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UNIFORMLY S-ARTINIAN RINGS AND MODULES

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Abstract. Let R be a commutative ring with identity and S a multiplicative subset of R. An R-module M is said to be a u-S-Artinian module if there is $s \in S$ such that any descending chain of submodules of M is S-stationary with respect to s. The notion of u-S-Artinian modules are characterized in terms of (S-MIN)-conditions and u-S-cofinite properties. A ring R is said to be u-S-Artinian if R itself is a u-S-Artinian module, and then show that any u-S-semisimple ring is u-S-Artinian. It is proved that a ring R is u-S-Artinian if and only if R is u-S-Noetherian, the u-S-Jacobson radical $\mathrm{Jac}_S(R)$ of R is S-nilpotent and $R/\mathrm{Jac}_S(R)$ is a u-$S/\mathrm{Jac}_S(R)$-semisimple ring. Besides, some examples are given to distinguish Artinian rings, u-S-Artinian rings and S-Artinian rings.

1. Introduction

Throughout this article, all rings are commutative rings with identity and all modules are unitary. Let R be a ring. We denote by $\mathrm{Max}(R)$ the set of all maximal ideals of R. A subset S of R is called a multiplicative subset of R if $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Early in 2002, Anderson and Dumitrescu [3] introduced the so-called S-Noetherian ring R, in which for any ideal I of R, there exists a finitely generated ideal K of R such that $sI \subseteq K \subseteq I$ for some $s \in S$. Note that Cohen's Theorem, Eakin-Nagata Theorem and Hilbert Basis Theorem for S-Noetherian rings are also given in [3]. The notion of S-Noetherian rings provides a good direction for S-generalizations of other classical rings (See, for example, [1, 4, 7, 8, 11]). However, it is often difficult to study these S-generalizations of classical rings via a module-theoretic approach. The essential difficulty is that the selected element $s \in S$ is often not "uniform" in their definitions. To overcome this difficulty for Noetherian properties, Qi et al. [16] recently introduced the notions of uniformly S-Noetherian rings which are S-Noetherian rings such that s is independent on I in the definition of S-Noetherian rings. They also introduced the notion of u-S-injective modules and then characterized uniformly S-Noetherian rings in terms of u-S-injective modules. Some other uniform S-versions of rings and modules, such as semisimple rings, von Neumann regular rings, projective modules and flat modules are introduced and studied by the named authors and coauthors in [19, 20].

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In 2020, Sengelen et al. [17] introduced the notions of S-Artinian rings for which any descending chain of ideals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m \supseteq \cdots$ of R satisfies S-stationary condition, i.e., then there exist $s \in S$ and $k \in \mathbb{Z}^+$ such that $sI_k \subseteq I_n$ for all $n \geq k$. In the definition of S-Artinian rings, it is easy to see that although the element $s \in S$ is independent on n but it is certainly dependent on the given descending chain of ideals. Recently, Özen et al [15] extended the notion of S-Artinian rings to that of S-Artinian modules by replacing descending chains of ideals to these of submodules. And then, Omid et al. [14] introduced the notion of weakly S-Artinian modules, for which every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_m \supseteq \cdots$ of submodules of M is weakly S-stationary, i.e., there exists $k \in \mathbb{Z}^+$ such that for each $n \geq k$, $s_n N_k \subseteq N_n$ for some $s_n \in S$. In the definition of weakly S-Artinian modules, it is easy to see the element s_n is dependent both on n and the descending chain of modules. So the notion of weakly S-Artinian modules is certainly a “weak” version of that of S-Artinian modules. In this article, we introduce and study the “uniform” S-version of Artinian rings and modules (we call them u-S-Artinian rings and modules) such that the element s given in the definition of S-Artinian rings and modules is both independent on n and the descending chain of ideals or submodules. Obviously, we have the following implications:

$$\boxed{\text{Artinian rings}} \Longrightarrow \boxed{\text{u-S-Artinian rings}} \Longrightarrow \boxed{\text{S-Artinian rings}}$$

But neither of implications can be reversed (see Example 2.7 and Example 3.4). Denote by $\mathrm{Jac}_S(R)$ the u-S-Jacobson radical of a ring R (see Definition 4.2). Then it is also worth to mention that a ring R is u-S-Artinian if and only if R is u-S-Noetherian, the u-S-Jacobson radical $\mathrm{Jac}_S(R)$ of R is S-nilpotent and $R/\mathrm{Jac}_S(R)$ is a u-$S/\mathrm{Jac}_S(R)$-semisimple ring (see Theorem 4.9).

The related notions of uniformly S-torsion theory originally emerged in [19], and we give a quick review below. An R-module T is called u-S-torsion (with respect to s) provided that there exists $s \in S$ such that $sT = 0$. A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of R-modules is called a short u-S-exact sequence (with respect to s), if $s\mathrm{Ker}(f) = s\mathrm{Coker}(f) = 0$, $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$ for some $s \in S$. An R-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) (with respect to s) provided $\mathrm{Ker}(f)$ is (resp., $\mathrm{Coker}(f)$ is, both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are) u-S-torsion (with respect to s). Recall from [20] an R-module P is called u-S-projective provided that the induced sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0$$

is u-S-exact for any u-S-exact sequence $0 \to A \to B \to C \to 0$. Suppose M and N are R-modules. We say M is u-S-isomorphic to N if there exists a u-S-isomorphism $f : M \to N$. A family $\mathcal{C}$ of R-modules is said to be closed under u-S-isomorphisms if M is u-S-isomorphic to N and M is in $\mathcal{C}$, then N is also in $\mathcal{C}$. Note that the class of u-S-projective modules is closed under u-S-isomorphisms. One can deduce from the following [20, Lemma 2.1] that the existence of u-S-isomorphisms of two R-modules is actually an equivalence relationship.

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2. UNIFORMLY S-ARTINIAN MODULES

Let R be a ring, S a multiplicative subset of R and M an R-module. Suppose $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ is a descending chain of submodules of M. The family $\{M_i\}_{i\in\mathbb{Z}^+}$ is said to be S-stationary (with respect to s) if there exists $s \in S$ and $k \in \mathbb{Z}^+$ such that $sM_k \subseteq M_n$ for every $n \geq k$. And M is called an S-Artinian module if each descending chain of submodules $\{M_i\}_{i\in\mathbb{Z}^+}$ of M is S-stationary (see [15, Definition 1]). Note that in the definition of S-Artinian module, the element s is dependent on the given descending chain of submodules. The main purpose of this section is to introduce and study a "uniform" version of S-Artinian modules.

Definition 2.1. Let R be a ring and S a multiplicative subset of R. An R-module M is called a u-S-Artinian (abbreviates uniformly S-Artinian) module (with respect to s) provided that there exists $s \in S$ such that each descending chain $\{M_i\}_{i\in\mathbb{Z}^+}$ of submodules of M is S-stationary with respect to s.

