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ON FRACTAL MULTIPLICATIVE DERIVATIVES AND THEIR APPLICATIONS

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ABSTRACT. We develop a fractal multiplicative calculus framework and investigate its main theoretical and applied aspects. We introduce a new fractal multiplicative derivative and establish its fundamental properties, including a Leibniz-type rule. In addition, we study a corresponding multiplicative integral operator and analyze its key features. The proposed framework is then applied to the study of fractal and multiplicative differential equations, as well as to multiplicative function spaces and problems in the calculus of variations. As a relevant application, we formulate a fractal multiplicative version of the Gompertz growth model. Furthermore, numerical experiments based on real data demonstrate that the proposed model provides an improved fit compared to other generalized approaches. These results highlight the effectiveness and flexibility of fractal multiplicative calculus in modeling complex systems with nonstandard growth dynamics.

1. INTRODUCTION

Multiplicative calculus, also referred to as non-Newtonian calculus, provides an alternative framework to classical calculus by emphasizing multiplicative (relative) changes instead of additive ones. Its origins can be traced back to the early ideas of Volterra [20], and it was later systematically developed by Grossman and Katz [14], who introduced a general theory of non-Newtonian calculi. In general, multiplicative calculus provides a powerful and complementary perspective to classical calculus, particularly in contexts where relative change is more natural than absolute variation.

Several variants of multiplicative calculus have been introduced, including geometric and bigeometric calculus, each emphasizing different aspects of multiplicative behavior. In recent years, multiplicative calculus has gained renewed attention due to its applications in numerical analysis, signal processing, and mathematical modeling. For instance, multiplicative numerical methods preserve relative errors more effectively than classical additive schemes, while in signal processing, multiplicative models are well suited for data exhibiting large dynamic ranges [18].

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In [7], the authors establish a novel class of generalized inequalities within the framework of multiplicative calculus. Their approach is based on the use of a multiplicative absolute value and the concept of multiplicative h -convexity, which allows them to extend classical results on inequalities to the multiplicative setting. The paper develops several new inequalities for twice multiplicatively differentiable functions and shows that these results encompass various well-known inequalities, including trapezoidal, midpoint, Simpson, and related types, as particular cases.

Recent developments also include the study of multiplicative differential equations, functional spaces adapted to multiplicative structures, and extensions to vector-valued and operator-valued settings [6, 12, 15, 17]. Despite these advances, some theoretical challenges remain open, such as the development of a comprehensive functional analytic framework and the systematic comparison with classical and other generalized calculi.

A central concept in multiplicative calculus is the multiplicative (or geometric) derivative. For a positive function f , it is defined by

$$\frac{{}^*df}{dt}(t_0) := \lim_{t \rightarrow t_0} \left(\frac{f(t)}{f(t_0)} \right)^{\frac{1}{t-t_0}}, \quad (1)$$

whenever the limit exists. This derivative measures the relative rate of change of f at the point t_0 , that is, how the value of $f(t)$ varies multiplicatively in a neighborhood of t_0 . If the multiplicative derivative (1) exists, we say that the function f is * differentiable at t_0 .

Comparing (1) with the classical definition of the derivative

$$\frac{df}{dt}(t_0) = \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t - t_0}, \quad (2)$$

we observe that the additive increment $f(t) - f(t_0)$ in (2) is replaced by the multiplicative ratio $\frac{f(t)}{f(t_0)}$ in (1), while the division by $t - t_0$ is replaced by exponentiation with the reciprocal power $\frac{1}{t-t_0}$. If $\frac{{}^*df}{dt}$ exists for every t in an open set $\Omega \subset \mathbb{R}$, then the function $\frac{{}^*df}{dt} : \Omega \rightarrow \mathbb{R}$ is well defined. The function $\frac{{}^*df}{dt}$ is called the multiplicative derivative of $f : \Omega \rightarrow \mathbb{R}$. If f is differentiable in the usual sense, then

$$\frac{{}^*df}{dt}(t) = \exp\left(\frac{d \ln f(t)}{dt}\right),$$

which highlights the intrinsic relation between multiplicative and classical derivatives. This formulation makes multiplicative calculus particularly suitable for describing exponential growth, proportional change, and scale-invariant phenomena.

Moreover, there has been growing interest in connecting multiplicative calculus with other generalized frameworks. In particular, links with q -calculus have been explored, where difference operators are defined via multiplicative shifts [1, 21], and with fractional calculus, where derivatives of non-integer order or defined on irregular structures are considered [4, 8]. These connections aim to unify different notions of non-classical differentiation and extend their applicability to complex systems. Alongside the multiplicative derivative, one defines the multiplicative integral, which serves as the inverse operation and is closely related to product

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integrals [11]. These tools provide a natural language for describing systems where growth or decay occurs in a proportional or percentage-based manner rather than through absolute increments.

In [2], the authors studied the following fractal derivative

$$\frac{d^\beta f}{dt^\alpha}(t_0) := \lim_{t \rightarrow t_0} \frac{f^\beta(t) - f^\beta(t_0)}{t^\alpha - t_0^\alpha}, \tag{3}$$

where $\alpha, \beta > 0$ and $f^\beta := f|f|^{\beta-1}$. If this limit exists and is finite, then f is said to be (α, β) -fractal differentiable at t_0 . It is worth emphasizing that the fractal derivative (3) was originally defined in [2] for general real-valued functions, not necessarily positive. This motivated the need for a suitable definition of f^β . Throughout this work, we restrict our attention to positive functions. Let f and g be positive functions. In what follows, the expression f^g denotes the pointwise exponentiation given by

$$f^g(t) := (f(t))^{g(t)}.$$

However, there may be situations in which the base of a power is a function that can take negative values. In such cases, we adopt the convention $f^g := f|f|^{g-1}$; ensuring that all operations remain well defined in the real setting.

Using the derivative (3), the authors in [2] analyzed the Gompertz and logistic models and showed that these models provide a better fit to real tuberculosis data in Mexico compared to other approaches. If $\beta = 1$, the operator $\frac{d^\beta}{dt^\alpha}$ coincides with the Hausdorff derivative associated with the fractal measure $m(\alpha, t) = t^\alpha$. More recently, in [13], a Schwarz fractal derivative was studied, and the obtained results were extended to the case of complex-valued functions. Furthermore, in [3], the authors studied a Bernoulli-type differential equation in which the usual derivative is replaced by the fractal derivative (3), and proved the goodness of fit of this model to real data by comparing it with traditional models. The fractal derivative (3) exhibits several noteworthy properties. In particular, functions with fractal graphs, such as the Weierstrass function, fail to admit an ordinary derivative at any point; nevertheless, if α is chosen to be smaller than the corresponding Hölder exponent, such functions admit a fractal derivative at certain points.

The fractal derivative (3) is constructed by mapping the conventional integer-dimensional space–time onto a fractal space–time [9]. It has proven to be a valuable tool, for example, in modeling the physical behavior of fractal soft materials [10]. Derivatives of this type, based on fractal geometry or non-smooth dynamics, are especially appropriate for describing phenomena in which classical or fractional calculus becomes inadequate due to irregularity or scale-dependent effects.

In this paper, we propose the following fractal multiplicative derivative or (α, β) -*derivative:

$$\frac{*d^\beta f}{dt^\alpha}(t_0) := \lim_{t \rightarrow t_0} \left(\frac{f f^{\beta-1}(t)}{f f^{\beta-1}(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}}, \tag{4}$$

where, as before, $\alpha, \beta > 0$, and f is an arbitrary positive function. If this limit exists and is finite, we say that f is (α, β) -*differentiable at t_0 . Note that when $\alpha = \beta = 1$, the derivative (4) coincides with the standard multiplicative derivative

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(1). Whenever one of these orders equals 1, we omit it from the notation; in such cases, $\frac{*d}{dt^\alpha} = \frac{*d^1}{dt^\alpha}$ and $\frac{*d^\beta}{dt} = \frac{*d^\beta}{dt^1}$.

The paper is organized as follows. In Section 2, we establish the basic properties for fractal multiplicative derivatives, including fundamental rules such as a Leibniz-type formula. Section 3 is devoted to the study of the associated fractal multiplicative integral, where its main properties and connections with the derivative are analyzed. Finally, in Section 4, we present several applications of the developed framework, including fractal and multiplicative differential equations, multiplicative function spaces, problems in the calculus of variations and Gompertz models.

2. BASIC PROPERTIES

In this section, we study some properties of the derivative (4), which will play a fundamental role throughout the remainder of the paper. In addition, we explore its relationship with the fractal derivatives (3), as well as the corresponding Leibniz and chain rules. From now on, we assume that I is an open interval of the real line containing the point t_0 . The symbol \cdot will be used to denote the usual product in certain formulas for the sake of clarity and readability.

Fractal differentiability is a sufficient condition for the continuity of a function. This is not the case for $*$ differentiability. Indeed, consider the piecewise-defined function

$$f(t) = \begin{cases} 1, & t = 0, \\ e^{-\frac{1}{t}}, & t \neq 0. \end{cases}$$

Clearly, f is discontinuous at $t = 0$. However, it admits a multiplicative derivative at that point:

$$\frac{*df}{dt}(0) = \lim_{t \rightarrow 0} \left(\frac{f(t)}{f(0)} \right)^{\frac{1}{t}} = \lim_{t \rightarrow 0} e^{-\frac{1}{t^2}} = 0.$$

Proposition 2.1. *Let $f : I \rightarrow \mathbb{R}^+$ be an (α, α) -fractal differentiable function at t_0 . Then f is $(\alpha, 1)$ - $*$ differentiable at t_0 and*

$$\frac{*df}{dt^\alpha}(t_0) = \exp \left[\frac{1}{\alpha} \cdot f^{-\alpha}(t_0) \cdot \frac{d^\alpha f}{dt^\alpha}(t_0) \right]. \tag{5}$$

In particular, for $\alpha = 1$

$$\frac{*df}{dt}(t_0) = \exp \left[\frac{d \ln f}{dt}(t_0) \right]. \tag{6}$$

Proof. We have that

$$\left(\frac{f(t)}{f(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} = \exp \left[\frac{\ln f(t) - \ln f(t_0)}{t^\alpha - t_0^\alpha} \right].$$

Applying the chain rule for fractal derivatives (see [2, Proposition 2.9, p. 10728]), we obtain

$$\frac{d \ln f}{dt^\alpha}(t_0) = \frac{d \ln(t)}{dt^\alpha}(f(t_0)) \cdot \frac{d^\alpha f}{dt^\alpha}(t_0) = \frac{1}{\alpha} \cdot f^{-\alpha}(t_0) \cdot \frac{d^\alpha f}{dt^\alpha}(t_0).$$

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Thus, the function f is $(\alpha, 1)$ -*differentiable at t_0 and

$$\frac{*df}{dt^\alpha}(t_0) = \exp \left[\frac{d \ln f}{dt^\alpha}(t_0) \right] = \exp \left[\frac{1}{\alpha} \cdot f^{-\alpha}(t_0) \cdot \frac{d^\alpha f}{dt^\alpha}(t_0) \right]. \tag{7}$$

□

Proposition 2.2. *Let $\beta > 1$, and let $f : I \rightarrow \mathbb{R}^+$ be an $(\alpha, \beta - 1)$ -fractal differentiable function at t_0 . Then f is (α, β) -*differentiable at t_0 and*

$$\frac{*d^\beta f}{dt^\alpha}(t_0) = f^{\frac{d\beta-1}{dt^\alpha}f}(t_0) \cdot \exp \left[\frac{1}{\beta-1} \cdot \frac{d^{\beta-1}f}{dt^\alpha}(t_0) \right]. \tag{8}$$

Proof. Firstly, note that

$$\begin{aligned} & \left(\frac{f^{f^{\beta-1}}(t)}{f^{f^{\beta-1}}(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} \\ &= \exp \left[\frac{\ln f^{f^{\beta-1}}(t) - \ln f^{f^{\beta-1}}(t_0)}{t^\alpha - t_0^\alpha} \right] \\ &= \exp \left[\frac{f^{\beta-1}(t) \ln f(t) - f^{\beta-1}(t_0) \ln f(t_0)}{t^\alpha - t_0^\alpha} \right] \\ &= \exp \left[\frac{(f^{\beta-1}(t) - f^{\beta-1}(t_0)) \ln f(t) + f^{\beta-1}(t_0)(\ln f(t) - \ln f(t_0))}{t^\alpha - t_0^\alpha} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \lim_{t \rightarrow t_0} \left(\frac{f^{f^{\beta-1}}(t)}{f^{f^{\beta-1}}(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} \\ &= \exp \left[\lim_{t \rightarrow t_0} \frac{(f^{\beta-1}(t) - f^{\beta-1}(t_0)) \ln f(t) + f^{\beta-1}(t_0)(\ln f(t) - \ln f(t_0))}{t^\alpha - t_0^\alpha} \right]. \end{aligned}$$

