ab \in R^{-1}$ if and only if $1 - ba \in R^{-1}$.*

Theorem 2.6. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, the following are equivalent:*

(i) a *is B-D* (e, f) *invertible.*
(ii) $u = fae + 1 - f$, $v = fae + 1 - e \in R^{-1}$.
In this case, the B-D (e, f) inverse $y = eu^{-1} = v^{-1}f$.

Proof. $(i) \Rightarrow (ii)$ In this context, our primary objective is to demonstrate that $f \in Rf(fae)f \cap f(fae)fR$ and $e \in e(fae)eR \cap Re(fae)e$. Once this is established, by Lemma 2.4, we can infer the invertibility of u and v in R. Let y denote the B-D (e, f) inverse of a. First, we show that $f \in f(fae)fR$. Given that $y = ey = yf$ and f is left semicentral (see Proposition 2.3), we have the following equalities:

$$f = fay = fa(ey) = fae(yf) = faefyf = faefy \in f(fae)fR.$$

This clearly indicates that $f \in f(fae)fR$. Furthermore, on account of $eR = fR$, by Lemma 2.1, we conclude $f = ef$ and $e = fe$. Then, we can show that $f \in Rf(fae)f$ as follows:

$$f = fay = fa(yf) = fay(ef) = fay(yae)f = fay(yf)aef \in Rfaef.$$

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Next, we turn our attention to proving that $e \in e(fae)eR \cap Re(fae)e$. For the left-hand side of the intersection, we note that

$$e = yae = yfae = y(ef)ae \in Re(fae)e.$$

This implies that $e \in Re(fae)e$. On the other hand, we have

$$e = yae = (fy)ae = (ef)yae = e(fay)yae = efa(ey)yae \in e(fae)eR.$$

This shows that $e \in e(fae)eR$. Combining these, we conclude that $e \in e(fae)eR \cap Re(fae)e$.

$(ii) \Rightarrow (i)$ Note that $fu = ve = fae$. Set $y = eu^{-1} = v^{-1}f$. Then, it immediately follows that $ey = yf = y$. Furthermore, through straightforward calculations, we have $yae = v^{-1}fae = e$ and $fay = faeu^{-1} = f$. Consequently, based on the definition of the B-D (e, f) inverse, a is B-D (e, f) invertible and its B-D (e, f) inverse is y. □

Remark 2.7. In the aforementioned theorem, it can be observed that $y = yf = fy$ and $y = ey$. Specifically, the equality $y = ey = yf$ aligns precisely with the definition of the B-D (e, f) inverse. Regarding the equality $y = fy$, since f is left semicentral, we have $y = yf = fyf = fy$. under the conditions of Theorem 2.6, it is thus a worthwhile inquiry to consider whether $y = ye$ holds. Actually, we know if $y = ye$, then $f = fay = faye = fe = e$. Conversely, when $e = f$, it necessarily follows that $y = yf = ye$. This interconnection can be more formally presented as follows.

Proposition 2.8. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a is Bott–Duffin (e, f) invertible with the B-D (e, f) inverse y, then y is also the B-D (f, e) inverse of a if and only if $e = f$.*

In actuality, when we solely focus on the B-D (e, f) invertibility, it can be proved that if R is e-symmetric and a is B-D (e, f) invertible, then a is also B-D (f, e) invertible. However, according to Proposition 2.8, these two inverses are generally not identical,except when the condition $e = f$ is satisfied.

Lemma 2.9. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a is B-D (f, e) invertible, then R is also f-symmetric and $eR = fR$.*

Proof. To begin with, it suffices for us to demonstrate that $fR \subseteq eR$. Once this is established, it is evident that R is f-symmetric. Indeed, let z be established as the B-D (f, e) inverse of a. Then, by the equalities $f = zaf = (ze)af = ezeaf$, we can firmly conclude that $ef = f$. Thus, it is observable that R is f-symmetric, from which it further follows that f is left semicentral. Moreover, when considering the equation $e = eaz = eafz = feafz$, it follows that $fe = e$. Based on this established fact, it is evident that eR and fR are equal sets, i.e., $eR = fR$. □

Similar to the proof of Theorem 2.6, we can obtain the following result.

Corollary 2.10. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, the following are equivalent:*

(i) a is B-D (f, e) invertible.