Trivially, if $0 \in S$, then every R-module is u-S-Artinian. If $S_1 \subseteq S_2$ are multiplicative subsets of R and M is u-S_1-Artinian, then M is obviously u-S_2-Artinian. Note that Artinian modules are exactly u-$\{1\}$-Artinian modules. So all Artinian modules are u-S-Artinian modules for any multiplicative set S. Next we give a u-S-Artinian module which is not Artinian.

Example 2.2. Let $R = \mathbb{Z}$ be the ring of integers, p a prime in R and $M = \mathbb{Z}_p[[x]]$ the set of all formal power series over $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$. Set $S = \{p^n \mid n \in \mathbb{N}\}$. Then M is not Artinian since the descending chain

$$\langle x\rangle \supseteq \langle x^2\rangle \supseteq \cdots \supseteq \langle x^n\rangle \supseteq \cdots$$

is not stationary. However, since M is obviously u-S-torsion, M is a u-S-Artinian module.

Let S be a multiplicative subset of R. We always denote by $S^* = \{r \in R \mid rt \in S$ for some $t \in R\}$ and call it to be the saturation of S. A multiplicative set S is said to be saturated if $S = S^*$. Trivially, we have $S \subseteq S^*$ for all multiplicative subset S of R.

Proposition 2.3. *Let R be a ring, S a multiplicative subset of R and M an R-module. Let S^* be the saturation of S. Then M is a u-S-Artinian R-module if and only if M is a u-S^*-Artinian R-module.*

Proof. Suppose M is a u-S-Artinian R-module. Then M is trivially a u-S^*-Artinian R-module since $S \subseteq S^*$. Now, suppose M is a u-S^*-Artinian R-module. Then there is $r \in S^*$ such that each descending chain of submodules $\{M_i\}_{i\in\mathbb{Z}^+}$ of M is S^*-stationary with respect to r, i.e., there exits $k \in \mathbb{Z}^+$ such that $rM_k \subseteq M_n$ for each $n \geq k$. Since $r \in S^*$, $rt \in S$ for some $t \in R$. Note that $rtM_k \subseteq rM_k \subseteq M_n$. Hence M is a u-S-Artinian R-module. □

Let R be a ring, M an R-module and S a multiplicative subset of R. For any $s \in S$, there is a multiplicative subset $S_s = \{1, s, s^2, ...\}$ of S. We denote by M_s the localization of M at S_s. Note that $M_s \cong M \otimes_R R_s$.

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Lemma 2.4. *Let R be a ring, S a multiplicative subset of R and M an R-module. If M is a u-S-Artinian R-module, then there exists an element $s \in S$ such that M_s is an Artinian R_s-module.*

Proof. Let s be an element in S such that each family of descending chain of submodules $\{M_i\}_{i\in\mathbb{Z}^+}$ of M is S-stationary with respect to s. Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ be a descending chain of R_s-submodules of M_s. Considering the natural homomorphism $f : M \to M_s$, we have a a descending chain of submodules of M as follows:

$$f^{-1}(M_1) \supseteq f^{-1}(M_2) \supseteq \cdots \supseteq f^{-1}(M_n) \supseteq \cdots$$

There is a $k \in \mathbb{Z}^+$ such that $sf^{-1}(M_k) = f^{-1}(sM_k) = f^{-1}(M_k) \subseteq f^{-1}(M_n)$ for each $n \geq k$ since M_k is an R_s-module. Hence $M_k = M_n$ for each $n \geq k$. Consequently, M_s is an Artinian R_s-module. □

Remark 2.5. The converse of Lemma 2.4 also does not hold in general. Let $R = k[[x]]$ the formal power series ring over a field k. Let $S = \{1, x, x^2, \dots\}$. Then R_S is a field, and so is an Artinian R_S-module. However, R is not a u-S-Artinian R-module as R is not Artinian (see Proposition 3.2).

Recall from [2, 12] that a multiplicative subset S of R is said to satisfy the maximal multiple condition if there exists an $s \in S$ such that $t|s$ for each $t \in S$. Both finite multiplicative subsets and the multiplicative subsets that consist of units satisfy the maximal multiple condition. Certainly, u-S-Artinian modules are S-Artinian modules. And the following result shows that the converse also holds for multiplicative sets satisfying maximal multiple condition.

Proposition 2.6. *Let R be a ring, S a multiplicative subset of R satisfying the maximal multiple condition, and M an R-module. Then M is a u-S-Artinian module if and only if M is an S-Artinian module.*

Proof. If M is a u-S-Artinian module, then trivially M is S-Artinian. Let $s \in S$ such that $t|s$ for each $t \in S$. Suppose M is an S-Artinian module and $\{M_i\}_{i\in\mathbb{Z}^+}$ a descending chain of submodules of M. Then there exists $t \in S$ such that $tM_k \subseteq M_n$ for each $n \geq k$. So $sM_k \subseteq tM_k \subseteq M_n$ for each $n \geq k$. Hence M is a u-S-Artinian module. □

However, the following example shows S-Artinian modules are not u-S-Artinian modules in general.

Example 2.7. Let R be a valuation domain whose valuation group is $G = \prod_{\aleph} \mathbb{Z}$ the Hahn product of $\aleph$-copies of $\mathbb{Z}$ with lexicographic order, where $\aleph$ is an uncountable regular cardinal (see [10]). Let $S = R \setminus \{0\}$ the set of all nonzero elements of R. Then R itself is an S-Artinian R-module but not u-S-Artinian.

Proof. First, we claim that R is an S-Artinian R-module. Indeed, let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of R. We may assume that each I_i is not equal to 0. Then for each I_i, there exists a nonzero element $r_i \in I_i$ such that $v(r_i) \in G$. Moreover, we can assume all r_i satisfy $v(r_i) \leq v(r_{i+1})$. Since $\aleph$ is an

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uncountable regular cardinal, $\lim_{\longrightarrow} v(r_i) < \aleph$. So there is an element $x \in G$ such that $x \geq v(r_i)$ for each i. Suppose $v(s) = x$. Then $0 \neq s \in \bigcap_{i=1}^{\infty} I_i$. Hence $sI_k \subseteq I_n$ for each $n \geq k$. Consequently, R is an S-Artinian R-module. Now, we claim that R is not a u-S-Artinian R-module. On contrary, suppose R is u-S-Artinian. Then, by Lemma 2.4, there is an $s \in S$ such that R_s is an artinian domain, which is exactly a field. Let r be a nonzero element such that $v(r) > nv(s)$ for all non-negative integer n. Then r is not a unit in R_s, which is a contradiction. Consequently, R is not a u-S-Artinian R-module. (Note that it can also be easily deduced by Proposition 3.2.) ☐