Since f is $(\alpha, \beta - 1)$ -fractal differentiable at t_0 , then

$$\begin{aligned} & \lim_{t \rightarrow t_0} \frac{\ln f(t) - \ln f(t_0)}{t^\alpha - t_0^\alpha} \\ &= \frac{1}{\beta-1} \lim_{t \rightarrow t_0} \frac{\ln f^{\beta-1}(t) - \ln f^{\beta-1}(t_0)}{t^\alpha - t_0^\alpha} \\ &= \frac{1}{\beta-1} \lim_{t \rightarrow t_0} \frac{1}{f^{\beta-1}(t_0)} \cdot \frac{f^{\beta-1}(t) - f^{\beta-1}(t_0)}{t^\alpha - t_0^\alpha} \cdot \ln \left[\frac{f^{\beta-1}(t) - f^{\beta-1}(t_0)}{f^{\beta-1}(t_0)} + 1 \right]^{\frac{f^{\beta-1}(t_0)}{f^{\beta-1}(t) - f^{\beta-1}(t_0)}} \\ &= \frac{1}{(\beta-1)f^{\beta-1}(t_0)} \cdot \frac{d^{\beta-1}f}{dt^\alpha}(t_0). \end{aligned}$$

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Hence,

$$\begin{aligned} \lim_{t \rightarrow t_0} \left(\frac{f^{f^{\beta-1}}(t)}{f^{f^{\beta-1}}(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} &= \exp \left[\left(\ln f(t_0) + \frac{1}{\beta - 1} \right) \frac{d^{\beta-1} f}{dt^\alpha}(t_0) \right] \\ &= f^{\frac{d^{\beta-1} f}{dt^\alpha}(t_0)} \cdot \exp \left[\frac{1}{\beta - 1} \cdot \frac{d^{\beta-1} f}{dt^\alpha}(t_0) \right]. \end{aligned}$$

□

Remark 2.3. The Weierstrass function

$$\mathcal{W}(t) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi t), \quad 0 < a < 1, \quad b \in \mathbb{Z}_+ \text{ odd}, \quad ab > 1 + \frac{3\pi}{2}, \quad (9)$$

is a typical example of a function that is everywhere continuous on \mathbb{R} yet nowhere differentiable. In 1916, Hardy proved that \mathcal{W} is Hölder continuous with exponent $\nu = -\frac{\log a}{\log b}$; that is, there exists a constant $C > 0$ such that for all $t, t_1 \in \mathbb{R}$,

$$|\mathcal{W}(t) - \mathcal{W}(t_1)| \leq C|t - t_1|^\nu.$$

If we take $\alpha < \nu$, then for all $t, t_1 \in \mathbb{R}$,

$$\frac{|\mathcal{W}(t) - \mathcal{W}(t_1)|}{|t - t_1|^\alpha} \leq C|t - t_1|^{\nu-\alpha}. \quad (10)$$

Setting $t_1 = 0$ in (10), we obtain

$$\frac{|\mathcal{W}(t) - \mathcal{W}(0)|}{|t|^\alpha} \leq C|t|^{\nu-\alpha}. \quad (11)$$

Since $t^\alpha := t|t|^{\alpha-1}$, it follows that $|t^\alpha| = |t||t|^{\alpha-1} = |t|^\alpha$. Hence, relation (11) is equivalent to

$$\frac{|\mathcal{W}(t) - \mathcal{W}(0)|}{|t^\alpha|} \leq C|t|^{\nu-\alpha}. \quad (12)$$

By the squeeze theorem, relation (12) implies that whenever $\alpha < \nu$,

$$\lim_{t \rightarrow 0} \left| \frac{\mathcal{W}(t) - \mathcal{W}(0)}{t^\alpha} \right| = 0,$$

and therefore,

$$\frac{d\mathcal{W}}{dt^\alpha}(0) = \lim_{t \rightarrow 0} \frac{\mathcal{W}(t) - \mathcal{W}(0)}{t^\alpha} = 0.$$

Using Proposition 2.2, we obtain that

$$\frac{{}^*d^2\mathcal{W}}{dt^\alpha}(0) = 1. \quad (13)$$

Thanks to the Hölder condition, this simple example illustrates the existence of fractal-type derivatives for functions that are nowhere classically differentiable. By considering translations of the function \mathcal{W} , one can construct examples where the Hausdorff derivative exists at points other than 0. Similar examples can also be obtained for other functions whose graphs are fractals and which are Hölder continuous with exponent $0 < \nu < 1$, such as the Cantor function and Brownian

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motion. Beyond their well-known applications in anomalous diffusion (which has motivated much of their recent study), the possibility of defining a local derivative that exists—even at a single point—for highly irregular functions is already of intrinsic mathematical interest.

If we take $\beta = 1 + \alpha$, then, according to Propositions 2.1 and 2.2, the (α, α) -fractal differentiable function f is both $(\alpha, 1)$ - and $(\alpha, 1 + \alpha)$ -*differentiable at t_0 . The following proposition provides the equivalence between both classes of functions.

Proposition 2.4. *Let f be a positive continuous function defined on the interval I , and assume that $f(t_0) > \exp(-\frac{1}{\alpha})$. Then f is $(\alpha, 1)$ -*differentiable at t_0 if and only if it is also $(\alpha, 1 + \alpha)$ -*differentiable at the same point.*

Proof. If the function $f : I \rightarrow \mathbb{R}^+$ is $(\alpha, 1)$ -*differentiable at t_0 then the limit

$$\lim_{t \rightarrow t_0} \left(\frac{f(t)}{f(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} = \lim_{t \rightarrow t_0} \exp \left[\frac{\ln f(t) - \ln f(t_0)}{t^\alpha - t_0^\alpha} \right]$$

exists, and it is finite. On the other hand, we have that

$$\begin{aligned} & \left(\frac{f f^\alpha(t)}{f f^\alpha(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} \\ &= \exp \left[\frac{\ln f(t) - \ln f(t_0)}{t^\alpha - t_0^\alpha} \left(f^\alpha(t_0) + \frac{\alpha f^\alpha(t_0) \ln f(t)}{\ln \left(\frac{f^\alpha(t) - f^\alpha(t_0)}{f^\alpha(t_0)} + 1 \right)^{\frac{f^\alpha(t_0)}{f^\alpha(t) - f^\alpha(t_0)}}} \right) \right]. \end{aligned}$$

Consequently, if $f(t_0) > \exp(-\frac{1}{\alpha})$, then

$$\lim_{t \rightarrow t_0} \left(f^\alpha(t_0) + \frac{\alpha f^\alpha(t_0) \ln f(t)}{\ln \left(\frac{f^\alpha(t) - f^\alpha(t_0)}{f^\alpha(t_0)} + 1 \right)^{\frac{f^\alpha(t_0)}{f^\alpha(t) - f^\alpha(t_0)}}} \right) = f^\alpha(t_0)(1 + \alpha \ln f(t_0)) > 0.$$

Hence, we can conclude that the limit

$$\lim_{t \rightarrow t_0} \left(\frac{f f^\alpha(t)}{f f^\alpha(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}}$$

exists and is finite. Therefore, f is $(\alpha, 1 + \alpha)$ -*differentiable at t_0 . The converse statement follows by an analogous argument. \square

The following example shows that condition $f(t_0) > \exp(-\frac{1}{\alpha})$ is necessary for the equivalence.

Example 2.5. Let $\alpha \in (0, 1)$. Consider the function

$$f(t) = c \exp(-\sqrt[3]{t - t_0}), \quad c \in (0, e^{-1}).$$

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It is easy to see that

$$\lim_{t \rightarrow t_0} \frac{\ln f(t) - \ln f(t_0)}{t^\alpha - t_0^\alpha} = \lim_{t \rightarrow t_0} \frac{-\sqrt[3]{t-t_0}}{t^\alpha - t_0^\alpha} = -\infty.$$

Thus,

$$\frac{*d}{dt^\alpha} f(t_0) = \lim_{t \rightarrow t_0} \exp \left[\frac{\ln f(t) - \ln f(t_0)}{t^\alpha - t_0^\alpha} \right] = 0.$$

The function f is $(\alpha, 1)$ -*differentiable at t_0 . However, if $c < \exp(-\frac{1}{\alpha})$, then we obtain

$$\begin{aligned} & \lim_{t \rightarrow t_0} \frac{f^\alpha(t) \ln f(t) - f^\alpha(t_0) \ln f(t_0)}{t^\alpha - t_0^\alpha} \\ &= \lim_{t \rightarrow t_0} \frac{c^\alpha \exp(-\alpha \sqrt[3]{t-t_0})(\ln c - \sqrt[3]{t-t_0}) - c^\alpha \ln c}{t^\alpha - t_0^\alpha} \\ &= +\infty. \end{aligned}$$

Therefore,

$$\lim_{t \rightarrow t_0} \left(\frac{f^{f^\alpha}(t)}{f^{f^\alpha}(t_0)} \right)^{\frac{1}{t^{\alpha-1} - t_0^\alpha}} = \lim_{t \rightarrow t_0} \exp \left[\frac{f^\alpha(t) \ln f(t) - f^\alpha(t_0) \ln f(t_0)}{t^\alpha - t_0^\alpha} \right] = +\infty,$$

which shows that f is not $(\alpha, 1 + \alpha)$ -*differentiable at t_0 . Note that this function is not (α, α) -fractal differentiable at t_0 :

$$\lim_{t \rightarrow t_0} \frac{f^\alpha(t) - f^\alpha(t_0)}{t^\alpha - t_0^\alpha} = \lim_{t \rightarrow t_0} \frac{c^\alpha \exp(-\alpha \sqrt[3]{t-t_0}) - c^\alpha}{t^\alpha - t_0^\alpha} = -\infty.$$

On the other hand, if $c > \exp(-\frac{1}{\alpha})$, then

$$\lim_{t \rightarrow t_0} \frac{f^\alpha(t) \ln f(t) - f^\alpha(t_0) \ln f(t_0)}{t^\alpha - t_0^\alpha} = -\infty,$$

whence

$$\lim_{t \rightarrow t_0} \left(\frac{f^{f^\alpha}(t)}{f^{f^\alpha}(t_0)} \right)^{\frac{1}{t^{\alpha-1} - t_0^\alpha}} = 0.$$

That is, the function f is $(\alpha, 1 + \alpha)$ -*differentiable at t_0 . If we now take a constant $\tilde{c} > 0$ such that $\tilde{c} < \exp(-\frac{1}{\alpha})$, then the function $\tilde{c}f$ is no longer $(\alpha, 1 + \alpha)$ -*differentiable at t_0 . This curious fact highlights significant differences with the usual derivatives.

Proposition 2.6. *Let $c \in \mathbb{R}^+$ and let f be an (α, β) -*differentiable function at t_0 . Then, the following statements hold:*

- i) $\frac{*d^\beta f}{dt^\alpha}(t_0) = \frac{*d(f^{f^{\beta-1}})}{dt^\alpha}(t_0)$, $\frac{*d^\beta f}{dt^\alpha}(t_0) \geq 0$.
- ii) *The constant function c is (α, β) -*differentiable at t_0 , and*

$$\frac{*d^\beta c}{dt^\alpha}(t_0) = 1.$$

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iii) Assume that $\beta > 1$. If f is also $(\alpha, \beta - 1)$ -fractal differentiable at t_0 , then the function cf is (α, β) -*differentiable at t_0 , and

$$\frac{{}^*d^\beta(cf)}{dt^\alpha}(t_0) = c^{c^{\beta-1} \cdot \frac{d^{\beta-1}f}{dt^\alpha}(t_0)} \cdot \left(\frac{{}^*d^\beta f}{dt^\alpha}(t_0)\right)^{c^{\beta-1}}.$$

In the particular case $\beta = 1$, we have

$$\frac{{}^*d(cf)}{dt^\alpha}(t_0) = \frac{{}^*df}{dt^\alpha}(t_0).$$

Proof. The first two statements are straightforward; therefore, we omit their proofs and prove only the third one.

$$\begin{aligned} & \frac{{}^*d^\beta(cf)}{dt^\alpha}(t_0) \\ &= \lim_{t \rightarrow t_0} \left(\frac{(cf)^{(cf)^{\beta-1}}(t)}{(cf)^{(cf)^{\beta-1}}(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} \\ &= \lim_{t \rightarrow t_0} \exp \left[\frac{(cf)^{\beta-1}(t) \ln(cf)(t) - (cf)^{\beta-1}(t_0) \ln(cf)(t_0)}{t^\alpha - t_0^\alpha} \right] \\ &= \lim_{t \rightarrow t_0} \exp \left[c^{\beta-1} \ln c \frac{f^{\beta-1}(t) - f^{\beta-1}(t_0)}{t^\alpha - t_0^\alpha} + c^{\beta-1} \frac{f^{\beta-1}(t) \ln(f)(t) - f^{\beta-1}(t_0) \ln(f)(t_0)}{t^\alpha - t_0^\alpha} \right] \\ &= c^{c^{\beta-1} \cdot \frac{d^{\beta-1}f}{dt^\alpha}(t_0)} \cdot \left(\frac{{}^*d^\beta f}{dt^\alpha}(t_0)\right)^{c^{\beta-1}}. \end{aligned}$$

□

Remark 2.7. The fractal derivatives introduced in [2] can also be defined for $\beta < 0$, provided that we set $\frac{d^0 f}{dt^\alpha} := 0$. Accordingly, the results in Propositions 2 and 4 remain valid for $\beta < 1$. However, it is not our goal to present this generalization here.

