BLOW-UP OF POSITIVE-INITIAL-ENERGY SOLUTIONS FOR
NONLINEARLY DAMPED SEMILINEAR WAVE EQUATIONS

MOHAMED AMINE KERKER

Abstract. We consider a class of semilinear wave equations with both strongly
and nonlinear weakly damped terms,
$$u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u,$$
associated with initial and Dirichlet boundary conditions. Under certain con-
ditions, we show that any solution with arbitrarily high positive initial energy
blows up in finite time if $m < p$. Furthermore, we obtain a lower bound for
the blow-up time.

1. Introduction

In this contribution, we study the blow-up of solutions of the following initial
boundary value problem of a semilinear wave equation:
$$\begin{cases}
  u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u, & x \in \Omega, \ t > 0, \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega.
\end{cases}$$

Here, $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial \Omega$. Additionally, we
assume that
$$u_0 \in H_0^1(\Omega), \ u_1 \in L^2(\Omega),$$
and $\omega, \mu, m$ and $p$ are positive constants, with
$$\begin{cases}
  2 < p \leq \frac{2n}{n-2}, & \text{for } n \geq 3, \\
  2 < p < \infty, & \text{for } n = 2.
\end{cases}$$

The linear strong damping term $-\omega \Delta u_t$ appears in models describing Kelvin–
Voigt materials that exhibit both elastic and viscous properties, while the nonlinear
frictional damping term $\mu |u_t|^{m-2} u_t$ usually models external friction forces such as
air resistance acting on the vibrating structures.

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damping; finite time blow-up.
In the absence of strong damping \((\omega = 0)\), the equation in (1.1) reduces to the nonlinearly damped wave equation
\[
 u_{tt} - \Delta u + \mu |u_t|^{m-2}u_t = |u|^{p-2}u. \tag{1.4}
\]
Eq. (1.4) was first studied by Levine \[8\] in the case of linear weak damping \((m = 2)\). By using the concavity method, he showed that solutions with negative initial energy blow up in finite time. Later, for the case \(m > 2\), by using a different method, Georgiev and Todorova \[4\] established a global existence result for Eq. (1.4) if \(m \geq p\) and finite time blow-up if \(p > m\) and the initial energy is sufficiently negative. In the presence of the strong damping term, i.e. \(\omega > 0\), Gazzola and Squassina \[3\] studied (1.4) for \(m = 2\). They gave a necessary and sufficient condition for blow-up if \(E(0) < d\), where \(d\) is the depth of the potential well. Recently, Yang and Xu \[16\] gave a sufficient condition for blow-up if \(E(0) > d\). In the case of \(\omega > 0\) and \(m > 2\), Yu \[17\] gave a necessary and sufficient condition for blow-up when \(E(0) < d\). Boukhatem and Benabderrahmane \[2\] extended the previous work to a semilinear hyperbolic equation for a uniformly elliptic operator with nonlinear damping and source terms. For results of the same nature, we refer the reader to \[1, 6, 5, 9, 12, 14, 15, 18\] and the references therein.

In related work, Messaoudi \[11\] considered
\[
 u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) \, d\tau + |u_t|^{m-2}u_t = |u|^{p-2}u. \tag{1.5}
\]
He proved, for \(m < p\), that solutions with \(E(0) < d\) blow up in finite time. Later, by using a modified concavity method, Kafini and Messaoudi \[7\] established a blow-up result for (1.5) when the damping term is linear. When \(m > 2\), by introducing a new technique, Song \[13\] obtained a finite time blow-up result for solutions of (1.5) with arbitrarily high initial energy.

In this paper, motivated by the above-cited works, we give sufficient conditions for the finite time blow-up of solutions of (1.1) in both cases: \(E(0) < 0\) and \(E(0) > 0\). Furthermore, we give a lower bound for the blow-up time.