Lemma 2.8. *Let R be a ring and S a multiplicative subset of R. Let M and N be R-modules. If M is u-S-isomorphic to N, then M is u-S-Artinian if and only if N is u-S-Artinian.*

Proof. Let M be u-S-Artinian with respect to $s \in S$ and $f : M \to N$ a u-S-isomorphism. Suppose $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ is a descending chain of submodules of N. Then $f^{-1}(N_1) \supseteq f^{-1}(N_2) \supseteq \cdots \supseteq f^{-1}(N_n) \supseteq \cdots$ is a descending chain of submodules of M. Then there is $k \in \mathbb{Z}^+$ such that $sf^{-1}(N_{k_s}) \subseteq f^{-1}(N_n)$ for any $n \geq k$. Hence $s(\mathrm{Im}(f) \cap N_k) = sf(f^{-1}(N_k)) \subseteq f(f^{-1}(N_n)) = \mathrm{Im}(f) \cap N_n$ for any $n \geq k$. Note that we have the following commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(f) \cap N_n \arrow[r] \arrow[d, hook] & N_n \arrow[r] \arrow[d, hook] & (\mathrm{Im}(f) + N_n)/\mathrm{Im}(f) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathrm{Im}(f) \cap N_k \arrow[r] & N_k \arrow[r] & (\mathrm{Im}(f) + N_k)/\mathrm{Im}(f) \arrow[r] & 0
\end{tikzcd}$$

with $(\mathrm{Im}(f) + N_n)/\mathrm{Im}(f)$ and $(\mathrm{Im}(f) + N_k)/\mathrm{Im}(f)$ submodules of $\mathrm{Coker}(f)$, and hence u-S-torsion. It follows by [9, Lemma 2.11] that $s^2 N_k \subseteq N_n$. Consequently, N is u-S-Artinian with respect to s^2.

Suppose N is u-S-Artinian. Then by [20, Lemma 2.1], there is a u-S-isomorphism $g : N \to M$. So we can show M is u-S-Artinian similarly. ☐

Proposition 2.9. *Let R be a ring and S a multiplicative subset of R. Let $0 \to A \to B \to C \to 0$ be an u-S-exact sequence. Then B is u-S-Artinian if and only if A and C are u-S-Artinian. Consequently, a finite direct sum $\bigoplus_{i=1}^{n} M_i$ is u-S-Artinian if and only if each M_i is u-S-Artinian $(i = 1, \ldots, n)$.*

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an u-S-exact sequence. Then there is an exact sequence $0 \to \mathrm{Im}(f) \to B \to \mathrm{Coker}(f) \to 0$, with $\mathrm{Im}(f)$ u-S-isomorphic to A, and $\mathrm{Coker}(f)$ u-S-isomorphic to C. So we can assume that $0 \to A \to B \to C \to 0$ is an exact sequence by Lemma 2.8. If B is u-S-Artinian, then it is easy to verify A and C are u-S-Artinian. Now suppose A and C are u-S-Artinian. Let $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$ be a descending chain of submodules of B. Then

$$B_1 \cap A \supseteq B_2 \cap A \supseteq \cdots \supseteq B_n \cap A \supseteq \cdots$$

is a descending chain of submodules of A, and

$$(B_1 + A)/A \supseteq (B_2 + A)/A \supseteq \cdots \supseteq (B_n + A)/A \supseteq \cdots$$

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is a descending chain of submodules of $C \cong B/A$. So there is $s_1, s_2 \in S$ and $k \in \mathbb{Z}^+$ such that $s_1 B_k \cap A \subseteq B_n \cap A$ and $s_2(B_k + A) \subseteq B_n + A$ for any $n \geq k$. Then one can verify that $s_1 s_2 B_k \subseteq B_n$ for any $n \geq k$. Hence, B is u-S-Artinian. □

Let $\mathfrak{p}$ be a prime ideal of R. We say an R-module M is *u-$\mathfrak{p}$-Artinian* provided that M is u-$(R \setminus \mathfrak{p})$-Artinian. The next result gives a local characterization of Artinian modules.

Proposition 2.10. *Let R be a ring and M an R-module. Then the following statements are equivalent:*

1. *M is Artinian;*
2. *M is u-$\mathfrak{p}$-Artinian for any $\mathfrak{p} \in \mathrm{Spec}(R)$;*
3. *M is u-$\mathfrak{m}$-Artinian for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. (1) ⇒ (2) ⇒ (3): Trivial.

(3) ⇒ (1): Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ be a descending chain of submodules of M. Then, for each $\mathfrak{m} \in \mathrm{Max}(R)$, there exist $s_{\mathfrak{m}} \notin \mathfrak{m}$ and $k_{\mathfrak{m}} \in \mathbb{Z}^+$ such that $s_{\mathfrak{m}} M_{k_{\mathfrak{m}}} \subseteq M_n$ for each $n \geq k_{\mathfrak{m}}$. Since R is generated by $\{s_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}(R)\}$. So there is a finite subset $\{s_{\mathfrak{m}_1}, \ldots, s_{\mathfrak{m}_t}\}$ that generates R. Let $k = \max\{k_{\mathfrak{m}_1}, \ldots, k_{\mathfrak{m}_t}\}$. Then $M_k = \langle s_{\mathfrak{m}_1}, \ldots, s_{\mathfrak{m}_t} \rangle M_k \subseteq \sum_{i=1}^{t} (s_{\mathfrak{m}_i} M_{k_{\mathfrak{m}_i}}) \subseteq M_n$ for all $n \geq k$. Hence M is Artinian. □

Let R be a ring and S a multiplicative subset of R. Recall from [15], an R-module M is said to be S-cofinite (called finitely S-cogenerated in [15, Definition 3]) if for each nonempty family of submodules $\{M_i\}_{i \in \Delta}$ of M, $\bigcap_{i \in \Delta} M_i = 0$ implies that $s(\bigcap_{i \in \Delta'} M_i) = 0$ for some $s \in S$ and a finite subset $\Delta' \subseteq \Delta$. If $S = \{1\}$, then S-cofinite modules are exactly the classical cofinite modules.

Definition 2.11. Let R be a ring and S a multiplicative subset of R. An R-module M is called u-S-cofinite (with respect to s) if there is an $s \in S$ such that for each nonempty family of submodules $\{M_i\}_{i \in \Delta}$ of M, $\bigcap_{i \in \Delta} M_i = 0$ implies that $s(\bigcap_{i \in \Delta'} M_i) = 0$ for a finite subset $\Delta' \subseteq \Delta$.

Obviously, cofinite R-modules are u-S-cofinite, and u-S-cofinite R-modules is S-cofinite.

Proposition 2.12. *Let R be a ring, M an R-module and S a multiplicative subset of R. Then the following statements hold.*

1. *If $S_1 \subseteq S_2$ are multiplicative subsets of R and M is u-S_1-cofinite. Then M is u-S_2-cofinite.*
2. *Suppose S^* is the saturation of S. Then M is u-S-cofinite if and only if M is u-S^*-cofinite.*

Proof. (1) is trivial.