Proposition 2.8 (Leibniz rule). *Let $\beta > 1$ and let $f, g : I \rightarrow \mathbb{R}^+$ be $(\alpha, \beta - 1)$ -fractal differentiable functions at t_0 . The following statements hold:*

i) fg is (α, β) -*differentiable at t_0 , and

$$\frac{{}^*d^\beta(fg)}{dt^\alpha}(t_0) = \left(\frac{{}^*d^\beta f}{dt^\alpha}\right)^{g^{\beta-1}}(t_0) \cdot \left(\frac{{}^*d^\beta g}{dt^\alpha}\right)^{f^{\beta-1}}(t_0) \cdot f^{f^{\beta-1} \cdot \frac{d^{\beta-1}g}{dt^\alpha}(t_0)} \cdot g^{g^{\beta-1} \cdot \frac{d^{\beta-1}f}{dt^\alpha}(t_0)}.$$

In particular,

$$\frac{{}^*d(fg)}{dt^\alpha}(t_0) = \left(\frac{{}^*df}{dt^\alpha}\right)(t_0) \cdot \left(\frac{{}^*dg}{dt^\alpha}\right)(t_0).$$

ii) $\frac{1}{g}$ is (α, β) -*differentiable at t_0 , and

$$\frac{{}^*d^\beta\left(\frac{1}{g}\right)}{dt^\alpha}(t_0) = g^{2 \cdot g^{-2(\beta-1)} \cdot \frac{d^{\beta-1}g}{dt^\alpha}(t_0)} \cdot \left(\frac{{}^*d^\beta g}{dt^\alpha}\right)^{-g^{-2(\beta-1)}}(t_0).$$

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iii) $\frac{f}{g}$ is (α, β) -*differentiable at t_0 , and

$$\frac{{}^*d^\beta}{dt^\alpha} \left(\frac{f}{g} \right) (t_0) = \left(\frac{{}^*d^\beta f}{dt^\alpha} \right)^{g^{-(\beta-1)}} (t_0) \cdot \left(\frac{{}^*d^\beta g}{dt^\alpha} \right)^{-g^{-2(\beta-1)} \cdot f^{\beta-1}} (t_0) \cdot f^{-f^{\beta-1} \cdot g^{-2(\beta-1)} \cdot \frac{d^{\beta-1}g}{dt^\alpha}} (t_0) \cdot g^{2 \cdot f^{\beta-1} \cdot g^{-2(\beta-1)} \cdot \frac{d^{\beta-1}g}{dt^\alpha} - g^{1-\beta} \cdot \frac{d^{\beta-1}f}{dt^\alpha}} (t_0).$$

Proof. i) We have

$$\begin{aligned} & \frac{{}^*d^\beta(fg)}{dt^\alpha}(t_0) \\ &= \lim_{t \rightarrow t_0} \left(\frac{(fg)^{(fg)^{\beta-1}}(t)}{(fg)^{(fg)^{\beta-1}}(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} \\ &= \lim_{t \rightarrow t_0} \exp \left[\frac{(fg)^{\beta-1}(t) \ln(fg)(t) - (fg)^{\beta-1}(t_0) \ln(fg)(t_0)}{t^\alpha - t_0^\alpha} \right] \\ &= \lim_{t \rightarrow t_0} \exp \left[\frac{(f^{\beta-1}(t) \ln f(t) - f^{\beta-1}(t_0) \ln f(t_0))g^{\beta-1}(t) + f^{\beta-1}(t_0) \ln f(t_0)(g^{\beta-1}(t) - g^{\beta-1}(t_0))}{t^\alpha - t_0^\alpha} \right. \\ & \quad \left. + \frac{(g^{\beta-1}(t) \ln g(t) - g^{\beta-1}(t_0) \ln g(t_0))f^{\beta-1}(t) + g^{\beta-1}(t_0) \ln g(t_0)(f^{\beta-1}(t) - f^{\beta-1}(t_0))}{t^\alpha - t_0^\alpha} \right]. \end{aligned}$$

Using Proposition 2.2, we ensure that f and g are (α, β) -*differentiable at t_0 , and therefore,

$$\frac{{}^*d^\beta(fg)}{dt^\alpha}(t_0) = \left(\frac{{}^*d^\beta f}{dt^\alpha} \right)^{g^{\beta-1}} (t_0) \cdot \left(\frac{{}^*d^\beta g}{dt^\alpha} \right)^{f^{\beta-1}} (t_0) \cdot f^{f^{\beta-1} \cdot \frac{d^{\beta-1}g}{dt^\alpha}} (t_0) \cdot g^{g^{\beta-1} \cdot \frac{d^{\beta-1}f}{dt^\alpha}} (t_0).$$

ii)

$$\begin{aligned} & \frac{{}^*d^\beta \left(\frac{1}{g} \right)}{dt^\alpha}(t_0) \\ &= \lim_{t \rightarrow t_0} \exp \left[\frac{- \left(\frac{1}{g} \right)^{\beta-1} (t) \ln g(t) + \left(\frac{1}{g} \right)^{\beta-1} (t_0) \ln g(t_0)}{t^\alpha - t_0^\alpha} \right] \\ &= \lim_{t \rightarrow t_0} \exp \left[\frac{-g^{\beta-1}(t_0) \ln g(t) + g^{\beta-1}(t) \ln g(t_0)}{g^{\beta-1}(t)g^{\beta-1}(t_0)(t^\alpha - t_0^\alpha)} \right] \\ &= \lim_{t \rightarrow t_0} \exp \left[\frac{-g^{\beta-1}(t) \ln g(t) + g^{\beta-1}(t_0) \ln g(t_0) + (g^{\beta-1}(t) - g^{\beta-1}(t_0)) \ln g(t)}{g^{\beta-1}(t)g^{\beta-1}(t_0)(t^\alpha - t_0^\alpha)} \right. \\ & \quad \left. + \frac{(g^{\beta-1}(t) - g^{\beta-1}(t_0)) \ln g(t_0)}{g^{\beta-1}(t)g^{\beta-1}(t_0)(t^\alpha - t_0^\alpha)} \right] \\ &= g^{2 \cdot g^{-2(\beta-1)} \cdot \frac{d^{\beta-1}g}{dt^\alpha}} (t_0) \cdot \left(\frac{{}^*d^\beta g}{dt^\alpha} \right)^{-g^{-2(\beta-1)}} (t_0). \end{aligned}$$

iii) The result is a simple application of i), ii) and [2, Proposition 2.8 ii, p. 10728]. □

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Proposition 2.9 (Chain rule). *Let $g : I \rightarrow \mathbb{R}^+$ be an (α, α) -fractal differentiable function at t_0 , and let $f : I \rightarrow \mathbb{R}^+$ be (α, β) -*differentiable at $g(t_0)$. If $\frac{{}^*d^\beta f}{dt^\alpha}(g(t_0)) \neq 0$, then $f \circ g$ is (α, β) -*differentiable at t_0 and*

$$\frac{{}^*d^\beta(f \circ g)}{dt^\alpha}(t_0) = \left(\frac{{}^*d^\beta f}{dt^\alpha}(g(t_0)) \right)^{\frac{d^\alpha g}{dt^\alpha}(t_0)}.$$

While, if $\frac{{}^*d^\beta f}{dt^\alpha}(g(t_0)) = 0$, the existence of the multiplicative fractal derivative of $f \circ g$ at t_0 depends on the sign of $\frac{d^\alpha g}{dt^\alpha}(t_0)$.

In particular,

$$\frac{{}^*d^\beta \exp(g)}{dt^\alpha}(t_0) = \exp \left[\left(\frac{\beta - 1}{\alpha} \cdot g^{2-\alpha}(t_0) + \frac{1}{\alpha} g^{1-\alpha}(t_0) \right) \cdot \exp((\beta - 1)g(t_0)) \cdot \frac{d^\alpha g}{dt^\alpha}(t_0) \right].$$

Proof. It is clear that

$$\begin{aligned} & \frac{{}^*d^\beta(f \circ g)}{dt^\alpha}(t_0) \\ &= \lim_{t \rightarrow t_0} \left(\frac{(f \circ g)^{(f \circ g)^{\beta-1}}(t)}{(f \circ g)^{(f \circ g)^{\beta-1}}(t_0)} \right)^{\frac{1}{t^\alpha - t_0^\alpha}} \\ &= \lim_{t \rightarrow t_0} \exp \left[\frac{(f \circ g)^{\beta-1}(t) \ln(f \circ g)(t) - (f \circ g)^{\beta-1}(t_0) \ln(f \circ g)(t_0)}{g^\alpha(t) - g^\alpha(t_0)} \cdot \frac{g^\alpha(t) - g^\alpha(t_0)}{t^\alpha - t_0^\alpha} \right] \\ &= \left(\frac{{}^*d^\beta f}{dt^\alpha}(g(t_0)) \right)^{\frac{d^\alpha g}{dt^\alpha}(t_0)}. \end{aligned}$$

□

Remark 2.10. If f and h are positive (α, α) -fractal differentiable functions on I , then f^h is (α, β) -*differentiable and

$$\frac{{}^*d^\beta(f^h)}{dt^\alpha} = \exp \left[\left(\frac{\beta - 1}{\alpha} \cdot (h \ln f)^{2-\alpha} + \frac{1}{\alpha} (h \ln f)^{1-\alpha} \right) \cdot f^{(\beta-1)h} \cdot \frac{d^\alpha(h \ln f)}{dt^\alpha} \right].$$

A simple calculation shows that

$$\frac{d^\alpha(h \ln f)}{dt^\alpha} = h^\alpha \frac{d^\alpha \ln f}{dt^\alpha} + (\ln f)^\alpha \frac{d^\alpha h}{dt^\alpha} = \left(\frac{h}{f} \right)^\alpha |\ln f|^{\alpha-1} \frac{d^\alpha f}{dt^\alpha} + (\ln f)^\alpha \frac{d^\alpha h}{dt^\alpha}.$$

Subsequently, we obtain

$$\begin{aligned} \frac{{}^*d^\beta(f^h)}{dt^\alpha} &= \exp \left[\frac{\beta - 1}{\alpha} h^2 f^{(\beta-1)h-\alpha} (\ln f) \frac{d^\alpha f}{dt^\alpha} + \frac{1}{\alpha} h f^{(\beta-1)h-\alpha} \operatorname{sgn}(\ln f) \frac{d^\alpha f}{dt^\alpha} \right. \\ &\quad + \frac{\beta - 1}{\alpha} h^{2-\alpha} (\ln f)^2 \operatorname{sgn}(\ln f) f^{(\beta-1)h} \frac{d^\alpha h}{dt^\alpha} \\ &\quad \left. + \frac{1}{\alpha} h^{1-\alpha} f^{(\beta-1)h} (\ln f) \operatorname{sgn}(\ln f) \frac{d^\alpha h}{dt^\alpha} \right]. \end{aligned}$$

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A number of classical results can be extended using the fractal multiplicative derivative (4). In the sequel, we establish Rolle Theorem and the Mean Value Theorem in this setting.