\section{Preliminaries}

We denote by \(\| \cdot \|_p\) the \(L^p(\Omega)\) norm \((2 \leq p < \infty)\), and by \((\cdot, \cdot)\) the \(L^2\) inner product. The notation \((\cdot, \cdot)\) is used in this paper to denote the duality paring between \(H^{-1}(\Omega)\) and \(H_0^1(\Omega)\). We introduce the energy functional
\[
 E(t) := \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 - \frac{1}{p} \|u\|_p^p.
\]
By simple calculation we have
\[
 E'(t) = -\mu \|u_t\|_m^m - \omega \|\nabla u_t\|_2^2 \leq 0,
\]
which implies that
\[
 E(t) \leq E(0) \quad \forall t \geq 0,
\]
and
\[
 -E'(t) \geq \mu \|u_t\|_m^m, \quad -E'(t) \geq \omega \|\nabla u_t\|_2^2. \tag{2.1}
\]
Definition 2.1. By solution of problem (1.1) over $[0, T]$ we mean a function $u \in C([0, T], H^1_0(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], H^{-1}(\Omega))$, with $u_t \in L^m([0, T], H^1_0(\Omega))$, such that $u(0) = u_0$, $u_t(0) = u_1$ and

$$
\langle u_{tt}(t), \eta \rangle + \int_\Omega \nabla u(t) \nabla \eta \, dx + \omega \int_\Omega \nabla u_t(t) \nabla \eta \, dx + \mu \int_\Omega |u(t)|^{m-2} u_t(t) \eta \, dx = \int_\Omega |u|^p-2 u \eta \, dx
$$

for all $\eta \in H^1_0(\Omega)$ and a.e. $t \in [0, T]$.

Theorem 2.2 ([17]). Assume that conditions (1.2) and (1.3) hold. Then the problem (1.1) admits a unique local solution defined on $[0, T]$. Moreover, if $T_{\text{max}} := \sup \{ T > 0 : u = u(t) \text{ exists on } [0, T] \} < \infty$, then

$$
\lim_{t \to T_{\text{max}}} \|u(t)\|_q = \infty \quad \text{for all } q \geq 1 \text{ such that } q > \frac{n(p-2)}{2}.
$$

3. Blow-up with negative initial energy

In this section we show that the solution of (1.1) blows up in finite time if $m < p$ and $E(0) < 0$. To prove the main result in this section, we define $H(t) := -E(t)$ and we use the following lemma. For the proof, see [10].

Lemma 3.1. Suppose (1.3) holds. Then we have

$$
\|u\|_p^s \leq C \left( |H(t)| + \|u_t\|_2^2 + \|u\|_p^p \right)
$$

for any $u \in H^1_0(\Omega)$ and $2 \leq s \leq p$.

Theorem 3.2. Suppose (1.2) and (1.3) hold. Assume further that $p > m \geq 2$ and $E(0) < 0$. Then the solution of the problem (1.1) blows up in finite time.

Proof. To obtain a contradiction, we suppose that the solution of (1.1) is global; then, for every fixed $T > 0$, there exists a constant $K > 0$ such that

$$
\max \left\{ \|\nabla u\|_2^2, \|u_t\|_2^2, \|u\|_p^p \right\} \leq K \quad \forall t \in [0, T].
$$

We have $H'(t) = -E'(t) \geq 0$, which together with $E(0) < 0$ shows that

$$
0 < H(0) \leq H(t) \leq \frac{1}{p} \|u\|_p^p;
$$

furthermore,

$$
\mu \|u_t\|_m^m \leq H'(t).
$$

We now define an auxiliary function

$$
L(t) := H^{1-\alpha}(t) + \varepsilon(u_t, u) + \frac{\omega}{2} \|\nabla u\|_2^2,
$$

for $\varepsilon$ small (to be chosen later) and

$$
0 < \alpha \leq \min \left\{ \frac{p-2}{2p}, \frac{p-m}{p(m-1)} \right\} < \frac{1}{2}.
$$
Therefore, in view of the last inequality, (3.8) becomes

\[ L'(t) = (1 - \alpha)H'(t)H^{-\alpha}(t) + \varepsilon(u_t, u) + \varepsilon||u_t||_2^2 + \varepsilon\omega(\nabla u_t, \nabla u). \quad (3.4) \]

Using (1.1), the equation (3.4) takes the form

\[ L'(t) = (1 - \alpha)H'(t)H^{-\alpha}(t) - \varepsilon||\nabla u||_2^2 + \varepsilon||u_t||_2^2 + \varepsilon||u||_p^p - \varepsilon\mu||u_t||_{m-2}^2(u_t, u). \quad (3.5) \]