(2) If M is u-S-cofinite, then M is also u-S^*-cofinite by (1). Suppose M is u-S^*-cofinite. Then there is an $r \in S^*$ such that for each nonempty family of submodules $\{M_i\}_{i \in \Delta}$ of M, $\bigcap_{i \in \Delta} M_i = 0$ implies that $r(\bigcap_{i \in \Delta'} M_i) = 0$ for a finite subset $\Delta' \subseteq \Delta$. Let $s := rt \in S$ for some $t \in R$. Then $s(\bigcap_{i \in \Delta'} M_i) = 0$. Hence M is u-S-cofinite. □

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Let $\mathfrak{p}$ be a prime ideal of R. We say an R-module M is *u-$\mathfrak{p}$-cofinite* provided that M is u-$(R\backslash\mathfrak{p})$cofinite. The next result gives a local characterization of cofinite modules.

Proposition 2.13. *Let R be a ring and M an R-module. Then the following statements are equivalent:*

1. *M is cofinite;*
2. *M is u-$\mathfrak{p}$-cofinite for any $\mathfrak{p} \in \operatorname{Spec}(R)$;*
3. *M is u-$\mathfrak{m}$-cofinite for any $\mathfrak{m} \in \operatorname{Max}(R)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $\{M_i\}_{i\in\Delta}$ be a family of submodules of M such that $\bigcap_{i\in\Delta} M_i = 0$. Then, for each $\mathfrak{m} \in \operatorname{Max}(R)$, there exist $s_{\mathfrak{m}} \notin \mathfrak{m}$ and $k_{\mathfrak{m}} \in \mathbb{Z}^+$ such that $s_{\mathfrak{m}}(\bigcap_{i\in\Delta'_{\mathfrak{m}}} M_i) = 0$ for a finite subset $\Delta'_{\mathfrak{m}} \subseteq \Delta$. Since R can be generated by a finite subset $\{s_{\mathfrak{m}_1}, \ldots, s_{\mathfrak{m}_t}\}$. Let $\Delta' = \bigcap_{i=1}^{t} \Delta'_{\mathfrak{m}_i}$. Then

$$\bigcap_{i\in\Delta'} M_i = \langle s_{\mathfrak{m}_1}, \ldots, s_{\mathfrak{m}_t}\rangle\Big(\bigcap_{i\in\Delta'} M_i\Big) \subseteq \sum_{i=1}^{t}\Big(s_{\mathfrak{m}_i}\bigcap_{i\in\Delta'_{\mathfrak{m}_i}} M_i\Big) = 0.$$

Hence, M is cofinite. □

Definition 2.14. Let $\mathcal{N}$ be a nonempty family of submodules of M. Then $N \in \mathcal{N}$ is called an S-minimal element of $\mathcal{N}$ with respect to s if whenever $N' \subseteq N$ for some $N' \in \mathcal{N}$ then $sN \subseteq N'$. We say M satisfies (S-MIN)-condition with respect to s if every nonempty family of submodules of M has an S-minimal element with respect to s.

It follows from [5, Proposition 10.10] that an R-module M is Artinian if and only if every factor module M/N is finitely cogenerated, if and only if M satisfies (MIN)-Condition. Recently, Özen et al. extended this result to s-Artinian modules in [15, Theorem 3]. Now we give a uniform S-version of [5, Proposition 10.10].

Theorem 2.15. *Let R be a ring, S a multiplicative subset of R and M an R-module. Let $s \in S$. Then the following statements are equivalent:*

1. *M is a u-S-Artinian module with respect to s.*
2. *M satisfies (S-MIN)-condition with respect to s.*
3. *For any nonempty family $\{N_i\}_{i\in\Gamma}$ of submodules of M, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $s\bigcap_{i\in\Gamma_0} N_i \subseteq \bigcap_{i\in\Gamma} N_i$.*
4. *Every factor module M/N is u-S-cofinite with respect to s.*

Proof. (1) $\Rightarrow$ (2) : Suppose that M is a u-S-Artinian module with respect to s. Let $\mathcal{N}$ be a nonempty set of submodules of M. On contrary, suppose $\mathcal{N}$ has no S-minimal element of $\mathcal{N}$ with respect to s. Take $N_1 \in \mathcal{N}$, and then there exists $N_2 \in \mathcal{N}$ such that $N_1 \supseteq N_2$ and $sN_1 \nsubseteq N_2$. Iterating these steps, we can obtain a descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ such that $sN_k \nsubseteq N_{k+1}$ for any k. This implies M is not a u-S-Artinian module with respect to s, which is a contradiction.

(2) $\Rightarrow$ (3) : Let $\{N_i\}_{i\in\Gamma}$ be a nonempty family of submodules of M. Let $\mathcal{A}$ be the set of all intersections of finitely many N_i. Then each $N_i \in \mathcal{A}$, and so $\mathcal{A}$ is

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nonempty. So there is an S-minimal element, say $A=\bigcap_{i\in\Gamma_0}N_i$, of $\mathcal{A}$. Choose $j\in\Gamma_0-\Gamma$ and note that $N_j\cap A\subseteq A$ and $N_j\cap A\in\mathcal{A}$. By the S-minimality of A, we have $sA\subseteq N_j\cap A$. Since j is arbitrary, we conclude that $sA\subseteq\bigcap_{i\in\Gamma}N_i$.

(3) ⇒ (1) : Let $N_1\supseteq N_2\supseteq\cdots\supseteq N_n\supseteq\cdots$ be a descending chain of submodules of M. Then there is a positive integer k such that $sN_k=s\bigcap_{i=1}^{k}N_i\subseteq\bigcap_{i=1}^{\infty}N_i$. So $sN_k\subseteq N_n$ for any $n\geq k$.

(2) ⇒ (4) : Suppose $\bigcap_{i\in\Gamma}N_i/N=0$ for some family of submodules $\{N_i/N\}_{i\in\Gamma}$ of M/N. Then $\bigcap_{i\in\Gamma}N_i=N$. Set $\mathcal{A}=\{\bigcap_{i\in\Gamma'}N_i\mid\Gamma'\subseteq\Gamma\text{ is a finite subset }\}$. By (3), $\mathcal{A}$ has an S-minimal element with respect to s, say $M_0=\bigcap_{i\in\Gamma_0}N_i$ for some finite subset $\Gamma_0\subseteq\Gamma$. So, for any $k\in\Gamma-\Gamma_0$, we have $sM_0\subseteq M_0\bigcap N_k$. Thus $sM_0\subseteq M_0\cap(\bigcap_{k\in\Gamma-\Gamma_0}N_k)\subseteq\bigcap_{k\in\Gamma}N_k=N$. Consequently, M/N is u-S-cofinite with respect to s.