Theorem 2.11 (Fractal Multiplicative Rolle Theorem). *Let f be a positive function that is continuous on the closed interval $[a, b]$ and (α, β) -*differentiable on the open interval (a, b) , with $f(a) = f(b)$. Then there exists at least one point $c \in (a, b)$ such that*

$$\frac{{}^*d^\beta f}{dt^\alpha}(c) = 1.$$

Proof. It is known that three possibilities may arise: either the function under consideration is constant, or there exists a point t_0 where the function attains a value greater than at the endpoints, or a point where its value is smaller than at the endpoints. In the first case, it is clear that the function has derivative identically equal to 1, by definition. We now present the key steps of the proof.

- Since f is continuous and positive, the function $f^{\beta-1} \ln f$ is also continuous. Hence, the image of $[a, b]$ under this function is a connected subset of \mathbb{R} , and therefore an interval (the image interval).
- The image of a compact set under a continuous function is compact; therefore, the image interval is closed and bounded. In particular, it is of the form $[m, M]$, where m is the minimum value of $f^{\beta-1} \ln f$ and M is its maximum value.
- If $m = M$, then the function is constant and any point $c \in (a, b)$ suffices. Excluding this case, $m \neq M$ implies that at least one of them is different from $f^{\beta-1}(a) \ln f(a) = f^{\beta-1}(b) \ln f(b)$. Suppose it is M . Then $M > f^{\beta-1}(a) \ln f(a) = f^{\beta-1}(b) \ln f(b)$, and therefore the maximum M is attained in the interior of the interval.
- Let $c \in (a, b)$ be such that $f^{\beta-1}(c) \ln f(c) = M$. By definition of the maximum, $M = f^{\beta-1}(c) \ln f(c) \geq f^{\beta-1}(t) \ln f(t)$ for all $t \in [a, b]$. Hence, the quotient

$$\frac{f^{\beta-1}(c) \ln f(c) - f^{\beta-1}(t) \ln f(t)}{c^\alpha - t^\alpha}$$

is positive when $t < c$ (since the numerator is nonnegative and, as the function $t^\alpha := t|t|^{\alpha-1}$ is increasing on \mathbb{R} , the denominator is positive and nonzero), and negative when $t > c$.

By definition, $\frac{{}^*d^\beta f}{dt^\alpha}(c)$ is the limit of

$$\exp \left[\frac{f^{\beta-1}(c) \ln f(c) - f^{\beta-1}(t) \ln f(t)}{c^\alpha - t^\alpha} \right]$$

as $t \rightarrow c$. The left-hand limit $\frac{{}^*d^\beta f}{dt^\alpha}(c^-)$ must coincide with the right-hand limit $\frac{{}^*d^\beta f}{dt^\alpha}(c^+)$. Therefore, this common limit is 1, that is,

$$\frac{{}^*d^\beta f}{dt^\alpha}(c) = 1.$$

The proof is entirely analogous if the minimum is attained in (a, b) . □

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Theorem 2.12 (Fractal Multiplicative Mean Value Theorem). *Let $f : [a, b] \rightarrow (0, \infty)$ be a positive function that is continuous on $[a, b]$ and $(\alpha, \beta - 1)$ -fractal differentiable on (a, b) . Then there exists a point $c \in (a, b)$ such that*

$${}^*d^\beta f / dt^\alpha (c) = \left(\frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{c^\alpha - a^\alpha}{b^\alpha - a^\alpha}} \right)^{\frac{1-\beta}{b^\alpha - a^\alpha} f^{\beta-1}(c) \ln \left(\frac{f(a)}{f(b)} \right) - \frac{d^{\beta-1} f}{dt^\alpha} (c)} \cdot (f(c))^{\frac{1-\beta}{b^\alpha - a^\alpha} f^{\beta-1}(c) \ln \left(\frac{f(a)}{f(b)} \right)}.$$

Proof. Define the auxiliary function

$$g(t) = \frac{f(t)}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}}.$$

Then g is positive, continuous on $[a, b]$, and (α, β) -*differentiable on (a, b) . This last conclusion follows from the fact that both f and $\frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}}$ are $(\alpha, \beta - 1)$ -fractal differentiable functions on (a, b) (see Proposition 2.8). Observe that for all $t_0 \in (a, b)$ we have

$$\begin{aligned} & \frac{d^{\beta-1} \frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}}}{dt^\alpha} (t_0) \\ &= f^{1-\beta}(a) \lim_{t \rightarrow t_0} \frac{\left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t^\alpha - a^\alpha)}{b^\alpha - a^\alpha}} - \left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t_0^\alpha - a^\alpha)}{b^\alpha - a^\alpha}}}{t^\alpha - t_0^\alpha} \\ &= f^{1-\beta}(a) \lim_{t \rightarrow t_0} \frac{\frac{(\beta-1)\alpha}{b^\alpha - a^\alpha} \left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t^\alpha - a^\alpha)}{b^\alpha - a^\alpha}} |t|^{\alpha-1} \ln \left(\frac{f(a)}{f(b)} \right)}{\alpha |t|^{\alpha-1}} \\ &= \frac{\beta - 1}{b^\alpha - a^\alpha} f^{1-\beta}(a) \left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t_0^\alpha - a^\alpha)}{b^\alpha - a^\alpha}} \ln \left(\frac{f(a)}{f(b)} \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{{}^*d^\beta \frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}}}{dt^\alpha} (t_0) \\ &= \left[\frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t_0^\alpha - a^\alpha}{b^\alpha - a^\alpha}} \right]^{\frac{\beta-1}{b^\alpha - a^\alpha} f^{1-\beta}(a) \left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t_0^\alpha - a^\alpha)}{b^\alpha - a^\alpha}} \ln \left(\frac{f(a)}{f(b)} \right)} \\ & \cdot \left(\frac{f(a)}{f(b)} \right)^{\frac{1}{b^\alpha - a^\alpha} f^{1-\beta}(a) \left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t_0^\alpha - a^\alpha)}{b^\alpha - a^\alpha}}} \end{aligned}$$

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We denote by $p(t_0)$ and $q(t_0)$ the expressions associated with $\frac{d^{\beta-1} \frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}}}{dt^\alpha}(t_0)$ and $\frac{{}^*d^\beta \frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}}}{dt^\alpha}(t_0)$, respectively. Moreover,

$$g(a) = 1, \quad g(b) = 1.$$

By Theorem 2.11, there exists $c \in (a, b)$ such that

$$\frac{{}^*d^\beta g}{dt^\alpha}(c) = 1.$$

Using the Leibniz rule for (α, β) -*derivatives, we obtain

$$\begin{aligned} & \frac{{}^*d^\beta g}{dt^\alpha}(t) \\ &= \left(\frac{{}^*d^\beta f}{dt^\alpha}(t) \right)^{f^{1-\beta}(a) \left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t^\alpha - a^\alpha)}{b^\alpha - a^\alpha}}} \cdot (q(t))^{f^{\beta-1}(t)} \cdot (f(t))^{f^{\beta-1}(t)} p(t) \\ & \quad \cdot \left[\frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}} \right]^{f^{1-\beta}(a) \left(\frac{f(a)}{f(b)} \right)^{\frac{(\beta-1)(t^\alpha - a^\alpha)}{b^\alpha - a^\alpha}}} \cdot \frac{d^{\beta-1} f}{dt^\alpha} \end{aligned}$$

Evaluating at $t = c$ and using the fact that $\frac{{}^*d^\beta g}{dt^\alpha}(c) = 1$, we conclude that

$$\begin{aligned} & \frac{{}^*d^\beta f}{dt^\alpha}(c) \\ &= (q(c))^{-f^{\beta-1}(c)} f^{\beta-1}(a) \left(\frac{f(a)}{f(b)} \right)^{\frac{(1-\beta)(c^\alpha - a^\alpha)}{b^\alpha - a^\alpha}} \cdot (f(c))^{-f^{\beta-1}(c)} p(c) f^{\beta-1}(a) \left(\frac{f(a)}{f(b)} \right)^{\frac{(1-\beta)(c^\alpha - a^\alpha)}{b^\alpha - a^\alpha}} \\ & \quad \cdot \left(\frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{c^\alpha - a^\alpha}{b^\alpha - a^\alpha}} \right)^{-\frac{d^{\beta-1} f}{dt^\alpha}(c)} \\ &= \left(\frac{1}{f(a)} \left(\frac{f(a)}{f(b)} \right)^{\frac{c^\alpha - a^\alpha}{b^\alpha - a^\alpha}} \right)^{\frac{1-\beta}{b^\alpha - a^\alpha} f^{\beta-1}(c) \ln \left(\frac{f(a)}{f(b)} \right) - \frac{d^{\beta-1} f}{dt^\alpha}(c)} \cdot (f(c))^{\frac{1-\beta}{b^\alpha - a^\alpha} f^{\beta-1}(c) \ln \left(\frac{f(a)}{f(b)} \right)}. \end{aligned}$$

□

3. FRACTAL MULTIPLICATIVE INTEGRAL

In this section, we introduce the concept of multiplicative integrals within the framework of fractal calculus. Unlike the classical additive integral, the multiplicative integral is designed to capture the cumulative effect of relative changes, making it particularly suitable for the analysis of processes governed by scaling and nonlocal behaviors.

Motivated by the (α, β) -fractal derivatives, we develop a corresponding notion of integration that is consistent with the underlying fractal structure. This approach

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allows us to extend fundamental results of classical calculus to a multiplicative-fractal setting, providing new tools for the study of functions exhibiting nonlinear growth and self-similar properties.

In [2], the integral operators

$$L_{t_0}^\alpha(f)(t) = \int_{t_0}^t \alpha |s|^{\alpha-1} f(s) ds \tag{14}$$

and

$$K_{t_0}^{\alpha,\beta}(f)(t) = (L_{t_0}^\alpha(f)(t))^{\frac{1}{\beta}} \tag{15}$$

were introduced and studied as inverse operators of the fractal derivative (3), corresponding to the cases $\beta = 1$ and $\beta \neq 1$, respectively. These operators provide a natural extension of the classical integral in the fractal setting and play a fundamental role in establishing the connection between differentiation and integration within this framework. In this work, we also derive integral operators, analogous to (14) and (15), that act as inverses of our fractal multiplicative derivative (4). These operators will provide a consistent multiplicative counterpart to the inverse relationship between differentiation and integration.

We now introduce our fractal multiplicative integral.

Definition 3.1 (Fractal multiplicative integral). Let $f : [a, b] \rightarrow (0, \infty)$ be a positive function. The (α, β) -*integral of f on $[a, b]$ is defined by

$$\int_a^b f(t) d_{\alpha}^{\beta} t := \begin{cases} \exp \left(\alpha \int_a^b |t|^{\alpha-1} \ln f(t) dt \right), & \beta = 1, \\ \exp \left[\frac{1}{\beta-1} W \left((\beta-1) \alpha \int_a^b |t|^{\alpha-1} \ln f(t) dt \right) \right], & \beta \neq 1, \end{cases}$$

provided that the integral exists. Here, W denotes the Lambert W function, also called the omega function or product logarithm. If the (α, β) -*integral of f on $[a, b]$ exists, we say that the function f is (α, β) -*integrable on $[a, b]$.