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(ii) $u = fae + 1 - f$, $v = fae + 1 - e \in R^{-1}$.

Proof. $(i) \Rightarrow (ii)$ Analogously to the proof of Theorem 2.6, we also can prove that $f \in Rf(fae)f \cap f(fae)fR$ and $e \in e(fae)eR \cap Re(fae)e$. Indeed, assuming that a is B-D (f, e) invertible, and denoting z as its the B-D (f, e) inverse, by Lemma 2.9, we derive the following equality:

$$\begin{aligned} f &= zaf = za(ef) = fzaef = f(ezae)f = f(eaz)zaef \\ &= faezzaef = faezzfaef = faefzzfaef. \end{aligned}$$

This clearly show that $f \in f(fae)fR \cap Rf(fae)f$. Similarly, we can also prove that $e \in e(fae)eR \cap Re(fae)e$.

$(ii) \Rightarrow (i)$ According to Theorem 2.6, the B-D (e, f) inverse of a is given by $y = eu^{-1} = v^{-1}f$. Let us define $z = ye$. Then it can be readily verified that $fz = fye = ye = z$ and $ze = z$. Additionally, through the following calculations, we can obtain that $eaz = eaye = eayfe = efayfe = efe = fe = e$, and $zaf = yeaf = yefaf = yfaf = yaf = yaef = ef = f$. This is sufficient to show that a is B-D (f, e) invertible and its B-D (f, e) inverse is z. □

Corollary 2.11. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, then a is B-D (e, f) invertible if and only if a is B-D (f, e) invertible. In this case, $z = ye$ and $y = zf$, where y is the B-D (e, f) inverse of a and z is the B-D (f, e) inverse of a.*

Remark 2.12. By Corollary 2.11, we can see the related properties of B-D (e, f) inverse also hold if we replaced e-symmetric condition with the f-symmetric condition.

Remark 2.13. Let $a \in N(R)$ and $e, f \in E(R)$. Suppose that R is e-symmetric and a is B-D (e, f) invertible. Then, it can be rigorously demonstrated that $e = f = 0$. Indeed, since R is e-symmetric and a is B-D (e, f) invertible, we have $fe = e$ and e is left semicentral. Thus, this implies $(fae)^2 = faefeae = faeae = fa^2e$. By induction, it can be systematically shown that $(fae)^k = fa^ke$ for any positive integer k. As $a \in N(R)$, we have $a^m = 0$ for some positive integer m. It leads to $u^m = (fae)^m + 1 - e = 1 - e \in R^{-1}$, and consequently, $e = 0$. Similarly, we can conclude that $f = 0$ as well.

Question 2.14. Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a is B-D (e, f) invertible with the B-D (e, f) inverse y, then we can obtain that $ay = ayf = fayf = f$, we may wonder if $ya = e$.

Before talking about the question, we first provide the following examples to show that the full matrix ring $M_2(\mathbb{C})$ over the complex field fails to be e-symmetric for any nonzero idempotents. Moreover, the upper triangular matrix ring $T_2(\mathbb{C})$ over the complex field is e-symmetric for certain idempotents.

Example 2.15. Set $R = M_2(\mathbb{C})$ and $e \in E(R)$. If e is left semicentral, then $xe = exe$ for any $x \in R$. Herein, to avoid the trivial case, $e \neq 0$ and $e \notin R^{-1}$, that is, $e \neq I_2$. Indeed, if $e \in R^{-1}$, since $xe = exe$ implies $x = ex$, and choosing $x = I_2$ gives $e = I_2$. Moreover, it is well known that $R = M_2(\mathbb{C})$ is not symmetric in the

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general sense. Based on the above mentioned non-trivial conditions and the non-symmetry of $R = M_2(\mathbb{C})$, one can conclude that the idempotent e has $rank(e) = 1$ and has the following forms:

$$e = \begin{pmatrix} \lambda\alpha & \lambda\beta \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix}$$

or

$$e = \begin{pmatrix} \gamma & \gamma u \\ \delta & \delta u \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}.$$

Case 1. Assume that $e = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix}$. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, by a computation, we have

$$xe = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix},$$
$$exe = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \lambda\alpha \\ \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix}.$$

Consequently, we deduce that $\lambda\alpha = 1$ and $\alpha = 0$, which is an obvious contradiction.