To estimate the last term in the right-hand side of (3.5), we use Young’s inequality

\[ (|u_t|^{m-2}u_t, u) \leq \frac{\delta}{m}||u||_m^m + \frac{m-1}{m}\delta^{m/(1-m)}||u_t||_m^m \quad \forall \delta > 0. \quad (3.6) \]

By taking

\[ \delta = \left[ kH^{-\alpha}(t) \right]^{1-m}, \]

for a large constant \( k \) to be chosen later, (3.6) becomes

\[ (|u_t|^{m-2}u_t, u) \leq \frac{1}{m} \left[ kH^{-\alpha}(t) \right]^{1-m}||u||_m^m + \frac{m-1}{m}kH^{-\alpha}||u_t||_m^m. \quad (3.7) \]

Combining (3.5) and (3.7), and using (3.3), yields

\[ L'(t) \geq \left[ 1 - \alpha - \varepsilon k \frac{m-1}{m} \right] H'(t)H^{-\alpha}(t) - \varepsilon||\nabla u||_2^2 + \varepsilon||u||_p^p \]
\[ + \varepsilon||u_t||_2^2 - \frac{\varepsilon\mu}{m} \left[ kH^{-\alpha}(t) \right]^{1-m}||u||_m^m. \quad (3.8) \]

By using (3.2), we obtain

\[ H^{\alpha(m-1)}(t)||u||_m^m \leq p^{-\alpha(m-1)}||u||_p^{\alpha(m-1)}||u||_m^m, \]

and hence by the inequality

\[ ||u||_m \leq C||u||_p, \]

we get

\[ H^{\alpha(m-1)}(t)||u||_m^m \leq C\alpha^{\alpha(m-1)}||u||_p^{\alpha(m-1)+m}. \quad (3.9) \]

Thus, by (3.9) and Lemma 3.1 for \( s = p\alpha(m-1) + m \leq p \), we obtain

\[ H^{\alpha(m-1)}(t)||u||_m^m \leq C\alpha^{\alpha(m-1)} \left[ H(t) + ||u_t||_2^2 + ||u||_p^p \right]. \]

Therefore, in view of the last inequality, (3.8) becomes

\[ L'(t) \geq \left[ 1 - \alpha - \varepsilon k \frac{m-1}{m} \right] H'(t)H^{-\alpha}(t) + \frac{\varepsilon}{2}(p-2)||\nabla u||_2^2 \]
\[ + \frac{\varepsilon}{2}(p+2)||u_t||_2^2 + \varepsilon \left\{ pH(t) - \lambda k^{1-m} \left[ H(t) + ||u_t||_2^2 + ||u||_p^p \right] \right\}, \quad (3.10) \]

where

\[ \lambda = Cp^{-\alpha(m-1)} \frac{\mu}{m}. \]

Writing \( p = (p+2)/2 + (p-2)/2 \) in (3.10) yields

\[ L'(t) \geq \gamma_1 H'(t)H^{-\alpha}(t) + \gamma_2 H(t) + \gamma_3||u_t||_2^2 + \gamma_4||u||_p^p + \gamma_5||\nabla u||_2^2, \quad (3.11) \]
where
\[ \gamma_1 = 1 - \alpha - \varepsilon k^{\frac{m - 1}{m}}, \quad \gamma_2 = \varepsilon \left( \frac{p + 2}{2} - \lambda k^{1-m} \right), \]
\[ \gamma_3 = \varepsilon \left( \frac{p + 6}{4} - \lambda k^{1-m} \right), \quad \gamma_4 = \varepsilon \left( \frac{p - 2}{2p} - \lambda k^{1-m} \right), \]
\[ \gamma_5 = \frac{\varepsilon}{4}(p - 2) > 0. \]

We choose now \( k \) large enough such that the coefficients \( \gamma_i \), for \( 2 \leq i \leq 4 \), are positive. Once \( k \) is fixed, we choose \( \varepsilon \) small enough such that \( \gamma_1 > 0 \) and \( L(0) > 0 \).

