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CONVOLUTION-FACTORABLE MULTILINEAR OPERATORS

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Abstract. We study multilinear operators defined on topological products of Banach algebras of integrable functions and Banach left modules with convolution product. The main theorem of the paper presents a factorization for multilinear operators through convolution that implies the property known as zero product preservation. By using this factorization we investigate properties of multilinear operators including integral representations and we give applications related to orthogonally additive homogeneous polynomials and Hilbert–Schmidt operators.

1. Introduction and preliminaries

The factorization of operators is a powerful technique widely used in functional analysis to analyse the structure of an operator as well as the spaces on which it acts. By decomposing an operator into simpler components, a great deal of information can be obtained about the behavior and properties of this operator. The factorization of operators through a generic map that we call *product* has been studied by several authors in different contexts, such as lattices and algebras. To obtain the squares of the Riesz spaces, Buskes and van Rooij proved such a factorization for the bilinear operators defined on the topological product of a Riesz space E which are zero valued for the pairs of orthogonal elements, and they called these bilinear operators *orthosymmetric maps* (see [9] and [10]). In subsequent works, Boulabiar and Buskes gave the definition of n-power of a vector lattice and presented its universal characterization by using orthosymmetric n-linear maps [8]. In their study of Banach algebras, some researchers have used the bilinear and multilinear operators called zero product preserving, which are zero valued for the elements whose algebraic product is zero (see [1, 3] and references therein). Recently, the factorization of bilinear and multilinear operators defined on Banach function spaces through pointwise product has been studied by the author with different collaborators in [14] and [16]. Though the structures of these spaces are varied, the factorizations are mainly based on the disjointness preserving property.

The purpose of this paper is to present a generalization for n-linear operators of the factorization given in [15] motivated by the convolution of three functions

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introduced by Arregui and Blasco in [4]. Although the main theorem is similar to the bilinear case, completely new results are given and different applications are presented.

The paper is organized as follows. In the introduction, we give some basic notions and terminology. In Section 2, we give the main definitions and requirements that will be the basis for the study. Section 3, which is the main section of the paper, concerns factorization through convolution of Banach-space-valued multilinear operators defined on the topological product of Banach algebras of integrable functions and Banach left modules. Section 4 is devoted to domination inequalities and integral representations obtained by using vector measures and some lattice geometric inequalities of convolution-factorable multinear operators. The paper finishes with some applications. In these applications, we show an isomorphism between orthogonally additive n-homogeneous polynomials and convolution-factorable operators and we give some integral and series representations by using Hilbert–Schmidt operators.

Throughout the paper, we will use the standard terminology and notions from functional analysis. Nevertheless, before going further, we remind the reader of some notations. The capital letters X, Y, Z denote Banach spaces defined on the same scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $\mathbb{Z}$ and $\mathbb{N}$ indicate the set of integer and natural numbers, respectively. $\mathbb{T}$ is the circle group of complex numbers, that is, the real line mod 2π. The topological dual space of a Banach space X is denoted by X'. B_X is the unit ball of the space X. The notation $X \cong Y$ means X and Y are isomorphic. $\times_n X$ denotes the n-fold Cartesian product of the Banach space X. The notation ℓ^p $(p \geq 1)$ denotes the Banach space of absolutely p-summable sequences endowed with the norm $\|(x_n)\|_{\ell^p} = (\sum |x_n|^p)^{1/p}$.

By *operator* (linear, multilinear or polynomial) we mean *continuous operator*. We will write $\mathfrak{L}^n(X_1 \times \cdots \times X_n, Y)$ for the Banach space of n-linear operators endowed with the norm

$$\|M\| = \sup\{\|M(x_1, \ldots, x_n)\| : x_i \in B_{X_i}, \, i \in \{1, \ldots, n\}\}.$$

$\mathfrak{L}^n(X_1 \times \cdots \times X_n)$ and $\mathfrak{L}(X, Y)$ denote the n-linear forms defined on $X_1 \times \cdots \times X_n$ and the space of linear operators from X to Y, respectively.