Case 2. Suppose $e = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}$. Since $e^2 = e$, we can infer that $\begin{pmatrix} 1 & u \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 1$, that is, $\gamma + u\delta = 1$.

Select $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. By a computation, we obtain

$$xe = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix},$$
$$exe = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix} \begin{pmatrix} \delta \\ 0 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix} = \begin{pmatrix} \delta\gamma \\ \delta^2 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}.$$

This implies that $\delta = 0$. Substituting $\delta = 0$ into the equation $\gamma + u\delta = 1$, we get $\gamma = 1$.

Now, Choose $x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we find that

$$xe = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}$$

and

$$exe = \begin{pmatrix} u \\ 0 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}.$$

Evidently, $exe \neq xe$.

In conclusion, within the full matrix ring $R = M_2(\mathbb{C})$, if an idempotent $e \in E(R)$ is left semicentral, then e must be either the zero matrix O_2 or the identity matrix I_2. Therefore, we can conclude that the ring $R = M_2(\mathbb{C})$ is not e-symmetric for any nonzero idempotent e.

Example 2.16. Let $R = T_2(\mathbb{C})$ be the subring of 2×2 complex uppertriangular matrices, and consider a nonzero idempotent $e \in R$. Easy computations show that e must take one of the following forms:

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$$e = I_2 \text{ or } e = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \text{ or } e = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{C}$. If R is e-symmetric, then e is left semicentral. This means that for every $x \in T_2(\mathbb{C})$, the equality $xe = exe$ holds true. It is readily verified that under this condition, the idempotent e is restricted to the forms $e = I_2$ or $e = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$. Indeed, if we take $e = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ and choose $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $xe \neq exe$. Furthermore, we can establish that R is e-symmetric whenever $e = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$. Indeed, for any elements $x, y, z \in R$ satisfying $xyz = 0$, we have $xzy = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. It follows that $xzye = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} = O_2$, which confirms the e-symmetric property. Moreover, it is well known that $T_2(\mathbb{C})$ is not symmetric. Indeed, set $A = I_2$, $B = e_{12}$ and $C = e_{11}$. Then we have $ABC = 0$ while $ACB \neq 0$. Based on the above-mentioned arguments, the following conclusion can be drawn: Let $R = T_2(\mathbb{C})$ and $e \in R$ be a nonzero idempotent. Then, R is e-symmetric if and only if $e = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$ for some $b \in \mathbb{C}$.

The subsequent discussion will present a counterexample to demonstrate that, in general, the assertion posed in Question 2.14 does not hold.

Example 2.17. It suffices to take $R = T_2(\mathbb{C})$ and $a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, e = f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Clearly, R is e-symmetric and a is B-D (e, f) invertible with $y = e$ being the B-D (e, f) inverse. But, we can check that $ay = f$ and $ya = a \neq e$.

Remark 2.18. Let $a \in R$ and $e, f \in E(R)$. Assume that R is e-symmetric and a is B-D (e, f) invertible. Then we can show that $v = fae + 1 - e \in R^{-1}$. By applying Lemma 2.5 and $ef = f$, we can deduce that $s = fa + 1 - e \in R^{-1}$. In light of these results, we propose the following conjectures: The elements $\tilde{s} = af + 1 - e$, $t = ea + 1 - f$ and $\tilde{t} = ae + 1 - f$ are all invertible in R. Furthermore, assume that $t = ea + 1 - f \in R^{-1}$. This assumption entails that $fea + 1 - f \in R^{-1}$ and further that $eaf + 1 - f \in R^{-1}$. Based on the above, it is reasonable to investigate the following result.

Proposition 2.19. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and $eR \subseteq fR$, then the following are equivalent:*
(1) $x_1 = eaf + 1 - f \in R^{-1}$.
(2) $x_2 = ea + 1 - f \in R^{-1}$.
(3) $x_3 = ae + 1 - f \in R^{-1}$.
(4) $x_4 = fae + 1 - f \in R^{-1}$.
(5) $x_5 = eae + 1 - f \in R^{-1}$.