Hence, the inequality (3.11) becomes
\[ L'(t) \geq A \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right] \tag{3.12} \]
for some constant \( A > 0 \). Consequently, we have
\[ L(t) \geq L(0) > 0 \quad \text{for all} \quad t \geq 0. \]

Next, by using Hölder’s and Young’s inequalities, we obtain
\[ |(u_t, u)|^{1/(1-\alpha)} \leq \|u_t\|_2^{1/(1-\alpha)} \|u\|_2^{1/(1-\alpha)} \leq C \left[ \|u_t\|_2^{s/(1-\alpha)} + \|u\|_2^{r/(1-\alpha)} \right] \]
for \( 1/s + 1/r = 1 \). We take \( s = 2(1-\alpha) \), which gives
\[ \frac{s}{1-\alpha} = 2 \quad \text{and} \quad \frac{r}{1-\alpha} = \frac{2}{1-2\alpha} \leq p. \]

Therefore, by using Lemma 3.1, we obtain
\[ |(u_t, u)|^{1/(1-\alpha)} \leq C \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]. \tag{3.13} \]

From (3.1) and (3.2), we have
\[ \|\nabla u\|_2^{2/(1-\alpha)} \leq K^{1/(1-\alpha)} \leq K^{1/(1-\alpha)} \frac{H(t)}{H(0)}. \tag{3.14} \]

So, by using Jensen’s inequality, we get
\[ L(t)^{1/(1-\alpha)} \leq C \left[ H(t) + |(u_t, u)|^{1/(1-\alpha)} + \|\nabla u\|_2^{2/(1-\alpha)} \right], \]
and by combining it with (3.13) and (3.14), we deduce
\[ L(t)^{1/(1-\alpha)} \leq B \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]. \tag{3.15} \]

From the inequalities (3.12) and (3.15), we finally obtain the differential inequality
\[ L'(t) \geq DL(t)^{1/(1-\alpha)} \tag{3.16} \]
for some \( D > 0 \). A simple integration of (3.16) over \((0, t)\) immediately yields
\[ L(t) \geq \left[ L^{-\alpha/(1-\alpha)}(0) - \frac{\alpha D}{1-\alpha} t \right]^{1-1/\alpha}, \tag{3.17} \]
which shows that the functional \( L(t) \) blows up in finite time. \( \square \)
Remark 3.3. From (3.17) we obtain the following upper bound of the blow-up time:

\[ T_{\text{max}} \leq \frac{1 - \alpha}{\alpha D} [L(0)]^{1-1/\alpha}. \]

4. Blow-up with positive initial energy

In this section, we consider the blow-up of solutions of the problem (1.1) when \( E(0) > 0 \). To prove the main theorem of this paper, we employ the following lemma.

Lemma 4.1. If \( 2 < m < p \) then

\[ \frac{1}{m} \|u\|_m^m \leq \frac{s}{2} \|u\|_2^2 + \frac{1 - s}{p} \|u\|_p^p, \quad \text{where} \ s = \frac{p - m}{p - 2}. \]

Proof. By the convexity of the function \( u^x/x \) for \( u \geq 0 \) and \( x > 0 \). \( \square \)

Theorem 4.2. Suppose [1.2] and [1.3] hold. Assume further that \( p > m \geq 2 \). If the solution of (1.1) satisfies

\[ (u_t(0), u(0)) > ME(0) > 0 \]

for some \( M > 0 \) to be specified later in the proof, then \( u(t) \) blows up in finite time.

Proof. Assume, towards a contradiction, that \( u(t) \) is a global solution of (1.1). Setting \( F(t) := \frac{1}{2} \|u(t)\|_2^2 \), it follows from (1.1) that

\[ F''(t) = \|u_t(t)\|_2^2 + \|u(t)\|_p^p - \|\nabla u(t)\|_2^2 - \omega(\nabla u_t, \nabla u) - \mu(|u_t|^{m-2}u_t, u). \quad (4.2) \]

By using Hölder’s and Young’s inequalities, we estimate the last two terms in the right-hand side of the previous equation as follows:

\[ (\nabla u_t, \nabla u) \leq \eta \|\nabla u\|_2^2 + \frac{1}{4\eta} \|\nabla u_t\|_2^2, \quad \eta > 0, \]

\[ (|u_t|^{m-2}u_t, u) \leq \frac{1}{m} \delta^m \|u\|_m^m + \frac{m - 1}{m} \delta^{m/(1-m)} \|u_t\|_m^m, \quad \delta > 0. \]