Theorem 3.2 (Fundamental Theorem of Fractal Multiplicative Calculus). *The following statements hold:*

- i) Let $f : [a, b] \rightarrow \mathbb{R}^+$ be (α, β) -*differentiable and let $\frac{*d^{\beta} f}{dt^{\alpha}}$ be (α, β) -*integrable. Then,

$$\int_a^b \left(\frac{*d^{\beta} f}{dt^{\alpha}}(t) \right) d_{\alpha}^{\beta} t = \begin{cases} \frac{f(b)}{f(a)}, & \beta = 1, \\ \exp \left[\frac{1}{\beta-1} W \left((\beta-1) (f^{\beta-1}(b) \ln f(b) - f^{\beta-1}(a) \ln f(a)) \right) \right], & \beta \neq 1. \end{cases}$$

- ii) Let $f : [a, b] \rightarrow \mathbb{R}^+$ be (α, β) -*integrable and let

$$F(t) = \int_a^t f(s) d_{\alpha}^{\beta} s, \quad a \leq t \leq b.$$

If $f(t)$ is continuous at $c \in [a, b]$, then F is (α, β) -*differentiable at c and

$$\frac{*d^{\beta} F}{dt^{\alpha}}(c) = f(c).$$

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Proof. i) For $\beta = 1$, we obtain that

$$\int_a^b \left(\frac{{}^*df}{dt^\alpha}(t) \right)^{d_\alpha t} = \exp \left(\alpha \int_a^b |t|^{\alpha-1} \ln \left(\frac{{}^*df(t)}{dt^\alpha} \right) dt \right).$$

Since f is a positive $(\alpha, 1)$ -*differentiable function on $[a, b]$ and $\frac{{}^*d^\beta f}{dt^\alpha}$ is $(\alpha, 1)$ -*integrable on the same interval, it follows that the function

$$t \mapsto |t|^{\alpha-1} \ln \left(\frac{{}^*df(t)}{dt^\alpha} \right)$$

is integrable on $[a, b]$. By the definition of the fractal multiplicative derivative, we have

$$\alpha |t|^{\alpha-1} \ln \left(\frac{{}^*df(t)}{dt^\alpha} \right) = \frac{d}{dt}(\ln f(t))$$

in the weak sense that the right-hand side is the density associated with the logarithmic variation of f . Integrating over $[a, b]$, we obtain

$$\alpha \int_a^b |t|^{\alpha-1} \ln \left(\frac{{}^*df(t)}{dt^\alpha} \right) dt = \ln f(b) - \ln f(a).$$

Hence,

$$\int_a^b \left(\frac{{}^*df}{dt^\alpha}(t) \right)^{d_\alpha t} = \frac{f(b)}{f(a)}.$$

For $\beta \neq 1$, we have that

$$\int_a^b \left(\frac{{}^*d^\beta f}{dt^\alpha}(t) \right)^{d_\alpha t} = \exp \left[\frac{1}{\beta-1} W \left((\beta-1)\alpha \int_a^b t^{\alpha-1} \ln \left(\frac{{}^*d^\beta f}{dt^\alpha} \right) dt \right) \right].$$

By definition, we obtain

$$\alpha |t|^{\alpha-1} \ln \left(\frac{{}^*d^\beta f(t)}{dt^\alpha} \right) = \frac{d}{dt}(f^{\beta-1}(t) \ln f(t)).$$

Thus,

$$\int_a^b \left(\frac{{}^*d^\beta f}{dt^\alpha}(t) \right)^{d_\alpha t} = \exp \left[\frac{1}{\beta-1} W((\beta-1)(f^{\beta-1}(b) \ln f(b) - f^{\beta-1}(a) \ln f(a))) \right].$$

ii) For $\beta = 1$, we have that

$$\begin{aligned} \frac{{}^*d \int_a^t f(s)^{d_\alpha s}}{dt^\alpha}(c) &= \lim_{t \rightarrow c} \exp \left(\frac{\alpha \int_a^t |s|^{\alpha-1} \ln f(s) ds - \alpha \int_a^c |s|^{\alpha-1} \ln f(s) ds}{t^\alpha - c^\alpha} \right) \\ &= f(c). \end{aligned}$$

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On the other hand, if $\beta \neq 1$ then

$$\begin{aligned} & \frac{{}^*d^\beta \int_a^t f(s) d_\alpha^\beta s}{dt^\alpha}(c) \\ &= \lim_{t \rightarrow c} \left[\frac{\exp\left(\frac{1}{\beta-1} W\left((\beta-1)\alpha \int_a^t |s|^{\alpha-1} \ln f(s) ds\right)\right) \exp\left(W\left((\beta-1)\alpha \int_a^t |s|^{\alpha-1} \ln f(s) ds\right)\right)}{\exp\left(\frac{1}{\beta-1} W\left((\beta-1)\alpha \int_a^c |s|^{\alpha-1} \ln f(s) ds\right)\right) \exp\left(W\left((\beta-1)\alpha \int_a^c |s|^{\alpha-1} \ln f(s) ds\right)\right)} \right]^{\frac{1}{t^\alpha - c^\alpha}} \\ &= \lim_{t \rightarrow c} \exp\left(\frac{\alpha \int_a^t |s|^{\alpha-1} \ln f(s) ds - \alpha \int_a^c |s|^{\alpha-1} \ln f(s) ds}{t^\alpha - c^\alpha}\right) \\ &= f(c), \end{aligned}$$

and we are done. □

Remark 3.3. Let $f : [a, b] \rightarrow \mathbb{R}^+$ be (α, β) -*differentiable and let $\frac{{}^*d^\beta f}{dt^\alpha}$ be (α, β) -*integrable. Set

$$F(t) = \int_a^t \left(\frac{{}^*d^\beta f(s)}{ds^\alpha} \right) d_\alpha^\beta s, \quad t \in (a, b).$$

Using Theorem 3.2 i), note that

$$F(t) = \begin{cases} \frac{f(t)}{f(a)}, & \beta = 1, \\ \exp\left[\frac{1}{\beta-1} W\left((\beta-1)(f^{\beta-1}(t) \ln f(t) - f^{\beta-1}(a) \ln f(a))\right)\right], & \beta \neq 1. \end{cases}$$

If $\frac{{}^*d^\beta f(t)}{dt^\alpha}$ is continuous at $c \in [a, b]$, then

$$\frac{{}^*d^\beta F}{dt^\alpha}(c) = \frac{{}^*d^\beta f}{dt^\alpha}(c).$$

Now, using Proposition 2.6, it can be verified that

- $\beta = 1$:

$$\frac{{}^*dF}{dt^\alpha}(c) = \frac{{}^*d\left(\frac{f(t)}{f(a)}\right)}{dt^\alpha}(c) = \frac{{}^*df}{dt^\alpha}(c),$$

- $\beta \neq 1$:

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$$\begin{aligned}
 & \frac{{}^*d^\beta F}{dt^\alpha}(c) \\
 &= \frac{{}^*d^\beta \left(\exp \left(\frac{1}{\beta-1} W((\beta-1)(f^{\beta-1}(t) \ln f(t) - f^{\beta-1}(a) \ln f(a))) \right) \right)}{dt^\alpha}(c) \\
 &= \lim_{t \rightarrow c} \exp \left[\frac{1}{\beta-1} \cdot \left(\frac{\exp \left(W((\beta-1)(f^{\beta-1}(t) \ln f(t) - f^{\beta-1}(a) \ln f(a))) \right)}{t^\alpha - c^\alpha} \right. \right. \\
 & \quad \left. \left. - \frac{\exp \left(W((\beta-1)(f^{\beta-1}(c) \ln f(c) - f^{\beta-1}(a) \ln f(a))) \right)}{t^\alpha - c^\alpha} \right) \right] \\
 &= \lim_{t \rightarrow c} \exp \left[\frac{f^{\beta-1}(t) \ln f(t) - f^{\beta-1}(a) \ln f(a) - f^{\beta-1}(c) \ln f(c) + f^{\beta-1}(a) \ln f(a)}{t^\alpha - c^\alpha} \right] \\
 &= \frac{{}^*df}{dt^\alpha}(c).
 \end{aligned}$$

The following proposition collects some basic properties of the fractal multiplicative integral.

Proposition 3.4. *The following rules hold:*

- i) $\int_a^b f(t) d_{\alpha+1}^\beta t = \int_a^b (f(t))^{\frac{\alpha+1}{\alpha}} |t| d_{\alpha}^\beta t,$
- ii) $\int_a^b (f(t)g(t)) d_{\alpha}^\beta t = \int_a^b f(t) d_{\alpha}^\beta t \cdot \int_a^b g(t) d_{\alpha}^\beta t,$
- iii) $\int_a^b f(t) d_{\alpha}^\beta t = \int_a^c f(t) d_{\alpha}^\beta t \cdot \int_c^b f(t) d_{\alpha}^\beta t,$
- iv) $\int_a^a f(t) d_{\alpha}^\beta t = 1,$
- v) $\int_a^b (cf(t)) d_{\alpha}^\beta t = c^{b^\alpha - a^\alpha} \int_a^b f(t) d_{\alpha}^\beta t, \quad c \in \mathbb{R}^+,$
- vi) $\int_{ca}^{cb} f(t) d_{\alpha}^\beta t = \int_a^b (f(ct)) c^\alpha d_{\alpha}^\beta t.$

4. APPLICATIONS

4.1. Fractal differential equations. The development of fractal multiplicative derivatives and integrals provides an effective framework for the analysis and solution of a broad class of fractal differential equations. These operators extend the classical concepts of differentiation and integration, allowing the treatment of problems where scaling behavior and nonlocal effects play a fundamental role. In particular, they offer a natural approach to solving equations in which the unknown function exhibits multiplicative or fractal-type dynamics.

In this context, fractal multiplicative calculus becomes a powerful tool for constructing explicit solutions and studying qualitative properties of such equations. To illustrate its applicability, we present below two representative examples of fractal differential equations that can be successfully solved using this approach.

Example 4.1. Consider the fractal differential equation

$$\frac{d^\alpha f(t)}{dt^\alpha} = \alpha g(t) f^\alpha(t), \quad t > 0, \quad f(0) = t_0 > 0, \tag{16}$$

where g is a prescribed continuous function.

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It is easy to see that

$$\frac{d \ln f(t)}{dt^\alpha} = \frac{d \ln z}{dz^\alpha} (f(t)) \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\alpha} |f(t)|^{1-\alpha} \frac{1}{f(t)} \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\alpha f^\alpha(t)} \frac{d^\alpha f(t)}{dt^\alpha}.$$

We can rewrite (16) in the form

$$\frac{d \ln f(t)}{dt^\alpha} = g(t),$$

hence

$$\exp \left[\frac{d \ln f(t)}{dt^\alpha} \right] = \exp[g(t)].$$

Therefore,

$${}^*df(t) = \exp[g(t)], \quad t > 0, \quad f(0) = t_0 > 0, \tag{17}$$

and we can express the solution of the above differential equation in terms of the fractal multiplicative integral as

$$f(t) = t_0 \int_0^t \exp[g(z)]^{d_\alpha z} = t_0 \exp \left(\alpha \int_0^t z^{\alpha-1} g(z) dz \right), \quad t \geq 0. \tag{18}$$

Example 4.2. Let $\beta > 1$. Consider the nonlinear fractal differential equation

$$(\ln f(t)) \frac{d^{\beta-1} f(t)}{dt^\alpha} + \frac{1}{\alpha} f^{\beta-\alpha-1}(t) \frac{d^\alpha f(t)}{dt^\alpha} = g(t), \quad t > 0, \quad f(0) = t_0 > 0, \tag{19}$$

where g is a prescribed continuous function.

Applying the Leibniz rule for fractal derivatives (see [2, Proposition 2.8, p. 10728]), we have

$$\begin{aligned} \frac{d^{\beta-1} f(t) (\ln f(t))^{\frac{1}{\beta-1}}}{dt^\alpha} &= (\ln f(t)) \frac{d^{\beta-1} f(t)}{dt^\alpha} + f^{\beta-1}(t) \frac{d^{\beta-1} (\ln f(t))^{\frac{1}{\beta-1}}}{dt^\alpha} \\ &= (\ln f(t)) \frac{d^{\beta-1} f(t)}{dt^\alpha} + f^{\beta-1}(t) \frac{d \ln f(t)}{dt^\alpha} \\ &= (\ln f(t)) \frac{d^{\beta-1} f(t)}{dt^\alpha} + \frac{1}{\alpha} f^{\beta-\alpha-1}(t) \frac{d^\alpha f(t)}{dt^\alpha}. \end{aligned}$$

Thus, equation (19) reduces to

$$\frac{d^{\beta-1} f(t) (\ln f(t))^{\frac{1}{\beta-1}}}{dt^\alpha} = g(t), \quad t > 0, \quad f(0) = t_0 > 0. \tag{20}$$

Then,

$$\frac{{}^*d^\beta f(t)}{dt^\alpha} = \exp[g(t)], \tag{21}$$

and, whence,

$$f(t) = t_0 \int_0^t \exp[g(z)]^{d_\alpha z} = t_0 \exp \left[\frac{1}{\beta-1} W \left((\beta-1)\alpha \int_0^t z^{\alpha-1} g(z) dz \right) \right], \quad t \geq 0. \tag{22}$$

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4.2. Fractal multiplicative differential equations. Fractal multiplicative differential equations constitute a novel class of differential equations introduced in this work through the concept of fractal multiplicative differentiation. This new framework extends classical and multiplicative calculus by incorporating fractal structures into the notion of relative change, thereby providing a more general setting for the analysis of dynamic processes.

Unlike classical differential equations, which rely on additive rates of variation, the proposed equations are governed by multiplicative mechanisms that naturally encode exponential growth, scaling effects, and nonlinear interactions. The presence of fractal parameters further enriches the model, allowing for the description of complex systems exhibiting irregular, non-smooth, or scale-dependent behavior.