Proof. Since R is e-symmetric, we have $fR \subseteq eR$ under all circumstances. Indeed, given that $x_i \in R^{-1}$ and e is left semicentral, we conclude that for some $r_i, \tilde{r}_i \in R$,

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$fx_i = fr_ie\tilde{r}_i = efr_ie\tilde{r}_i$, which implies that $f = efr_ie\tilde{r}_ix_i^{-1}$ and $fR \subseteq eR$. When combined with $eR \subseteq fR$, it follows that $ef = f$, $fe = e$ and R is also f-symmetric. This indicates that both e and f are left semicentral. Consequently, we can derive the equalities $ae = afe = fafe = fae = feae = eae$, which gives that $x_3 = x_4 = x_5$. Next, it suffices to prove (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). First, we start with the following equivalence relation based on Lemma 2.5:

$$eaf + 1 - f \in R^{-1} \Leftrightarrow fea + 1 - f = ea + 1 - f \in R^{-1}.$$

This leads to (1) $\Leftrightarrow$ (2). Moreover, by $eaf = eaef$, we obtain

$$eaf + 1 - f = eaef + 1 - f \in R^{-1} \Leftrightarrow feae + 1 - f \in R^{-1}$$

Note that $fe = e$ and e is left semicentral. Then we can conclude that $feae = eae = ae$, and then we arrive at the equivalence

$$eaf + 1 - f \in R^{-1} \Leftrightarrow ae + 1 - f \in R^{-1}.$$

This means (2) $\Leftrightarrow$ (3). The proof is completed. □

Analogously, we are able to derive the subsequent result.

Proposition 2.20. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and $fR \subseteq eR$, then*

$$\begin{aligned} &y_1 = eaf + 1 - e \in R^{-1} \Leftrightarrow y_2 = fa + 1 - e \in R^{-1} \Leftrightarrow y_3 = af + 1 - e \in R^{-1} \\ &\Leftrightarrow y_4 = fae + 1 - e \in R^{-1} \Leftrightarrow y_5 = faf + 1 - e \in R^{-1} \end{aligned}$$

Proof. Since R is e-symmetric and $fR \subseteq eR$, then we can obtain that R is also f-symmetric, and then f is left semicentral. Moreover, by checking all conditions, we can see $e \in fR$. Indeed, for any invertible element y_i, $ey_i \in fR$ and then $e \in fR$, which implies that $e = fe$, $f = ef$ and both e, f are left central. The following proof is similar to that of Proposition 2.19, so we omit it. □

Corollary 2.21. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, the following are equivalent:*

(1) *a is Bott–Duffin (e, f) invertible;*
(2) $u_1 = eaf + 1 - f$, $v_1 = eaf + 1 - e \in R^{-1}$.
(3) $u_2 = ae + 1 - f$, $v_2 = af + 1 - e \in R^{-1}$.
(4) $u_3 = ea + 1 - f$, $v_3 = fa + 1 - e \in R^{-1}$.
(5) $u_4 = ae + 1 - f$, $v_4 = fa + 1 - e \in R^{-1}$.
(6) $u_5 = ea + 1 - f$, $v_5 = af + 1 - e \in R^{-1}$.

Proof. As R is e-symmetric, e is left semicentral. So by conditions (2)-(6), it can be inferred that $fu_i \in eR$ and $ev_i \in fR$, and then $eR = fR$. Hence, the proof can be completed similarly to Theorem 2.6, Proposition 2.19 and Proposition 2.20. □

Remark 2.22. (1) Let $a \in R$ and $e, f \in E(R)$. If $fR \subseteq eR$, we can prove that

$$u_1 = fae + 1 - f \in R^{-1} \Leftrightarrow af + 1 - f \in R^{-1}.$$

Indeed, since $fR \subseteq eR$, we obtain $ef = f$, and consequently, $af + 1 - f = aef + 1 - f \in R^{-1} \Leftrightarrow fae + 1 - f \in R^{-1}$.

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(2) Let $a \in R$ and $e, f \in E(R)$. Assuming that R is e-symmetric and $eR \subseteq fR$. Then we can show that

$$v_1 = fae + 1 - e \in R^{-1} \Leftrightarrow ea + 1 - e \in R^{-1}.$$

Indeed, from $eR \subseteq fR$, we can deduce that $e = fe$. Then, $ea+1-e = efea+1-e \in R^{-1} \Leftrightarrow feae + 1 - e = fae + 1 - e \in R^{-1}$. Herein, $feae = fae$, because e is left semicentral.