So, by Lemma 4.1 we get

\[ \frac{\delta^m}{m} \|u\|_m^m \leq \frac{s}{2} \delta^m \|u\|_2^2 + \frac{1 - s}{p} \delta^m \|u\|_p^p. \]

Hence, (4.2) becomes

\[ F''(t) \geq \|u_t(t)\|_2^2 - (1 + \omega\eta) \|\nabla u(t)\|_2^2 + \left[ 1 - \frac{\mu(1 - s)}{p} \delta^m \right] \|u(t)\|_p^p \]

\[ - \frac{\mu s}{2} \delta^m \|u(t)\|_2^2 - \frac{\omega}{4\eta} \|\nabla u_t(t)\|_2^2 - \mu \frac{m - 1}{m} \delta^{-\frac{m}{m-1}} \|u_t(t)\|_m^m. \]
Adding and subtracting \( p(1 - \varepsilon)E(t) \) for \( \varepsilon \in (0, 1) \) in the right-hand side of the last inequality, and using (2.1) and the Poincaré inequality, we obtain

\[
\frac{d}{dt} \left\{ \frac{dF(t)}{dt} - \left[ \frac{1}{4\eta} + \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \right] E(t) \right\} \\
\geq F''(t) + \frac{\omega}{4\eta} \| \nabla u_t(t) \|^2 + \mu \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \| u_t(t) \|^m \\
\geq \| u_t(t) \|^2 - (1 + \omega \eta) \| \nabla u(t) \|^2 \\
+ \left[ 1 - \frac{\mu(1-s)}{p} \delta^m \right] \| u(t) \|^p - \frac{\mu}{2} \delta^m \| u(t) \|^2 \\
\geq \left[ 1 + \frac{p}{2}(1 - \varepsilon) \right] \| u_t(t) \|^2 + \left[ \frac{p}{2}(1 - \varepsilon) - (1 + \omega \eta) \right] \| \nabla u(t) \|^2 \\
+ \left[ \varepsilon - \frac{\mu(1-s)}{p} \delta^m \right] \| u(t) \|^p - \mu s^2 \| u(t) \|^2 \\
- \frac{\mu}{2} \delta^m \| u(t) \|^2 \\
\geq \left[ 1 + \frac{p}{2}(1 - \varepsilon) \right] \| u_t(t) \|^2 + \left\{ \alpha(\varepsilon) B - \frac{\mu s^2}{2} \delta^m \right\} \| u(t) \|^2 \\
+ \left[ \varepsilon - \frac{\mu(1-s)}{p} \delta^m \right] \| u(t) \|^p - \frac{\mu}{2} \delta^m \| u(t) \|^2 \\
- \frac{\mu}{2} \delta^m \| u(t) \|^2 \\
\geq \left[ 1 + \frac{p}{2}(1 - \varepsilon) \right] \| u_t(t) \|^2 + \left\{ \alpha(\varepsilon) B - \frac{\mu s^2}{2} \delta^m \right\} \| u(t) \|^2 \\
- \frac{\mu}{2} \delta^m \| u(t) \|^2 \\
- \frac{\mu}{2} \delta^m \| u(t) \|^2 \\
\geq \frac{\alpha(\varepsilon) B - \frac{\mu s^2}{2} \delta^m}{2} \| u(t) \|^2 - \frac{\mu}{2} \delta^m \| u(t) \|^2 \\
= \frac{\alpha(\varepsilon) B - \frac{\mu s^2}{2} \delta^m}{2} \| u(t) \|^2 - \frac{\mu}{2} \delta^m \| u(t) \|^2.
\]

where

\[
\alpha(\varepsilon) = \frac{p}{2}(1 - \varepsilon) - (1 + \omega \eta)
\]

and \( B \) is the best constant of the Poincaré inequality

\[
\| \nabla u \|^2 \geq B \| u \|^2.
\]