This formulation not only generalizes existing approaches but also reveals new qualitative features in the dynamics of solutions. In particular, fractal multiplicative differential equations offer a unified perspective that bridges multiplicative calculus and fractal analysis, opening new avenues for both theoretical developments and applications in complex systems.

Example 4.3. Consider the fractal multiplicative differential equation

$$\frac{{}^*d^\beta f(t)}{dt^\alpha} = t, \quad t > 0, \quad f(0) = 1. \tag{23}$$

From Theorem 3.2, it follows that

$$f(t) = \int_0^t z^{d_\alpha^\beta z} = \begin{cases} t^{t^\alpha} \exp\left(-\frac{t^2}{\alpha}\right), & \beta = 1, \\ \exp\left[\frac{1}{\beta-1} W\left((\beta-1)t^\alpha \left(\ln t - \frac{1}{\alpha}\right)\right)\right], & \beta \neq 1. \end{cases}$$

Example 4.4. Consider the equation

$$\frac{{}^*d^\beta f(t)}{dt^\alpha} = f(t), \quad t > 0. \tag{24}$$

The analysis is divided into two cases: $\beta = 1$ and $\beta \neq 1$.

- $\beta = 1$:

$$\frac{{}^*df(t)}{dt^\alpha} = f(t), \quad t > 0, \quad f(0) = 1. \tag{25}$$

Using Proposition 2.1, we obtain

$$\exp\left[\frac{1}{\alpha} \cdot f^{-\alpha}(t_0) \cdot \frac{d^\alpha f(t)}{dt^\alpha}\right] = f(t), \tag{26}$$

or, equivalently,

$$\frac{1}{\alpha} \cdot f^{-\alpha} \cdot \frac{d^\alpha f(t)}{dt^\alpha} = \ln f(t). \tag{27}$$

By [2, Proposition 2.2, p. 10727], we have

$$\frac{1}{\alpha} t^{1-\alpha} \frac{df(t)}{dt} = f(t) \ln f(t), \tag{28}$$

whose general solution is given by

$$f(t) = \exp(C \exp(t^\alpha)). \tag{29}$$

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- $\beta \neq 1$:

By means of Proposition 2.2, equation (24) can be rewritten in the form

$$f^{\frac{d^{\beta-1}f}{dt^\alpha}}(t) \cdot \exp \left[\frac{1}{\beta-1} \cdot \frac{d^{\beta-1}f(t)}{dt^\alpha} \right] = f(t) \tag{30}$$

or, equivalently,

$$\exp \left[\frac{d^{\beta-1}f(t)}{dt^\alpha} \ln f(t) + \frac{1}{\beta-1} \frac{d^{\beta-1}f(t)}{dt^\alpha} \right] = \exp [\ln f(t)]. \tag{31}$$

Therefore,

$$\frac{d^{\beta-1}f(t)}{dt^\alpha} \ln f(t) + \frac{1}{\beta-1} \frac{d^{\beta-1}f(t)}{dt^\alpha} = \ln f(t). \tag{32}$$

Using [2, Proposition 2.2, p. 10727], we have

$$\frac{\beta-1}{\alpha} \left(\ln f(t) + \frac{1}{\beta-1} \right) t^{1-\alpha} f^{\beta-2}(t) \frac{df(t)}{dt} = \ln f(t). \tag{33}$$

The general solution of (33) is

$$f(t) = F^{-1}(t^\alpha + C), \tag{34}$$

where

$$F(x) = \text{Ei}((\beta-1) \ln x) + x^{\beta-1}.$$

Here, Ei denotes the exponential integral function, defined by

$$\text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt.$$

It is easy to verify that

$$\frac{d}{dt} F^{-1}(t^\alpha + C) = \frac{\alpha t^{\alpha-1}}{(F^{-1}(t^\alpha + C))^{b-2} \left(\frac{1}{\ln(F^{-1}(t^\alpha + C))} + b - 1 \right)}$$

and hence, (34) satisfies (33).

4.3. Multiplicative spaces. In recent years, there has been increasing interest in the use of nonclassical time scales in the modeling of complex systems, particularly within the frameworks of fractal calculus and fractional dynamics. One of the most natural and widely used modifications consists in replacing the standard time variable t with a nonlinear scaling of the form t^α , where $\alpha \in (0, 1]$. This transformation reflects the presence of irregular, heterogeneous, or fractal temporal structures that cannot be adequately described by the classical linear flow of time.

The introduction of t^α is motivated by the need to capture memory effects, anomalous diffusion, and scaling behaviors observed in many physical, biological, and economic systems. In fractal media, time evolution may occur on sets with non-integer dimension, leading to a distortion of the usual temporal metric. The power-law transformation $t \mapsto t^\alpha$ provides a simple yet effective way to encode such fractal characteristics, allowing the formulation of differential and integral operators adapted to these contexts.

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Moreover, the use of t^α is closely related to the concept of self-similarity, which is a fundamental property of fractal structures. This scaling preserves the invariance of the system under appropriate dilations and leads naturally to generalized notions of differentiation and integration, such as fractal derivatives and multiplicative operators. Therefore, the introduction of t^α is not merely a formal modification, but a meaningful extension of classical calculus that enables the analysis of dynamical processes evolving in fractal time and complex geometrical settings.

The concepts of the $*$ derivative and $*$ integral are based on the classical notion of limit. In an analogous way, one can introduce the concept of a multiplicative limit, or $*$ limit.

Let $t \in \mathbb{R}^+$. The multiplicative absolute value of t , denoted by $*|t|_\alpha$, is defined by

$$*|t|_\alpha = \begin{cases} t^\alpha, & \text{if } t \geq 1, \\ \frac{1}{t^\alpha}, & \text{if } 0 < t < 1. \end{cases}$$

Using this notion, we define the multiplicative distance between $x, y \in \mathbb{R}^+$ as

$$*d(x, y) = *|x/y|_\alpha.$$

The multiplicative distance satisfies the following properties:

- i) For all $x, y \in \mathbb{R}^+$, $*d(x, y) \geq 1$.
- ii) $*d(x, y) = 1$ if and only if $x = y$.
- iii) For all $x, y \in \mathbb{R}^+$, $*d(x, y) = *d(y, x)$.
- iv) ($*$ triangle inequality) For all $x, y, z \in \mathbb{R}^+$,

$$*d(x, z) \leq *d(x, y) *d(y, z).$$

Based on these properties, one can define multiplicative metric spaces as an alternative to classical metric spaces. In particular, \mathbb{R}^+ is a multiplicative metric space. A sequence $\{x_n\} \subset \mathbb{R}^+$ is said to converge multiplicatively to $x \in \mathbb{R}^+$ if, for every $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that

$$*d(x_n, x) < \varepsilon \quad \text{for all } n \geq N.$$

In fact, convergence in \mathbb{R}^+ in the multiplicative sense is equivalent to convergence in the usual sense, although this equivalence may fail in more general settings.

Another example of a multiplicative metric space can be constructed from the set M_n^+ of positive $(n \times n)$ -matrices. A matrix A is said to be positive if $x^T A x > 0$ for every nonzero vector $x \in \mathbb{R}^n$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of a positive matrix A , then $\lambda_i > 0$ for all i . The multiplicative norm of A is defined by

$$*||A||_\alpha = \prod_{i=1}^n *|\lambda_i|_\alpha,$$

and the multiplicative distance between two matrices $A, B \in M_n^+$ is given by

$$*d(A, B) = *||AB^{-1}||_\alpha.$$

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4.4. Fractal multiplicative calculus of variations. Consider the problem of minimizing the functional

$$J(y) = \int_a^b f \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right)^{d_\alpha^\beta t}, \tag{35}$$

over all continuously differentiable functions $y(t)$ on $[a, b]$ satisfying the fixed boundary conditions $y(a) = y_1$ and $y(b) = y_2$.

By Definition 3.1, this problem is equivalent to minimizing the functional

$$J_0(y) = \ln J(y) = \begin{cases} \alpha \int_a^b |t|^{\alpha-1} \ln f \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right) dt, & \beta = 1, \\ \frac{1}{\beta-1} W \left((\beta-1)\alpha \int_a^b |t|^{\alpha-1} \ln f \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right) dt \right), & \beta \neq 1, \end{cases}$$

for which the classical methods of the calculus of variations are directly applicable.

However, our aim is to develop and employ a purely multiplicative framework. To this end, we begin by introducing a multiplicative analogue of the fundamental lemma of the calculus of variations.

Lemma 4.5. *Let $f : [a, b] \rightarrow \mathbb{R}^+$ be a positive continuous function such that*

$$\int_a^b (f(t)^{h(t)})^{d_\alpha^\beta t} = 1$$

for every infinitely differentiable function $h : [a, b] \rightarrow \mathbb{R}$. Then $f(t) \equiv 1$ for all $t \in [a, b]$.

Proof. From

$$1 = \int_a^b (f(t)^{h(t)})^{d_\alpha^\beta t} = \begin{cases} \exp \left(\alpha \int_a^b |t|^{\alpha-1} h(t) \ln f(t) dt \right), & \beta = 1, \\ \exp \left[\frac{1}{\beta-1} W \left((\beta-1)\alpha \int_a^b |t|^{\alpha-1} h(t) \ln f(t) dt \right) \right], & \beta \neq 1, \end{cases}$$

we deduce in both cases that

$$\int_a^b |t|^{\alpha-1} h(t) \ln f(t) dt = 0.$$

By the fundamental lemma of the calculus of variations, it follows that

$$\ln f(t) = 0 \quad \text{for all } t \in [a, b],$$

and hence

$$f(t) \equiv 1 \quad \text{on } [a, b].$$

□

Returning to the functional (35), assume that $f \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right)$ possesses continuous second-order partial (α, β) -*derivatives with respect to both variables y and $\frac{d^\beta y}{dt^\alpha}$.

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Let $h : [a, b] \rightarrow \mathbb{R}$ be an arbitrary infinitely differentiable function such that $h(a) = h(b) = 0$, and let $\varepsilon \in \mathbb{R}$. In particular, h is (α, β) -fractal differentiable. Then we consider

$$\frac{J(y + \varepsilon h)}{J(y)} = \frac{\int_a^b f(y(t) + \varepsilon h(t), \frac{d^\beta y(t)}{dt^\alpha} + \varepsilon \frac{d^\beta h(t)}{dt^\alpha}) d_\alpha^\beta t}{\int_a^b f(y(t), \frac{d^\beta y(t)}{dt^\alpha}) d_\alpha^\beta t}.$$

Using the multiplicative structure, this can be written as

$$\frac{J(y + \varepsilon h)}{J(y)} = \int_a^b \left(\frac{f(y(t) + \varepsilon h(t), \frac{d^\beta y(t)}{dt^\alpha} + \varepsilon \frac{d^\beta h(t)}{dt^\alpha})}{f(y(t), \frac{d^\beta y(t)}{dt^\alpha})} \right)^{d_\alpha^\beta t}.$$

We obtain

$$\frac{J(y + \varepsilon h)}{J(y)} = {}^*o(\varepsilon) \left[\int_a^b \left(\left(\frac{{}^*d^\beta f}{dy^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right) \right)^{h(t)} \left(\frac{{}^*d^\beta f}{d \left(\frac{d^\beta y}{dt^\alpha} \right)^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right) \right)^{\frac{d^\beta h(t)}{dt^\alpha}} \right)^{d_\alpha^\beta t} \right]^\varepsilon,$$

where ${}^*o(\varepsilon)^{1/\varepsilon} \rightarrow 1$ as $\varepsilon \rightarrow 0$. Raising both sides to the power $1/\varepsilon$ and letting $\varepsilon \rightarrow 0^\pm$, we obtain

$$\int_a^b \left(\left(\frac{{}^*d^\beta f}{dy^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right) \right)^{h(t)} \left(\frac{{}^*d^\beta f}{d \left(\frac{d^\beta y}{dt^\alpha} \right)^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right) \right)^{\frac{d^\beta h(t)}{dt^\alpha}} \right)^{d_\alpha^\beta t} = 1.$$

Integrating by parts and suitably applying the properties of the fractal multiplicative derivative, we arrive at

$$\int_a^b \left(\left(\frac{\frac{{}^*d^\beta f}{dy^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right)}{C(\alpha, \beta, a, b) \frac{{}^*d^\beta {}^*d^\beta f}{dt^\alpha d \left(\frac{d^\beta y}{dt^\alpha} \right)^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right)} \right)^{h(t)} \right)^{d_\alpha^\beta t} = 1,$$

where $C(\alpha, \beta, a, b)$ is a constant that depends on α, β, a , and b , and satisfies $C(\alpha, 1, a, b) = 1$. By Lemma 1, we obtain

$$\frac{{}^*d^\beta f}{dy^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right) = C(\alpha, \beta, a, b) \frac{{}^*d^\beta {}^*d^\beta f}{dt^\alpha d \left(\frac{d^\beta y}{dt^\alpha} \right)^\alpha} \left(y(t), \frac{d^\beta y(t)}{dt^\alpha} \right),$$

which may be viewed as a fractal multiplicative analogue of the Euler-Lagrange equation. The procedure followed in this subsection is similar to that carried out in [5] by Bashirov *et al.*

It is worth noting that further generalizations can be developed within the framework of non-Newtonian calculus. Indeed, by selecting different generators, one can construct alternative arithmetic structures and corresponding notions of differentiation and integration, leading to a wide variety of generalized calculi. These approaches allow the extension of classical results to settings where the standard

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additive structure is replaced by more suitable operations adapted to the nature of the problem. Consequently, non-Newtonian calculus provides a flexible and unifying perspective for formulating and analyzing generalized models, opening the door to new theoretical developments and potential applications. In this work, we have employed the notions of fractal multiplicative differentiation and integration to relate them with the concept of (α, β) -differentiability introduced in [2].