Remark 2.23. Based on the above analysis, we find that the conditions: (i) e is left semicentral; (ii) $eR = fR$ are the key to describing the existence of B-D (e, f) inverses. Note that if the condition $u = fae+1-f \in R^{-1}$ holds in an e-symmetric ring, we obtain $fu = fae = efae$, and this gives that $fR \subseteq eR$ and R is also f-symmetric. Hence, we may conjecture that if R is e-symmetric, a is Bott–Duffin (e, f) invertible if and only if $u = fae + 1 - f \in R^{-1}$.

Example 2.24. (1) Let D be a division ring and $R = \mathbb{T}_2(D)$ be an upper triangular matrix ring over D. Choose $E = \mathrm{diag}(1, 0)$ and $F = O_2$. Firstly, we know D is symmetric, and R is E-symmetric. By a computation, one can see that $FAE + I_2 - F = I_2$, where $A = E$. However, $FAE + I_2 - E = \mathrm{diag}(0, 1)$ is not invertible in R. This means that A is not Bott–Duffin (E, F) invertible.

In other words, for an arbitrary ring R, given that $e, f \in E(R)$ and R is e-symmetric, the invertibility of the element $u = fae + 1 - f$ does not necessarily imply that a is Bott–Duffin (e, f) invertible.

(2) Consider the ring $R = \mathbb{Z}_6$ and let $a = e = 1$ and $f = 3$. Clearly, R is e-symmetric. Upon computation, it is found that $fae+1-f = 1$ and $fae+1-e = 3$ is not invertible in R.

In fact, it is well-established that if a is Bott–Duffin (e, f) invertible, and either of the idempotents e or f is equal to 0, then the remaining idempotent is necessarily 0. An element $e \in E(R)$ is called left minimal idempotent if Re is a minimal left ideal. Write $ME_l^*(R)$ to denote the set of all nonzero left minimal idempotents of R, and $E^*(R)$ to denote the set of all nonzero idempotents of R. If e and $f \in E^*(R)$ and R is e-symmetric, the fact that $u = fae + 1 - f \in R^{-1}$ cannot imply a is Bott–Duffin (e, f) invertible, as shown in Example 2.24 (2).

Theorem 2.25. *Let $a \in R$ and $e, f \in ME_l^*(R)$. If R is e-symmetric, the following are equivalent:*

(1) a is Bott–Duffin (e, f) invertible;

(2) $u = fae + 1 - f \in R^{-1}$.

Proof. (1) $\Rightarrow$ (2). See Theorem 2.6.

(2) $\Rightarrow$ (1). Firstly, we claim R is also f-symmetric. Indeed, since R is e-symmetric, e is left semicentral, it implies $fu = fae = efae$ and $f = faeu^{-1} = efaeu^{-1}$. Set $y = eu^{-1}f$. Evidently, $y = ey = yf$. Moreover, a straightforward calculation reveals that $fay = faeu^{-1}f = f$. Next, we aim to demonstrate that $(1-f)e = 0$. If $(1-f)e \neq 0$, because $e \in ME_l^*(R)$, we conclude that $R(1-f)e = Re$, which implies $e = t(1 - f)e$ for some $t \in R$. Multiplying it on the right by f gives $ef = t(1 - f)ef$, and then, by $ef = f$, we have $f = t(1 - f)f = 0$, which is a

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contradiction. Hence, it follows from $(1-f)e=0$ that $ue=fae$ and $e=u^{-1}fae$. This shows that $yae=eu^{-1}fae=e$, and consequently, a is Bott–Duffin (e,f) invertible. □

3. Bott–Duffin inverse in e-symmetric $*$-rings

In this section, let R be a $*$-ring. An idempotent $e\in R$ is called a self-adjoint idempotent, if $e^*=e$. We use $P(R)$ to denote the set of all self-adjoint idempotents in R. Recall that an element $a\in R$ is EP if $a^\dagger$ and $a^\sharp$ exist and satisfy $a^\dagger=a^\sharp$. It is clear that a is EP if and only if $a^\sharp$ exists and $aa^\sharp\in P(R)$. We now introduce some notations that will be used throughout this section. The symbol $C(R)$ denotes the center of R. That is, $C(R)=\{x : rx=xr, \text{ for all } r\in R\}$. In particular, $C(a)=\{x\in R : ax=xa\}$.