Therefore, taking \( \eta = \varepsilon \) and

\[
\delta = \left[ \frac{p \varepsilon}{\mu(1-s)} \right]^{1/m},
\]

setting

\[
\gamma_1(\varepsilon) = \frac{1}{4\varepsilon} + \frac{m-1}{m} \left( \frac{1-s}{p \varepsilon} \right)^{-\frac{1}{m-1}}
\]

and substituting in (4.3), we arrive at

\[
\frac{d}{dt} \left[ F'(t) - \gamma_1(\varepsilon) E(t) \right] \geq \left[ 1 + \frac{p}{2}(1 - \varepsilon) \right] \| u_t(t) \|^2 \\
+ \left\{ \alpha(\varepsilon) B - \frac{\mu s^2}{2(1-s)} \varepsilon \right\} \| u(t) \|^2 - \frac{\mu}{2} \delta^m \| u(t) \|^2 - \frac{\mu}{2} \delta^m \| u(t) \|^2.
\]

Hence, we choose \( \varepsilon \) small enough such that

\[
\alpha(\varepsilon) B - \frac{\mu s^2}{2(1-s)} \varepsilon > 0.
\]
By using the Schwarz inequality, we have
\[ 2 \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right]^{1/2} \left[ \alpha(\varepsilon) B - \frac{ps}{2(1 - s)} \varepsilon \right]^{1/2} (u_t, u) \leq \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right] \| u(t) \|_2^2 + \left[ \alpha(\varepsilon) B - \frac{ps}{2(1 - s)} \varepsilon \right] \| u(t) \|_2^2. \]
Consequently, we obtain
\[ \frac{d}{dt} [F'(t) - \gamma_1(\varepsilon) E(t)] \geq \beta(\varepsilon) (u_t, u) - p(1 - \varepsilon) E(t) = \beta(\varepsilon) [F'(t) - \gamma_2(\varepsilon) E(t)], \]
where
\[ \beta(\varepsilon) = 2 \left[ 1 + \frac{p}{2} (1 - \varepsilon) \right]^{1/2} \left[ \alpha(\varepsilon) B - \frac{ps}{2(1 - s)} \varepsilon \right]^{1/2}, \]
\[ \gamma_2(\varepsilon) = \frac{p(1 - \varepsilon)}{\beta(\varepsilon)}. \]
Since
\[ \alpha(\varepsilon) B - \frac{ps}{2(1 - s)} \varepsilon \to \begin{cases} \frac{2B}{p - 2} > 0 & \text{as } \varepsilon \to 0^+, \\ - \frac{B}{1 + \omega} - \frac{ps}{2(1 - s)} < 0 & \text{as } \varepsilon \to 1^-, \end{cases} \]
there exists \( \varepsilon_0 \in (0, 1) \) such that
\[ \beta(\varepsilon_0) = 0 \quad \text{and} \quad \beta(\varepsilon) > 0 \quad \forall \varepsilon \in (0, \varepsilon_0). \]
Hence, we have
\[ \gamma_1(\varepsilon) - \gamma_2(\varepsilon) \to \begin{cases} +\infty & \text{as } \varepsilon \to 0^+, \\ -\infty & \text{as } \varepsilon \to \varepsilon_*^- \end{cases}. \]
Therefore, there exists \( \varepsilon_0 \in (0, \varepsilon_0) \) such that \( \gamma_1(\varepsilon_0) = \gamma_2(\varepsilon_0) > 0 \). So, by setting
\[ L(t) = F'(t) - \gamma_1(\varepsilon_0) E(t), \]
\[ M = \gamma_1(\varepsilon_0), \]
and by using [4.1], we obtain
\[ L(0) = (u_t(0), u(0)) - \gamma_1(\varepsilon_0) E(0) > (u_t(0), u(0)) - ME(0) > 0. \]
Moreover, with this choice of \( \varepsilon_0 \), [4.4] becomes
\[ \frac{d}{dt} L(t) \geq \beta(\varepsilon_0) L(t), \]
which gives
\[ L(t) \geq L(0) e^{\beta(\varepsilon_0) t} \quad \forall t \geq 0. \]
Since \( u(t) \) is global and \( E(0) > 0 \), by Theorem 3.2 we have that \( E(t) > 0 \) for all \( t \geq 0 \). Hence, we arrive at the inequality
\[ F'(t) \geq L(0) e^{\beta(\varepsilon_0) t} \quad \forall t \geq 0. \]
By integrating this inequality over \((0, t)\), we get
\[
|u(t)|_2^2 = 2F(t) \geq 2F(0) + 2 \frac{L(0)}{\beta(\varepsilon_0)} \left[ e^{\beta(\varepsilon_0)t} - 1 \right] \quad \forall t \geq 0. \tag{4.5}
\]
On the other hand, by using Hölder’s inequality and \((2.1)\), we have
\[
|u(t)|_2 \leq |u(0)|_2 + \int_0^t |u_\tau(\tau)|_2 d\tau \leq |u(0)|_2 + C \int_0^t |u_\tau(\tau)|_m d\tau \leq |u(0)|_2 + C t^{\frac{m-1}{m}} \int_0^t |u_\tau(\tau)|_m^m d\tau \leq |u(0)|_2 + C t^{\frac{m-1}{m}} \left[ \frac{E(0) - E(t)}{\mu} \right]^{1/m} \leq |u(0)|_2 + C t^{\frac{m-1}{m}},
\]
which clearly contradicts \((4.5)\). \(\square\)