4.5. Gompertz growth. In this subsection, we examine the formulation of the Gompertz growth model within the framework of fractal multiplicative calculus. This model plays a fundamental role in describing growth processes across diverse areas such as biology, population dynamics, and epidemiology. Traditionally expressed in the setting of classical calculus, it captures nonlinear growth behavior characterized by saturation effects and a decreasing growth rate.

By employing fractal multiplicative differentiation and integration, we extend this classical model to a more general setting that naturally incorporates scaling properties and multiplicative dynamics. This approach provides a refined description of growth phenomena, particularly in situations where relative changes and fractal-like structures are more appropriate than additive ones.

The resulting formulation not only generalizes the standard Gompertz equation but also provides new insights into its qualitative behavior. In particular, we show how the interplay between fractal parameters and multiplicative operators leads to modified growth laws that may be better suited for modeling complex systems with non-uniform or anomalous dynamics.

Throughout this subsection, $y(t)$ denotes the population size at time t , while the parameters a and K represent, respectively, the intrinsic growth rate and the carrying capacity of the system.

Definition 4.6. Let $a > 0$ and $K > 1$. The fractal multiplicative Gompertz model is given by the following equation:

$${}^*d^\beta y(t) = \left(1 + \ln\left(\frac{\ln K}{y(t)}\right)\right)^{ay(t)}. \tag{36}$$

In the case $\beta = 1$, Proposition 2.1 implies that (36) can be rewritten in the form

$$\exp\left[\frac{1}{\alpha} y^{-\alpha}(t) \frac{d^\alpha y(t)}{dt^\alpha}\right] = \left(1 + \ln\left(\frac{\ln K}{y(t)}\right)\right)^{ay(t)},$$

provided that the solution $y(t)$ is (α, α) -fractal differentiable. Therefore,

$$\frac{1}{\alpha} \cdot y^{-\alpha}(t) \cdot \frac{d^\alpha y(t)}{dt^\alpha} = ay(t) \ln\left(1 + \ln\left(\frac{\ln K}{y(t)}\right)\right). \tag{37}$$

Assume, in addition, that $y(t)$ is differentiable in the usual sense. Using [2, Proposition 2.2, p. 10727], we have

$$\frac{1}{\alpha} \cdot t^{1-\alpha} \cdot \frac{dy(t)}{dt} = ay^2(t) \ln\left(1 + \ln\left(\frac{\ln K}{y(t)}\right)\right). \tag{38}$$

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Equation (38) is nonlinear and may be solved numerically by means of various methods available in the literature. In particular, iterative schemes such as finite difference methods or Runge–Kutta techniques can be employed to approximate the solution. The choice of the numerical method depends on the qualitative behavior of the solution and the stability requirements of the problem.

In the case $\beta \neq 1$, Proposition 2.2 gives

$${}^*d^\beta y(t) = y^{\frac{d^{\beta-1}y}{dt^\alpha}}(t) \cdot \exp \left[\frac{1}{\beta-1} \cdot \frac{d^{\beta-1}y(t)}{dt^\alpha} \right].$$

If we want to find an $(\alpha, \beta - 1)$ -fractal differentiable solution, then equation (36) can be written as

$$y^{\frac{d^{\beta-1}y}{dt^\alpha}}(t) \cdot \exp \left[\frac{1}{\beta-1} \cdot \frac{d^{\beta-1}y(t)}{dt^\alpha} \right] = \left(1 + \ln \left(\frac{\ln K}{y(t)} \right) \right)^{ay(t)}, \tag{39}$$

or, equivalently,

$$\exp \left[\frac{d^{\beta-1}y(t)}{dt^\alpha} \ln y(t) + \frac{1}{\beta-1} \frac{d^{\beta-1}y(t)}{dt^\alpha} \right] = \exp \left[ay(t) \ln \left(1 + \ln \left(\frac{\ln K}{y(t)} \right) \right) \right]. \tag{40}$$

Therefore,

$$\frac{d^{\beta-1}y(t)}{dt^\alpha} \left(\ln y(t) + \frac{1}{\beta-1} \right) = ay(t) \ln \left(1 + \ln \left(\frac{\ln K}{y(t)} \right) \right). \tag{41}$$

Assuming differentiability, it follows from [2, Proposition 2.2, p. 10727] that, for $t > 0$,

$$\frac{d^{\beta-1}y(t)}{dt^\alpha} = \frac{\beta-1}{\alpha} t^{1-\alpha} y^{\beta-2}(t) \frac{dy(t)}{dt}.$$

Hence,

$$\frac{\beta-1}{\alpha} t^{1-\alpha} y^{\beta-2}(t) \frac{dy(t)}{dt} \left(\ln y(t) + \frac{1}{\beta-1} \right) = ay(t) \ln \left(1 + \ln \left(\frac{\ln K}{y(t)} \right) \right). \tag{42}$$

Equation (42) is also nonlinear and must likewise be solved numerically using one of the methods mentioned above.

In [16], the authors introduced a generalized conformable fractional derivative G_T^α defined by

$$(G_T^\alpha f)(t) = \lim_{h \rightarrow 0} \frac{f(t) - f(t - he^{(\alpha-1)t})}{h},$$

and applied it to the analysis of a Gompertz growth model. Furthermore, they considered a real data set on tuberculosis in Mexico to address an inverse problem aimed at estimating the order of the proposed fractional derivative. Their approach also included a comparison with the classical derivative and other fractional operators, such as the Khalil and Caputo derivatives. Remarkably, the results showed that the proposed conformable fractional model provides a significantly improved fit, yielding smaller errors in the estimation of the parameters associated with the Gompertz model. This highlights the effectiveness of the generalized operator in capturing the underlying dynamics of the data.

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In what follows, we show that a particular fractal multiplicative Gompertz model provides the best fit to the data set studied in [16]. For this study we used the same data of the percentage of people with tuberculosis in Mexico between the years 1990 and 2015 [19]. Figure 1 shows the data associated with the percentage of people with tuberculosis in Mexico (black asterisk) and the corresponding fits to the Gompertz model

$$\frac{dy(t)}{dt} = 0.2008 \cdot y(t) \ln \left(\frac{1}{y(t)} \right), \tag{43}$$

and logistic model

$$\frac{dy(t)}{dt} = 0.2008 \cdot y(t) (1 - y(t)), \tag{44}$$

both with initial condition $y(0) = 0.335$. Using MATLAB, we developed a script to compute numerical approximations of the solutions to fractal multiplicative Gompertz models (see Appendix). For its implementation, we employed the efficient `ode45` solver, which is based on an explicit Runge–Kutta method, namely the Dormand–Prince pair. We arrive that the following fractal multiplicative Gompertz model:

$$\begin{aligned} \frac{{}^*d^{0.401}y(t)}{dt^{1.101}} &= \left(1 + \ln \left(\frac{\ln 2.65}{y(t)} \right) \right)^{0.42 \cdot y(t)}, \\ y(0) &= 0.335, \end{aligned} \tag{45}$$

best fits the data of the percentage of people with tuberculosis in Mexico with respect to the previous models (see Figure 2). Table 1 reports the fitting errors of the models under consideration, namely the Mean Absolute Error (MAE) and the Mean Squared Error (MSE). It is worth noting that the errors associated with the generalized Gompertz model proposed in [16] are relatively larger than those obtained with our model (45). In particular, the MAE and MSE for the generalized Gompertz model proposed in [16] are 0.0416 and 0.0027, respectively.

Models	Fitting errors	
	MAE	MSE
Gompertz growth (43)	0.0495	0.0047
Logistic growth (44)	0.0614	0.0057
Fractal Multiplicative Gompertz model (45)	0.0310	0.0019

TABLE 1. Fitting errors for the studied models.

5. ADDITIONAL EXAMPLES AND GRAPHICS

The use of examples such as the Weierstrass function and the Gompertz model is very appropriate. However, we present some cases that illustrate the strength and breadth of our results.

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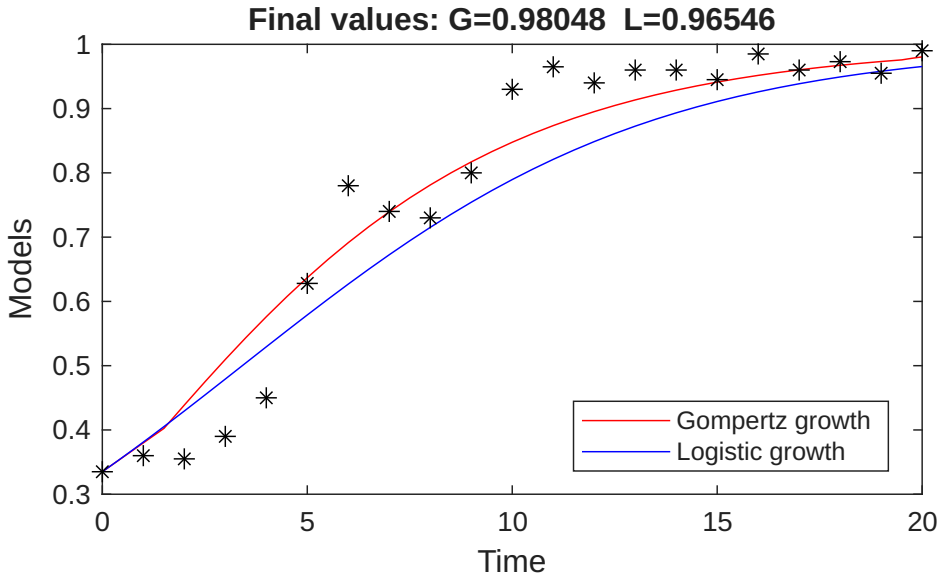


FIGURE 1. Data and estimates of tuberculosis infectious using Gompertz and logistic models.

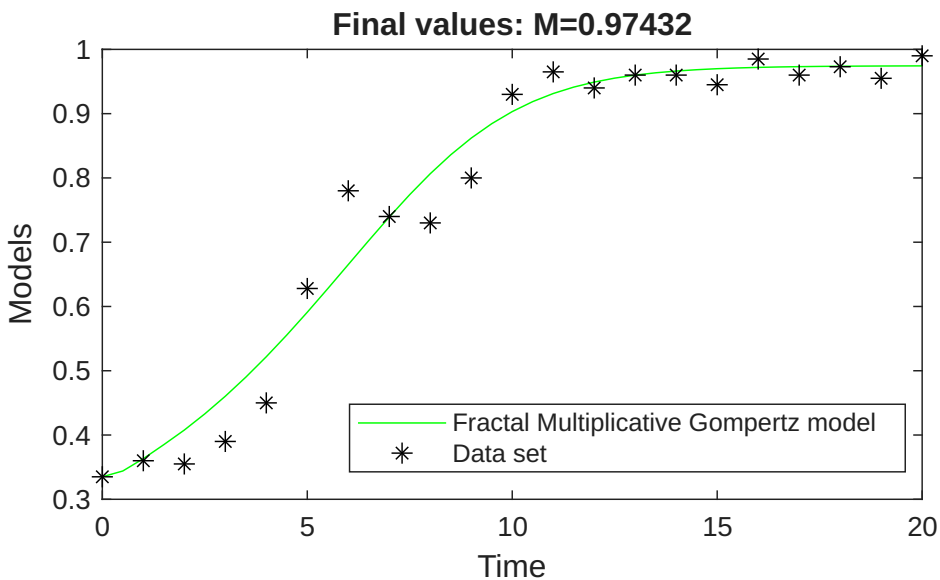


FIGURE 2. Data and estimates of tuberculosis infectious using the fractal multiplicative Gompertz model (45).