Proposition 3.1. *Let $e\in P(R)$. If R is e-symmetric, then $e\in C(R)$.*

Proof. Since R is e-symmetric, we know that e is left semicentral, which means $re=ere$ for any $r\in R$. Taking the adjoint of both sides, we obtain $(re)^*=(ere)^*$, and consequently, $er^*=er^*e=r^*e$. This leads to $re=er$ and $e\in C(R)$. □

Corollary 3.2. *If $a\in R^\dagger$ and R is $aa^\dagger$-symmetric, then a is EP.*

Proof. By the hypothesis, we can prove that R is also $a^\dagger a$-symmetric. Indeed, it can be readily observed that $a^\dagger a=a^\dagger aa^\dagger a=aa^\dagger a^\dagger a\in aa^\dagger R$. By Lemma 2.2, this observation, combined with the lemma, allows us to conclude that R is $a^\dagger a$-symmetric. In accordance with Proposition 3.1, we can conclude that $aa^\dagger, a^\dagger a\in C(R)$, and then $a=a^2a^\dagger=a^\dagger a^2$. Consequently, this implies that $a\in R^\sharp$. Furthermore, since $a^\sharp a=a^\sharp a^2a^\dagger=aa^\dagger$, we can conclude that a is EP. □

Moreover, similar to Corollary 3.2, we can obtain the following results.

Proposition 3.3. *Suppose that a is regular and a^- is any regular inverse of a. If R is aa^--symmetric, then $a\in R^\sharp$.*

Proof. Similar to the proof of Corollary 3.2, we can obtain that R is also a^-a-symmetric. Hence, a^-a is also left semicentral, and then $a=aa^-(aa^-)a=a(aa^-)a^-(aa^-)a\in a^2R$. Moreover, $a=aa^-a(a^-a)=aa^-(a^-a)a(a^-a)=aa^-(a^-a)a\in Ra^2$. This leads to $a\in R^\sharp$. □

Corollary 3.4. *If $a\in R\{1,3\}$ and R is $aa^{(1,3)}$-symmetric, then a is EP.*

Proof. By Proposition 3.3, we have $a\in R^\sharp$. Moreover, since R is $aa^{(1,3)}$-symmetric, by Proposition 3.1, we know that $aa^{(1,3)}\in C(R)$, and it leads to $a=aa^{(1,3)}a=a^2a^{(1,3)}$. Multiplying it on the left by $a^\sharp$ gives that $a^\sharp a=aa^{(1,3)}$. This shows that a is EP. □

Corollary 3.5. *If $a\in R\{1,4\}$ and R is $aa^{(1,4)}$-symmetric, then a is EP.*

Proof. By Proposition 3.3, we have $a\in R^\sharp$. Moreover, as R is $aa^{(1,4)}$-symmetric, $aa^{(1,4)}$ is left semicentral, and it leads to $a^{(1,4)}a=a^{(1,4)}aa^{(1,4)}a=aa^{(1,4)}a^{(1,4)}a\in aa^{(1,4)}R$. So, we obtain R is $a^{(1,4)}a$-symmetric, and then, $a^{(1,4)}a\in C(R)$ by

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Proposition 3.1. Based on the fact that $a^{(1,4)}a \in C(R)$, we further conclude that $a = aa^{(1,4)}a = a^{(1,4)}a^2$. Multiplying it on the right by $a^\sharp$ gives that $aa^\sharp = a^{(1,4)}a$. This shows that a is EP. □

Similar to the proof of Corollary 3.4 and 3.5, we can obtain the following results.

Corollary 3.6. *If $a \in R\{1,3\}$ (or $a \in R\{1,4\}$) and R is $a^{(1,3)}a$- (or $a^{(1,4)}a$-) symmetric, then a is EP.*

We say that a is normal, if $a = aa^*a$. And in this case, $aa^*, a^*a \in E(R)$. The following result will be presented. It will show that when R is aa^*- or a^*a-symmetric, a is not just group invertible.