5. Lower bound for the blow-up time

In this section, we give a lower bound for the blow-up time \(T_{\text{max}}\). To this end, we define
\[
G(t) := \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2.
\]

**Theorem 5.1.** Assume that \((1.2)\) and \((1.3)\) hold, and let \(u\) be the solution of \((1.1)\), which blows up at a finite \(T_{\text{max}}\). Then
\[
T_{\text{max}} \geq \int_{G(0)}^{+\infty} \left\{ A_1 + A_2 \sigma^{\beta/2} \right\}^{-1} d\sigma,
\]
where \(\beta, A_1\) and \(A_2\) are positive constants to be determined later in the proof.

**Proof.** By differentiating \(G(t)\) and using \((1.1)\), we obtain
\[
G'(t) = (\nabla u_t, \nabla u) + (u_{tt}, u_t) = (\nabla u_t, \nabla u) + (\Delta u, u_t) + \omega(\Delta u_t, u_t) - \mu |u_t|^m + (|u|^{p-2}u, u_t) = -\omega |\nabla u_t|^2 - \mu |u_t|^m + (|u|^{p-2}u, u_t).
\]

Thus,
\[
G'(t) \leq -\omega |\nabla u_t|^2 + (|u|^{p-1}, |u_t|).
\]
Using Hölder’s inequality, Young’s inequality and the Sobolev inequality
\[
|v|_q \leq B_q |\nabla v|_q \quad \forall q \in [1, 2^*], \forall v \in H_0^1(\Omega),
\]

we get
\[
(\|u\|_{p-1}^{p-1}, |u_t|) \leq \|u_t\|_p \|u\|_{p-1}^p \\
\leq \|u_t\|_p^\alpha + C_1\|u\|_p^\beta \\
\leq B_p^\alpha \|\nabla u_t\|_p^\alpha + C_2\|\nabla u\|_2^\beta \\
\leq A_1 + \|\nabla u_t\|_2^\beta + A_2(G(t))^{\beta/2},
\]
(5.2)
where $1 < \alpha < 2$ is some positive constant, $\beta = \alpha(p-1)/(\alpha-1)$ and
\[
C_1 = (\alpha - 1)\alpha^{-\alpha/(\alpha-1)}, \quad C_2 = C_1 B_p^\beta, \\
A_1 = (2-\alpha)2^{-2/(2-\alpha)}B_p^{2\alpha/(2-\alpha)}\alpha^{\alpha/(2-\alpha)}, \quad A_2 = C_2 2^{\beta/2}.
\]
Combining (5.2) and (5.1) gives
\[
G'(t) \leq A_1 + A_2(G(t))^{\beta/2}.
\]
(5.3)
Finally, integrating inequality (5.3) over $(0, T_{\text{max}})$ we get
\[
T_{\text{max}} \geq \int_0^{T_{\text{max}}} \left\{ A_1 + A_2(G(\tau))^{\beta/2} \right\}^{-1} G'(\tau) \, d\tau,
\]
and so
\[
T_{\text{max}} \geq \int_{G(0)}^{+\infty} \left\{ A_1 + A_2\sigma^{\beta/2} \right\}^{-1} d\sigma.
\]
□

REFERENCES


Mohamed Amine Kerker
Laboratory of Applied Mathematics, Badji Mokhtar-Annaba University, P.O. Box 12, Annaba, 23000, Algeria
mohamed-amine.kerker@univ-annaba.dz; a_kerker@yahoo.com

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