(4) ⇒ (1) : Let $N_1\supseteq N_2\supseteq\cdots\supseteq N_n\supseteq\cdots$ be a descending chain of submodules of M. Set $N=\bigcap_{i=1}^{\infty}N_i$. By assumption, M/N is u-S-cofinite with respect to s. Note that $\bigcap_{i=1}^{\infty}N_i/N=0$ in M/N. Then there is a positive integer k such that $s\bigcap_{i=1}^{k}N_i/N=0$ by (4). So $sN_k\subseteq N_n$ for all $n\geq k$. So M is a u-S-Artinian module with respect to s. ☐

3. BASIC PROPERTIES OF u-S-ARTINIAN RINGS

Recall from [17, Definition 2.1] that a ring R is called an S-Artinian ring if any descending chain of ideals $\{I_i\}_{i\in\mathbb{Z}^+}$ of R is S-stationary with respect to some $s\in S$. Note that the s is determined by the descending chain of ideals in the definition of S-Artinian rings. Now we introduce a “uniform” version of S-Artinian rings.

Definition 3.1. Let R be a ring and S a multiplicative subset of R. Then R is called a u-S-Artinian (abbreviates uniformly S-Artinian) ring (with respect to s) provided that R is a u-S-Artinian R-module (with respect to s), that is, there exists $s\in S$ such that each descending chain $\{I_i\}_{i\in\mathbb{Z}^+}$ of ideals of R is S-stationary with respect to s.

Since u-S-Artinian rings are u-S-Artinian modules over themselves, the results in Secton 2 also hold for u-S-Artinian rings. In particular, u-S-Artinian rings are S-Artinian. However, S-Artinian rings are not u-S-Artinian in general. A counterexample was given in Example 2.7. If $0\in S$, then every ring R is u-S-Artinian. So u-S-Artinian rings are not Artinian in general. A multiplicative set S is said to be regular if every element in S is a non-zero-divisor. The following Proposition shows that u-S-Artinian rings are exactly Artinian for any regular multiplicative set S.

Proposition 3.2. *Let R be a ring and S a regular multiplicative subset of R. If R is a u-S-Artinian ring, then R is an Artinian ring.*

Proof. Let s be an element in S. Consider the descending chain $Rs\subseteq Rs^2\subseteq\cdots$ of ideals of R. Then there exists k such that $sRs^k\subseteq Rs^n$ for any $n\geq k$. In particular, we have $s^{k+1}=rs^{k+2}$ for some $r\in R$. Since s is a non-zero-divisor, we have $1=rs$, and thus s is a unit. So R is an Artinian ring. ☐

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In order to give a non-trivial u-S-Artinian ring which is not Artinian, we consider the direct product of u-S-Artinian ring.

Proposition 3.3. *Let $R = R_1 \times R_2$ be a direct product of rings R_1 and R_2, and $S = S_1 \times S_2$ a direct product of multiplicative subsets of R_1 and R_2. Then R is a u-S-Artinian ring if and only if R_i is a u-S_i-Artinian ring for each $i = 1, 2$.*

Proof. Suppose R is a u-S-Artinian ring with respect to (s_1, s_2). Let $\{I^1_i\}_{i\in\mathbb{Z}^+}$ be a descending chain of ideals of R_1. Then $\{I^1_i \times 0\}_{i\in\mathbb{Z}^+}$ be a descending chain of ideals of R. Then there exists an integer k such that $(s_1, s_2)(I^1_k \times 0) \subseteq I^1_n \times 0$ for all $n \geq k$. Hence $s_1 I^1_k \subseteq I^1_n$ for all $n \geq k$. So R_1 is a u-S_1-Artinian ring with respect to s_1. Similarly, R_2 is a u-S_2-Artinian ring with respect to s_2. On the other hand, suppose R_i is a u-S_i-Artinian ring with respect to s_i for each $i = 1, 2$. Let $\{I_i = I^1_i \times I^2_i\}_{i\in\mathbb{Z}^+}$ be a descending chain of ideals of R. Then there exists an integer k_i such that $s_i I^i_{k_i} \subseteq I^i_n$ for all $n \geq k_i$. Set $k = \max\{k_1, k_2\}$. Then $(s_1, s_2)I_k = s_1 I^1_k \times s_2 I^2_k \subseteq I^1_n \times I^2_n = I_n$ for all $n \geq k$. So R is a u-S-Artinian ring. □

The promised non-Artinian u-S-Artinian rings are given as follows.

Example 3.4. Let $R = R_1 \times R_2$ be a product of R_1 and R_2, where R_1 is an Artinian ring but R_2 is not Artinian. Set $S = \{1\} \times \{1, 0\}$. Then R is a u-S-Artinian ring but not Artinian by Proposition 3.3.

Let R be a ring and S a multiplicative subset of R. Recall from [20] that an R-module M is called u-S-semisimple (with respect to s) provided that any u-S-short exact sequence $0 \to A \xrightarrow{f} M \xrightarrow{g} C \to 0$ is u-S-split (with respect to s), i.e., there exists an R-homomorphism $h : M \to A$ such that $h \circ f = s\mathrm{Id}_A$ for some $s \in S$. A ring R is called a u-S-semisimple ring if every free R-module is u-S-semisimple. The rest of this section is devoted to show any u-S-semisimple ring is u-S-Artinian. First, we introduce the notion of u-S-simple modules.

Definition 3.5. An R-module M is said to be u-S-simple (with respect to s) provided that M is not u-S-torsion with respect to some $s \in S$, and any proper submodule of M is u-S-torsion with respect to s.

Since any proper submodule of a u-S-simple R-module is u-S-torsion, we have any u-S-simple R-module is u-S-semisimple by [20, Lemma 2.1]. Moreover, we have the following result.

Proposition 3.6. *Suppose M is a u-S-simple R-module. Then $M^{(\aleph)}$ is a u-S-semisimple R-module for any ordinal $\aleph$.*

Proof. Suppose M is a u-S-simple R-module with respect to s and N is a submodule of $M^{(\aleph)}$ such that $M^{(\aleph)}/N$ is not u-S-torsion with respect to s. Set $\Gamma = \{\alpha \subseteq \aleph \mid s(N \cap M^{(\alpha)}) = 0\}$. For each $i \in \aleph$, we set M^i to be the i-th component of $M^{(\aleph)}$. Then we claim there is $i \in \aleph$ such that $s(N \cap M^i) = 0$, and hence Γ is not empty. Indeed, on contrary, suppose $N \cap M^i$ is not u-S-torsion for any $i \in \aleph$. Since M is u-S-simple with respect to s, then $N \cap M^i = M^i \cong M$,