Proposition 3.7. *If a is normal and R is aa^*- (or a^*a-) symmetric, then a is EP.*

Proof. Firstly, we claim that if R is aa^*-symmetric, then R is also a^*a-symmetric. Indeed, as $a = aa^*a$, then we have $a^*a = a^*aa^*a = aa^*a^*a \in aa^*R$. By Lemma 2.2, we conclude that R is a^*a-symmetric. In accordance with Proposition 3.1, this follows that $aa^*, a^*a \in C(R)$. Hence, we can obtain that $a = (aa^*)a = a(aa^*) \in a^2R$, and $a = a(a^*a) = (a^*a)a \in Ra^2$. Thus, this implies $a \in R^\sharp$. Since $a = a^2a^*$, when we premultiply both sides of the equation by $a^\sharp$, we have $aa^\sharp = aa^*$, and consequently, $a^\sharp = a^\dagger = a^*$. □

Definition 3.8 (Bott–Duffin Decomposition [6])**.** Let R be a ring and $\alpha \in R$. If $\alpha = g + u$ where $g \in E(R) \cap C(u)$ and u is B-D invertible relative to g, it is called to be a B-D decomposition of α. And g is the spectral idempotent of the B-D decomposition $\alpha = g + u$.

Furthermore, it is proven in [6] that if y is the B-D inverse of u relative to g, then we can obtain that $yu = uy = g$. In the sequel, we investigate the B-D decompositions of idempotents in rings.

Theorem 3.9. *Let $e = g + u$ be a B-D decomposition. Then $g = -ug$ and $u^3 = u$.*

Proof. Clearly $eg = ge$, as $ge = ege$, we have $g + gu = g + gu + ug + ugu$, and then $ug = -ugu$. Let y be the Bott–Duffin inverse of u relative to g. Then $guy = g$, and then $g = -ug$. Furthermore, $u + g = e = e^2 = (u + g)(u + g) = u^2 + gu + gu + g = u^2 - g$, this gives that $u^2 - u = g + g$ and $u^3 - u^2 = -g - g$. Hence, we can conclude that $u^2 - u = -(u^3 - u^2)$, and consequently, $u = u^3$. □

In particular, if $u \in R^{-1}$, then $u^2 = 1$ and $e = e^2 = (u + g)(u + g) = u^2 + gu + gu + g = 1 - g$. Moreover, in this case, u is B-D invertible relative to g with the inverse $y = u^{-1}g$, and $e = (1 - e) + (2e - 1)$.

Definition 3.10 (Strongly Clean Decomposition [14])**.** Let R be a ring and $\alpha \in R$. If $\alpha = g + u$, where $g \in E(R) \cap C(u)$ and $u \in R^{-1}$, it is called to be a strongly clean decomposition of α.

In fact, whenever $u \in R^{-1}$, the B-D decomposition is the strongly clean decomposition. So, we can obtain the result as follows: For an idempotent, the strongly clean decomposition is unique, that is, $e = (1 - e) + (2e - 1)$. Furthermore, it is

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well known that e is left semicentral if and only if $1-e$ is right semicentral. Then the following result can be provided.

Corollary 3.11 ([10, Theorem 4.3])**.** *Let $e \in E(R)$ and $e = u+g$ be a strongly clean decomposition. Then e is left semicentral if and only if g is right semicentral.*

Assume $e = u+g$ is a B-D decomposition, one can see that $u^2 \in E(R)$, and then the following result can be obtained.

Corollary 3.12. *Let $e = g+u$ be a Bott–Duffin decomposition. Then, R is u^2-symmetric iff R is e- and g-symmetric.*

Proof. By Theorem 3.9, we know that $g = -ug$, then $eg = (u+g)g = 0 = ge$, and consequently, $eu = e(e-g) = e = ue$. Hence, $e = u^2e$ and $g = u^2g$. This implies that R is e- and g-symmetric if R is u^2-symmetric. Conversely, by checking the proof of Theorem 3.9, we have $u^2 = e+g$, which implies that R is u^2-symmetric if R is e- and g-symmetric. □

Remark 3.13. Let R be a $*$-ring and $e \in E(R)$. Clearly, e^* is an idempotent. Assume that $e = (1-e^*)+u$ is a clean decomposition (A clean decomposition is defined as an expression where every element $\alpha = g+u$, where $g \in E(R)$ and $u \in R^{-1}$ [13]). If $eu = ue$, then we have $e^*u = ue^*$. Indeed, $e^* = 1-e+u$ implies that $e^*u = ue^*$. Hence, in this case, the clean decomposition is a strongly clean decomposition. Moreover, since it is proved that the strongly clean decomposition of idempotents is unique, we can conclude that $1-e^* = 1-e$, and thus $e = e^*$. In fact, we can also prove that if $ue = eue$, then $eu = ue$ and $e^* = e$.