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5.1. Growth dynamics. This example compares the solution of a linear growth equation under three regimes: Ordinary, Fractal, and Fractal Multiplicative. The base equation is $y'(t) = ky(t)$.

The respective solutions are:

- Ordinary: $y(t) = y_0e^{kt}$,
- Fractal: $y(t) = y_0e^{k\frac{t^\alpha}{\alpha}}$,
- Fractal Multiplicative: $y(t) = y_0e^{k\frac{t^{\alpha\beta}}{\alpha\beta}}$ (considering the interaction of both parameters in the exponential rate).

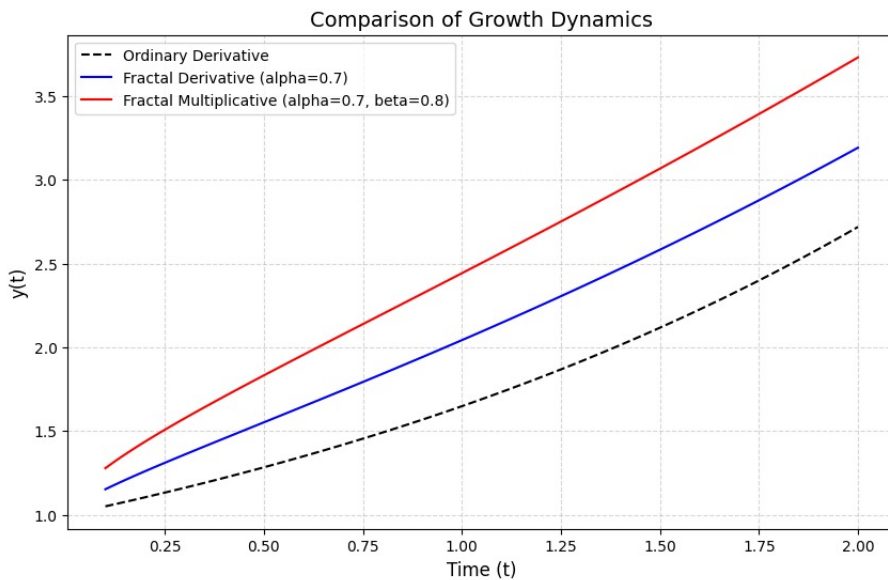


FIGURE 3. Comparison

Figure 3 illustrates that fractal multiplicative calculus allows for more flexible acceleration or deceleration of growth than simple power models.

Remark 5.1. The use of $y(t) = y_0e^{k\frac{t^{\alpha\beta}}{\alpha\beta}}$ for the fractal multiplicative derivative is illustrative (it clearly shows the interaction of the parameters). It allows us to observe how the product $\alpha\beta$ acts as an “acceleration index”. We should note that while the classical fractal derivative alters the time scale, the multiplicative version affects the exponential growth rate nonlinearly, which is ideal for systems with memory and proportional growth.

5.2. Fractal derivative of a rough function. We will use the Weierstrass function to visualize how the fractal derivative smooths or captures the rate of change in a function that is not differentiable in the classical sense.

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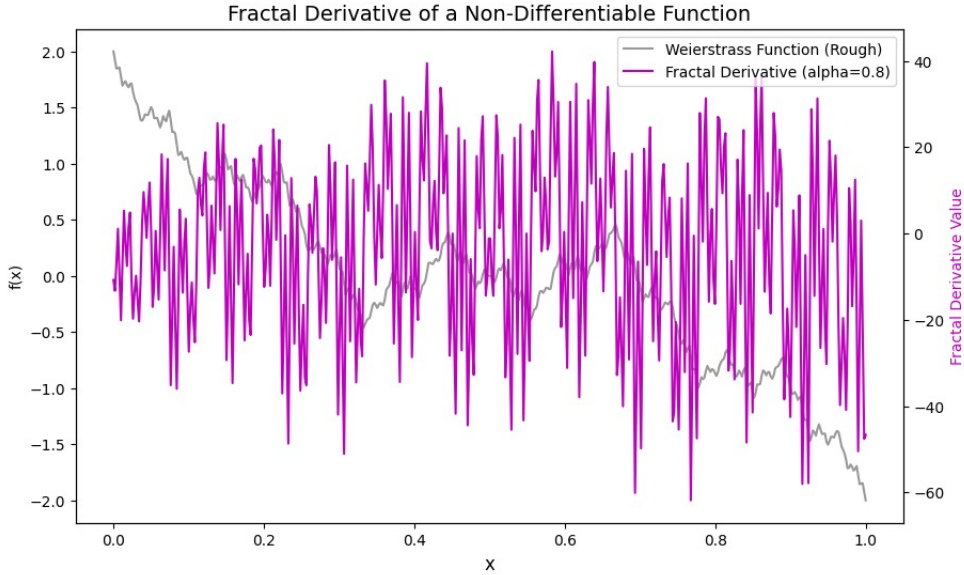


FIGURE 4. A rough function: Weierstrass

Figure 4 allows us to clarify that the fractal derivative is calculated with respect to t^α , which allows us to define a local rate of change even where the conventional slope tends to infinity.

Remark 5.2. It is essential to link this graphical representation with Proposition 2 and Remark 1. The figure visually demonstrates that, although the Weierstrass function is non-differentiable in the Newtonian sense, its (α, β) -*derivative exists and remains bounded, provided that $\alpha < \nu$ (where ν denotes the Hölder exponent), that is, the plot illustrates the existence and boundedness of the (α, β) -fractal derivative for the Weierstrass function when $\alpha < \nu$, contrasting with its classical non-differentiability.

5.3. Error surface analysis. This visualization is crucial for demonstrating the accuracy of the fractal multiplicative Gompertz model compared to real data, showing how the Mean Squared Error (MSE) is minimized in a specific combination of (α, β) .

The error surface justifies why certain values were chosen for α and β in the Gompertz model, adding statistical validity to our work (see Figure 5).

Remark 5.3. This example justifies the numerical experiments we can develop with other cases. By presenting the error surface for the fractal multiplicative Gompertz model, we provide visual evidence of the existence of a global minimum for the parameters (α, β) .

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Error Surface for Parameter Optimization

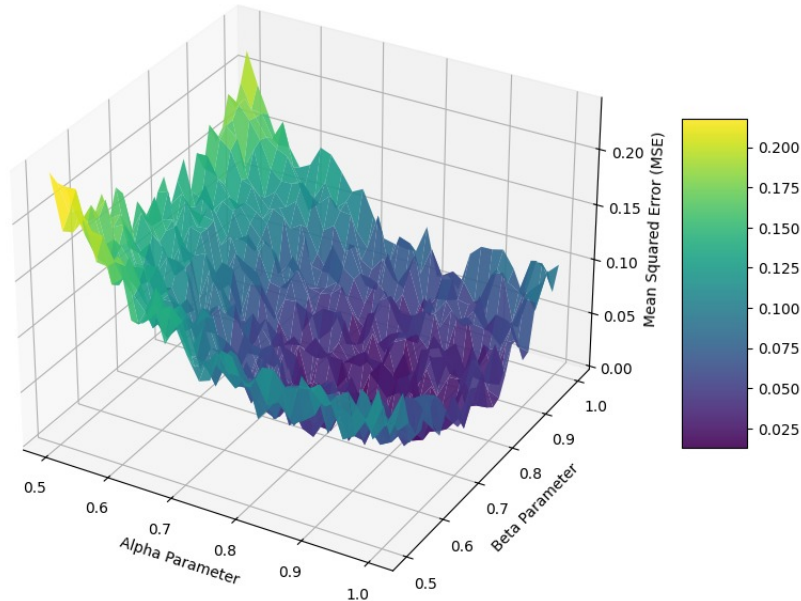


FIGURE 5. The Error: 3D vision

CONCLUSIONS

In this paper, we have introduced and developed a novel theoretical framework: fractal multiplicative calculus. By merging the principles of non-Newtonian (multiplicative) calculus with fractal derivatives, we have established a robust operator, the (α, β) -*derivative, which offers superior flexibility in modeling complex phenomena.

The theoretical analysis, supported by Proposition 2 and Remark 1, confirms that this new derivative is well-defined even for non-differentiable functions in the classical sense, such as the Weierstrass function. Our graphical results visually validate that as long as the fractal order α is lower than the Hölder exponent ν , the multiplicative fractal derivative remains bounded and captures the local dynamics of rough systems that standard calculus fails to describe.

Furthermore, the application of this framework to the Gompertz growth model demonstrates a significant advancement in applied mathematics. The numerical experiments and the subsequent error surface analysis show that the dual-parameter approach (α, β) allows for a more precise characterization of growth rates and saturation levels. The minimization of the Mean Squared Error (MSE) in our optimization plots provides empirical evidence that fractal multiplicative models provide a better fit for real-world data than their ordinary or purely fractal counterparts.

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In conclusion, the fractal multiplicative calculus not only generalizes previous approaches but also opens new avenues in the study of differential equations, the calculus of variations, and complex system modeling. Future research should focus on the stability of these solutions in stochastic environments and their potential application in financial mathematics and biological signaling.

Overall, these results highlight the potential of fractal multiplicative calculus as a powerful and unifying tool, future work may include the extension of these methods to broader classes of equations and further applications in biology, epidemiology, and related fields.

APPENDIX

MATLAB script for fractal multiplicative Gompertz models.

```
function fmultiplicative

global alpha beta
%% Set parameters
alpha=1.101 ; % FILL IN A VALUE FOR ALPHA
beta=0.401; % FILL IN A VALUE FOR BETA

%% Solve equations
pt = linspace(0,20,100); % Generate t for p
p = (alpha/(beta-1))*pt.^(alpha-1); % Generate p(t)
Tspan = [0 20]; % Solve from t=0 to t=20
IC = 0.335; % y(t=0)=1
[T L] = ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC)
%%Errors
err2=immse([0.335,0.36,0.355,0.39,0.45,0.628,0.78,0.74,0.73,0.8,
0.93,0.965,0.94,0.96,0.96,0.945,0.985,0.96,0.973,0.955,0.99],[0.335,
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),1),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),2),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),3),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),4),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),5),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),6),deval(ode45(@(t,l)
myode2(t,l,pt,p),
Tspan,IC),7),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),8),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),9),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),10),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),11),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),12),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),13),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),14),
deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),15),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),16),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),17),
deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),18),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),19),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),20)])
mae2=mae([0.335,0.36,0.355,0.39,0.45,0.628,0.78,0.74,0.73,0.8,0.93,
0.965,0.94,0.96,0.96,0.945,0.985,0.96,0.973,0.955,0.99],[0.335,deval(ode45(@(t,l)
myode2(t,l,pt,p),
Tspan,IC),1),deval(ode45(@(t,l)
```

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```

myode2(t,l,pt,p),Tspan,IC),2),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),3),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),4),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),5),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),6),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),7),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),8),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),9),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),10),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),11),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),12),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),13),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),14),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),15),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),16),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),17),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),18),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),19),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),20)]])
%% Plot results
figure;
plot(T,L,'green',0,0.335,'k*',1,0.36,'k*',2,0.355,'k*',3,0.39,'k*',4,0.45,'k*',5,
0.628,'k*',6,0.78,'k*',7,0.74,'k*',8,0.73,'k*',9,0.8,'k*',10,0.93,'k*',11,0.965,'k*',12,
0.94,'k*',13,0.96,'k*',14,0.96,'k*',15,0.945,'k*',16,0.985,'k*',17,0.96,'k*',18,
0.973,'k*',19,0.955,'k*',20,0.99,'k*');
title('Plot of y as a function of time');
title(strcat('Final values: B=',num2str(L(end))))
xlabel('Time');
ylabel('Models');
legend('Fractal Multiplicative Gompertz model','Data set')

function dldt = myode2(t,l,pt,p)
global beta
f = interp1(pt,p,t); % Interpolate the data set (pt,p) at time t
dldt = f.*(abs(1))^(2-beta)*1*0.42*log(1+log(log(2.65)/1))*(log(1)+1/(beta-1))^(-1);

```

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