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and hence $N = M^{(\aleph)}$ which is a contradiction. Let Λ be a chain in Γ. Then $s(N \cap \bigcup_{\alpha\in\Lambda} M^{(\alpha)}) = \bigcup_{\alpha\in\Lambda} s(N \cap M^{(\alpha)}) = 0$. So Λ has a upper bound. By Zorn's Lemma, there is a maximal element, say β, in Γ. Set $L = M^{(\beta)}$. We claim that $M^{(\aleph)}$ is u-S-isomorphic to $N + L$. Otherwise, since $M^j \cong M$ is u-S-simple, there is an $M^j \nsubseteq N + L$ for some $j \in \aleph$. Hence $N \cap M^{(\beta\cup\{j\})}$ is u-S-torsion with respect to s, which contradicts the maximality of β. Since $N \cap L$ is u-S-torsion, N is u-S-isomorphic to $N/(N \cap L)$ and $(N + L)/(N \cap L)$ is u-S-isomorphic to $(N + L)$ which is also u-S-isomorphic to $M^{(\aleph)}$. Considering the split monomorphism $g : N/(N \cap L) \to (N + L)/(N \cap L)$, we have the natural embedding map $N \to M$ is also a u-S-split monomorphism. □

Theorem 3.7. *Let R be a ring and S a multiplicative subset of R. Suppose R is a u-S-semisimple ring. Then R is a u-S-artinian ring.*

Proof. Suppose R is a u-S-semisimple ring. Let $\aleph$ be a cardinal larger than $2^{\sharp(R)} \cdot \aleph_0$, where $\sharp(R)$ is the cardinal of R. Then the free R-module $R^{(\aleph)}$ is u-S-semisimple with respect to some $s \in S$. And so every subquotient of $R^{(\aleph)}$ is also u-S-semisimple with respect to some $s \in S$. Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of R. Note that there are at most $2^{\sharp(R)} \cdot \aleph_0$ such chains. Set $R = I_0$. Consider the exact sequence $\xi_i : 0 \to I_i \to I_{i-1} \to I_{i-1}/I_i \to 0$ for any positive integer i. Since R is a u-S-semisimple ring, then each I_{i-1}/I_i is u-S-projective by [20, Theorem 3.5]. So, by [20, Corollary 2.10], each ξ_i is u-S-split with respect to s. Hence, by [20, Lemma 2.4], there is a u-S-isomorphism $f_i : I_{i-1} \to I_i \oplus I_{i-1}/I_i$ with respect to s for each i. So there are u-S-isomorphisms

$$R \xrightarrow{f_1} I_1 \bigoplus R/I_1 \xrightarrow{f_2\oplus \mathrm{Id}} I_2 \bigoplus I_1/I_2 \bigoplus R/I_1 \to \cdots$$

$$\xrightarrow{f_k\oplus \mathrm{Id}} I_k \bigoplus \Big(\bigoplus_{i=0}^{k} I_i/I_{i-1}\Big) \to \cdots$$

Assume $f(1) \in \bigoplus_{i=1}^{k} I_{i-1}/I_i \subseteq \bigoplus_{i=1}^{\infty} I_{i-1}/I_i$ where $f = \lim_{\to_k}((f_k \oplus \mathrm{Id}) \circ \cdots \circ f_1)$. Then R is u-S-isomorphic to $\bigoplus_{i=0}^{k} I_i/I_{i-1}$ with respect to s. And so I_k is u-S-torsion with respect to s. Hence $sI_k/I_n = 0$, i.e., $sI_k \subseteq I_n$ for all $n \geq k$. So R is a u-S-artinian ring. □

Corollary 3.8. *Let R be a ring and S a multiplicative subset of R. Suppose R is a u-S-semisimple ring. Then any S-finite R-module is u-S-artinian.*

Proof. Since the class of u-S-artinian modules is closed under u-S-isomorphisms, we just need to show any finitely generated R-module is u-S-artinian, which can easily be deduced by Proposition 2.9 and Theorem 3.7. □

4. A CHARACTERIZATION OF u-S-ARTINIAN RINGS

It is well-known that a ring R is Artinian if an only if R is a Noetherian ring with its Jacobson radical $\mathrm{Jac}(R)$ nilpotent and $R/\mathrm{Jac}(R)$ a semisimple ring (see [18, Theorem 4.1.10]). In this section, we will give a"uniform" S-version of this

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result. We begin with the notion of u-S-maximal submodules and the u-S-Jacobson radical of a given R-module.

Definition 4.1. Let R be a ring, S a multiplicative subset of R and $s \in S$. Then a submodule N is said to be u-S-maximal in an R-module M with respect to s provided that

(1) M/N is not u-S-torsion with respect to s;
(2) if $N \subsetneqq H \subseteq M$, then M/H is u-S-torsion with respect to s.

Note that an R-module N is a u-S-maximal submodule of M (with respect to s) if and only if M/N is u-S-simple (with respect to s). If M does not have any u-S-maximal submodule with respect to s, then we denote by $\mathrm{Jac}_s(M) = M$. Otherwise, we denote by $\mathrm{Jac}_s(M)$ the intersection of all u-S-maximal submodules of M with respect to s. The submodule $\mathrm{Jac}_s(M)$ of M is called the u-S-Jacobson radical of M with respect to s.

Definition 4.2. Let R be a ring, S a multiplicative subset of R and M an R-module. Then the u-S-Jacobson radical of M is defined to be $\mathrm{Jac}_S(M) = \bigcap_{s\in S} \mathrm{Jac}_s(M)$ under the above notions.

First, we have the following observation.

Lemma 4.3. *Let M be an R-module. Then*

$$\mathrm{Jac}_s(M/\mathrm{Jac}_s(M)) = 0, \text{ and } \mathrm{Jac}_S(M/\mathrm{Jac}_S(M)) = 0.$$

Proof. We just note that an R-module $N/\mathrm{Jac}_s(M) \subseteq M/\mathrm{Jac}_s(M)$ is u-S-maximal with respect to s if and only if $N + \mathrm{Jac}_s(M) \subseteq M$ is u-S-maximal with respect to s. ☐

Proposition 4.4. *Suppose M is a u-S-Artinian R-module with $\mathrm{Jac}_S(M)$ u-S-torsion. Then there exists $T \subseteq M$ such that $sT = 0$ and $M/T \subseteq \bigoplus_{i=1}^{t} S_i$ where each S_i is u-S-simple with respect to s for some $s \in S$. Consequently, M is a u-S-Noetherian R-module.*

Proof. If M has no u-S-maximal submodule, then we may assume $T = \mathrm{Jac}_S(M) = M$ is u-S-torsion. So the assertion trivially holds. Now suppose M has a u-S-maximal submodule. Then the intersection of all u-S-maximal submodules of M is u-S-torsion. Since M is u-S-Artinian, then there exists a finite family of u-S-maximal submodules, say $\{M_1, \ldots, M_n\}$, of M such that $T := \bigcap_{i=1}^{n} M_i$ is u-S-torsion with respect to some $t \in S$ by Theorem 2.15. Note that M/T is a submodule of $\bigoplus_{i=1}^{n} M/M_i$, where M/M_i is u-S-simple with respect to some $s_i \in S$. Set $s = ts_1 \cdots s_n$. Then each $S_i := M/M_i$ is u-S-simple with respect to s. One can easily check that M/T, as a submodule of $\bigoplus_{i=1}^{n} M/M_i$, is u-S-Noetherian with respect to s. Thus M is u-S-Noetherian with respect to s. ☐