Proposition 3.14. *Let $e \in E(R)$ and $e = (1-e^*)+u$ be a clean decomposition. If $ue = eue$, then $eu = ue$ and $e^* = e$.*

Proof. As $e = (1-e^*)+u$, we have $u = e-1+e^*$, and then $u^2 = 1-e-e^*+ee^*+e^*e$. Set $v = u^2$. It is easy to find that $ev = ve = ee^*e$ and $v^* = v$. Hence, we can see that $ev^{-1} = v^{-1}e$ and $e^*v^{-1} = v^{-1}e^*$. Set $f = e^*ev^{-1}$. Then one can see that $f = v^{-1}e^*e$ and $f^* = f$. As $ue = eue$, then we have $e^*e = (1-e+u)e = ue = eue$, and consequently, $ee^*e = e^*e$. It gives that $f = e^*ev^{-1} = ee^*ev^{-1} = e$. So, $e^* = e$, and then $u = 2e-1$ and $eu = ue$. □

Corollary 3.15. *Let $e \in E(R)$ and R be an e-symmetric ring. If $e = (1-e^*)+u$ is a clean decomposition, then $e^* = e \in C(R)$.*

Proof. Indeed, we know that e is left semicentral if and only if e^* is right semicentral. □

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References

[1] R. BOTT and R. DUFFIN, On the algebra of networks, Trans. Amer. Math. Soc. **74** (1953), 99-109.

[2] M. DRAZIN, A class of outer generalized inverses, Linear Algebra Appl. **436** (2012), 1909-1923.

[3] M. DRAZIN, Left and right generalized inverses, Linear Algebra Appl. **510** (2016), 64-78.

[4] G. KAFKAS, B. UNGOR, S. HALICIOGLU, and A. HARMANCI, Generalized symmetric rings, Algebra Discrete Math. **12** (2011), 78-84.

[5] Y. KE and J. CHEN, The Bott–Duffin (e, f)-inverses and their applications, Linear Algebra Appl. **489** (2016), 61-74.

[6] T. LAM and P. NIELSEN, Jacobson pairs and Bott–Duffin decompositions in rings, Contemp. Math. **727** (2019), 249-267.

[7] J. LAMBEK, On the representation of modules by sheaves of factor modules, Canad. Math. Bull. **14** (1971), 359-368.

[8] G. MARKS, Reversible and symmetric rings, J Pure Appl Algebra **174** (2002), 311-318.

[9] F. MENG and J. WEI, Some properties of e-symmetric rings, Turk. J. Math. **42** (2018), 2389-2399.

[10] F. MENG and J. WEI, (g, e)-symmetric rings, Algebra Colloq. **31** (2024), 263-270.

[11] F. MENG and J. WEI, e-symmetric rings, Commun. Contemp. Math. **20** (2018), Paper No. 3, 8 pp.

[12] F. MENG, J. WEI, and R. CHEN, Weak e-symmetric rings, Comm. Algebra **51** (2023), 3042-3050.

[13] W. NICHOLSON, Lifting idempotents and exchange rings, Trans. Am. Math. Soc. **229** (1977), 269-278.

[14] W. NICHOLSON, Strongly clean rings and Fitting's lemma, Commun. Algebra **27**(8) (1999), 3583-3592.

[15] L. OUYANG and H. CHEN, On weak symmetric rings, Comm. Algebra **38** (2010), 697-713.

[16] N. CASTRO-GONZÁLEZ, J. CHEN, and L. WANG, Further results on generalized inverses in rings with involution, Electron. J. Linear Algebra **30** (2015), 118-134.

[17] C. WU and J. CHEN, Left core inverses in rings with involution, Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemticas **116** (2022), Paper No. 67, 15 pp.

[18] L. WANG, J. WEI, and R. ZHAO, The characterizations of one-sided generalized inverses, Commun. Algebra **47** (2019), 4160-4166.

[19] J. WEI, Generalized weakly symmetric rings, J. Pure. Appl. Algebra **218** (2014), 1594-1603.

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