Proposition 4.5. *Suppose R is a u-S-Artinian ring. Then $R/\mathrm{Jac}_S(R)$ is a u-$S/\mathrm{Jac}_S(R)$-semisimple ring.*

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Proof. Write $J = \mathrm{Jac}_S(R)$, $\overline{R} = R/J$ and $\overline{S} = S/\mathrm{Jac}_S(R)$. Since R is a u-S-Artinian ring, $\overline{R}$ is a u-S-Artinian R-module by Proposition 2.9. By Lemma 4.3, $\mathrm{Jac}_S(\overline{R}) = 0$. It follows from by Proposition 4.4 that there exists an element $s \in S$ and a submodule T of $\overline{R}$ such that $sT = 0$ and $\overline{R}/T \subseteq \bigoplus_{i=1}^{t} S_i$ with each S_i u-S-simple with respect to s. Now let $F = \overline{R}^{(\aleph)}$ be a free $\overline{R}$-module with $\aleph$ an arbitrary cardinal. Then there is a short exact sequence $0 \to T^{(\aleph)} \to \overline{R}^{(\aleph)} \to (\overline{R}/T)^{(\aleph)} \to 0$. By Proposition 4.4, $(\overline{R}/T)^{(\aleph)}$ is a submodule of $(\bigoplus_{i=1}^{t} S_i)^{(\aleph)} \cong \bigoplus_{i=1}^{t} (S_i)^{(\aleph)}$ which is u-S-semisimple by Proposition 3.6. Hence $(\overline{R}/T)^{(\aleph)}$ is also u-S-semisimple by [20, Proposition 3.3]. Since $sT^{(\aleph)} = 0$, $(\overline{R})^{(\aleph)}$ is a u-S-semisimple R-module. So $(\overline{R})^{(\aleph)}$ is actually a u-$\overline{S}$-semisimple $\overline{R}$-module, that is, $\overline{R}$ is a u-$\overline{S}$-semisimple ring. ☐

Lemma 4.6. *Let R be a ring, S a multiplicative subset of R and S^* the saturation of S. Suppose $r \in R$ and $s \in S$. If $r - s \in \mathrm{Jac}_S(R)$, then $r \in S^*$.*

Proof. Assume on contrary $r \notin S^*$. Then we claim R/Rr is not u-S-torsion. Indeed, if $t(R/Rr) = 0$ for some $t \in S$, then $t = rr'$ for some $r' \in R$. So $r \in S^*$, which is a contradiction. We also claim that there exists a u-S-maximal ideal I of R such that $r \in I$. Indeed, let Λ be the set of ideals J of R that contains r satisfying R/J is not u-S-torsion. One can check that the union of any ascending chain in Λ is also in Λ. So there is a maximal element I in Λ by Zorn Lemma. Hence I is a u-S-maximal ideal of R. Since $\mathrm{Jac}_S(R) \subseteq I$, we have $s \in I$. Then $s(R/I) = 0$, which is a contradiction. So $r \in S^*$. ☐

Proposition 4.7 (Nakayama Lemma for S-finite modules)**.** *Let R be a ring, $I \subseteq \mathrm{Jac}_S(R)$, S a multiplicative subset of R and M an S-finite R-module. If $sM \subseteq IM$ for some $s \in S$, then M is u-S-torsion.*

Proof. Let F be a finitely generated submodule, say generated by $\{m_1, \dots, m_n\}$, of M satisfying $s'M \subseteq F$ and $sM \subseteq IM$ for some $s', s \in S$. Then $ss'F \subseteq IF$. So we can assume M itself is generated by $\{m_1, \dots, m_n\}$. Then we have $a := s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$ where $a_i \in I^i$ by [13, Theorem 2.1]. Note that $aM = 0$ and $a - s^n \in I$. By Lemma 4.6, $a \in S^*$, where S^* is the saturation of S. It follows that there is $r \in R$ such that $ar \in S$. Hence $arM = 0$, and so M is u-S-torsion. ☐

Let I be an ideal of R. We say I is S-nilpotent if $sI^k = 0$ for some integer k and $s \in S$.

Proposition 4.8. *Let R be a ring and S a multiplicative subset of R. Suppose R is a u-S-Artinian ring. Then $\mathrm{Jac}_S(R)$ is S-nilpotent.*

Proof. Suppose R is a u-S-Artinian ring with respect to some $t \in S$. Write $J = \mathrm{Jac}_S(R)$. Consider the descending chain

$$J \supseteq J^2 \supseteq \cdots \supseteq J^n \supseteq J^{n+1} \supseteq \cdots$$

Then there exists an integer k such that $tJ^k \subseteq J^n$ for some $n \geq k$. We claim that $sJ^k = 0$ for some $s \in S$. On contrary, set $\Gamma = \{I \subseteq R \mid sJ^kI \neq 0$ for all $s \in S\}$. Since $R \in \Gamma$, Γ is nonempty. So there is an S-minimal element I

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in Γ by Theorem 2.15. Let $x \in I$ such that $sJ^kx \neq 0$ for all $s \in S$. Then $0 \neq stJ^kx \subseteq sJ^{k+1}x$. So $sJ^{k+1}x \neq 0$ for any $s \in S$. Hence $Jx \in \Gamma$. Since $Jx \subseteq Rx \subseteq I$, there exists $s_1 \in S$ such that $s_1Rx \subseteq s_1I \subseteq Jx \subseteq Rx$ by the S-minimality of I. So there exists $s_2 \in S$ such that $s_2Rx = 0$ by Proposition 4.7, which contradicts $sJ^kx \neq 0$ for all $s \in S$. ☐

Recently, Qi et al. [16, Definition 2.1] introduced the notion of u-S-Noetherian rings. A ring R is called a u-S-Noetherian (abbreviates uniformly S-Noetherian) ring provided there exists an element $s \in S$ such that for any ideal I of R, $sI \subseteq K$ for some finitely generated sub-ideal K of I. Finally, we will show the promised result.

Theorem 4.9. *Let R be a ring and S a multiplicative subset of R. Then the following statements are equivalent:*

(1) *R is a u-S-Artinian ring;*

(2) *R is a u-S-Noetherian ring, $\mathrm{Jac}_S(R)$ is S-nilpotent and $R/\mathrm{Jac}_S(R)$ is a u-$S/\mathrm{Jac}_S(R)$-semisimple ring.*

Proof. (1) ⇒ (2) Suppose R is a u-S-Artinian ring with respect to some $s \in S$. We just need to prove R is u-S-Noetherian because the other two statements are showed in Proposition 4.8 and Proposition 4.5 respectively. Write $J = \mathrm{Jac}_S(R)$. By Proposition 4.8, there exits an integer m such that $tJ^m = 0$ for some $t \in S$. We will show R is u-S-Noetherian by induction on m. Let $m = 1$. It follows by Proposition 4.4 that R is u-S-Noetherian. Now, let $m > 1$. Set $\overline{R} = R/J^{m-1}$. Then $\overline{R}$ is also u-S-Artinian by Proposition 2.9. Note that $\mathrm{Jac}_S(\overline{R}) = J/J^{m-1}$. So $\mathrm{Jac}_S(\overline{R})^{m-1}$ is also u-S-torsion. Hence $\overline{R}$ is u-S-Noetherian by induction. Since $tJ^m = 0$, tJ^{m-1} can also be seen as an ideal of R/J. Since R/J is a u-S/J-semisimple ring, R/J is also u-S/J-Noetherian by [20, Corollary 3.6]. So tJ^{m-1}, and hence J^{m-1}, are both u-S-Noetherian R-modules since J^{m-1} is u-S-isomorphic to tJ^{m-1}. Considering the exact sequence $0 \to J^{m-1} \to R \to R/J^{m-1} \to 0$, we have R is also u-S-Noetherian by [16, Lemma 2.12].

(2) ⇒ (1) Write $J = \mathrm{Jac}_S(R)$. We may assume R is u-S-Noetherian with respect to s such that $sJ^m = 0$ and R/J is a u-S-semisimple R-module with respect to s. We claim that J is a u-S-Artinian R-module. Since sJ^{m-1} is an S-finite R/J-module and R/J is a u-S-semisimple ring, we have sJ^{m-1} is u-S-Artinian by Corollary 3.8. Consider the sequence $0 \to sJ^{m-1} \to sJ^{m-2} \to sJ^{m-2}/sJ^{m-1} \to 0$. Since sJ^{m-2}/sJ^{m-1} is an S-finite R/J-module, we have sJ^{m-2}/sJ^{m-1} is u-S-Artinian by Corollary 3.8. Since sJ^{m-1} is u-S-Artinian, sJ^{m-2} is also u-S-Artinian by Proposition 2.9. Iterating these steps, we have sJ is also u-S-Artinian. So J is also u-S-Artinian since J is u-S-isomorphic to sJ.

Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of R. Consider the following natural commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I_i \cap J \arrow[r] \arrow[d, hook] & I_i \arrow[r] \arrow[d, hook] & (I_i + J)/J \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I_{i+1} \cap J \arrow[r] & I_{i+1} \arrow[r] & (I_{i+1} + J)/J \arrow[r] & 0.
\end{tikzcd}$$

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Since J is u-S-Artinian, there is an integer k_1 such that $s(I_{k_1} \cap J) \subseteq I_n \cap J$ for $n \geq k_1$. Since R/J is a u-S/J-semisimple ring, R/J is also a u-S/J-artinian ring by Theorem 3.7. Hence there is an integer k_2 such that $s((I_{k_2} + J)/J) \subseteq (I_n + J)/J$ for $n \geq k_2$. Taking $k = \max\{k_1, k_2\}$, we can easily deduce $sI_k \subseteq I_n$ for $n \geq k$. Hence R is a u-S-Artinian ring. ☐

Let R be a commutative ring and M an R-module. Then the idealization of R by M, denoted by $R(+)M$, is equal to $R \bigoplus M$ as R-modules with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. It is easy to verify that $R(+)M$ is a commutative ring with identity $(1, 0)$(see [6] for more details). Note that there is a natural exact sequence of $R(+)M$-modules:

$$0 \to 0(+)M \to R(+)M \xrightarrow{\pi} R \to 0.$$

Let S be a multiplicative subset of R. Then it is easy to verify that $S(+)M = \{(s, m) | s \in S, m \in M\}$ is a multiplicative subset of $R(+)M$. Now, we give a u-S-Artinian property on idealizations.

Proposition 4.10. *Let R be a commutative ring, S a multiplicative subset of R and M an R-module. Then $R(+)M$ is a u-$S(+)M$-Artinian ring if and only if R is a u-S-Artinian ring and M is an S-finite R-module.*

Proof. Suppose $R(+)M$ is a u-$S(+)M$-Artinian ring. Then $R \cong R(+)M/0(+)M$ is a u-S-Artinian ring essentially by Proposition 2.9. By Theorem 4.9, $R(+)M$ is a u-$S(+)M$-Noetherian ring. So $0(+)M$ is an $S(+)M$-finite ideal of $R(+)M$, which implies that M is an S-finite R-module.

Suppose R is a u-S-Artinian ring and M is an S-finite R-module. Then M is u-S-Artinian R-module by Proposition 2.9. Let $I^{\bullet} : I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of ideals of $R(+)M$. Then there is a descending chain of ideals of R: $\pi(I^{\bullet}) : \pi(I_1) \supseteq \pi(I_2) \supseteq \cdots$, where $\pi : R(+)M \twoheadrightarrow R$ is the natural epimorphism. Thus there is an element $s' \in S$ which is independent of $I^{\bullet}$ satisfying that there exists $k' \in \mathbb{Z}^+$ such that $s'\pi(I_{k'}) \subseteq \pi(I_n)$ for any $n \geq k'$. Similarly, $I^{\bullet} \cap 0(+)M : I_1 \cap 0(+)M \supseteq I_2 \cap 0(+)M \supseteq \cdots$ is a descending chain of sub-ideals of $0(+)M$ which is equivalent to a descending chain of submodules of M. So there is an element $s'' \in S$ satisfying that there exists $k'' \in \mathbb{Z}^+$ such that $s''(I_{k''} \cap 0(+)M) \subseteq I_n \cap 0(+)M$ for any $n \geq k''$. Let $k = \max(k', k'')$ and $n \geq k$. Consider the following natural commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I_n \cap 0(+)M \arrow[r] \arrow[d, hook] & I_n \arrow[r] \arrow[d, hook] & \pi(I_n) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I_k \cap 0(+)M \arrow[r] & I_k \arrow[r] & \pi(I_k) \arrow[r] & 0.
\end{tikzcd}$$

Set $s = s's''$. Then we have $sI_k \subseteq I_n$ for any $n \geq k$. So $R(+)M$ is a u-$S(+)M$-Artinian ring. ☐

Taking $S = \{1\}$ in Proposition 4.10, we can easily deduce the following classical result.

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Corollary 4.11. *Let R be a commutative ring and M an R-module. Then $R(+)M$ is an Artinian ring if and only if R is an Artinian ring and M is a finitely generated R-module.*

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