$\mathcal{W}(\mathbb{T})$ is the unital algebra known as Wiener algebra under multiplication operations that comprises the continuous positive definite functions. Since $\mathbb{T}$ is a compact Abelian group, $\mathcal{W}(\mathbb{T})$ corresponds to the space of functions having absolutely convergent Fourier series and it is a Banach space with the norm $\|f\| = \|\hat{f}\|_{\ell^1}$, where $\hat{f}$ is the sequence of the coefficients of the Fourier series of the function f. It is known that $\mathcal{W}(\mathbb{T})$ is isometrically isomorphic to the sequence space $\ell^1(\mathbb{Z})$ by the Fourier transform (see [17, §32 and §34] and also [7, p. 7]). $\mathcal{I}(\mathbb{T})$ denotes the Banach algebra of all the trigonometric polynomials on $\mathbb{T}$. $\mathcal{C}(\mathbb{T})$ is the Banach space endowed with the usual supremum norm of the scalar-valued continuous functions defined on the compact group $\mathbb{T}$. $\mathcal{M}(\mathbb{T})$ is the space of regular Borel signed measures on $\mathbb{T}$.

A linear operator $T : X \to Y$ is called *(p,q)-summing* if there exists a constant $c > 0$ such that, for every choice of elements $x_1, \ldots, x_m \in X$ and for all positive integers m,

$$\left(\sum_{i=1}^{m}\left\|T(x_i)\right\|_Y^p\right)^{1/p} \leq k \sup_{x'\in B_{X'}} \left(\sum_{i=1}^{m}|\langle x_i, x'\rangle|^q\right)^{1/q}.$$

The space of (p,q)-summing operators from X to Y is denoted by $\Pi_{p,q}(X,Y)$, and by $\Pi_p(X,Y)$ if $p=q$.

For the locally compact Abelian group $\mathbb{T}$, it is known that $(\mathbb{T},\mathcal{B}(\mathbb{T}),m)$ is a finite measure space, where m is the Haar measure that is simply the normalized Lebesgue measure and $\mathcal{B}(\mathbb{T})$ is the Borel σ-algebra defined on $\mathbb{T}$ (see [19, Section VIII.2]). We denote all m-a.e. equivalence classes of measurable functions on $\mathbb{T}$ by $\mathcal{L}^0(\mathbb{T})$. $\mathcal{L}^p(\mathbb{T})$ $(p\geq 1)$ denotes the Banach space of p-integrable functions equipped with its standard norm $\|f\|_{\mathcal{L}^p(\mathbb{T})} = (\int_{\mathbb{T}}|f|^p\,dm)^{1/p}$. For $p=\infty$, $\mathcal{L}^\infty(\mathbb{T})=\mathcal{L}^\infty(m)$ denotes the Banach space of m-a.e. bounded measurable functions equipped with the essential supnorm $\|.\|_{\mathcal{L}^\infty(\mathbb{T})}$.

Throughout the paper we will use the convolution operator defined on the Banach space $\mathcal{L}^1(\mathbb{T})$ by the formula

$$f*g(x)=\int_{\mathbb{T}} f(x-y)g(y)\,dm(y).$$

Recall that $\mathcal{L}^1(\mathbb{T})$ is a Banach algebra by its natural convolution product.

A net $\{e_\tau\}_{\tau\in\Lambda}$ in a non-unital normed algebra A is called a *k-bounded left approximate identity* for A if $e_\tau x\to x$ $(x\in A)$ and there exists a positive constant k such that $\|e_\tau\|\leq k$ for all $\tau\in\Lambda$ (see [7, §11]).

A (linear, multilinear or polynomial) operator is called *(weakly) compact* if it maps the unit ball to a relatively (weakly) compact set.

2. Product-factorable multilinear maps

Let $X_1, X_2, \dots, X_n$, and Z be Banach spaces. Consider a Banach-space-valued n-linear map $\circledast : X_1\times X_2\times\cdots\times X_n\to Z$ defined by

$$(x_1,x_2,\dots,x_n)\rightsquigarrow \circledast(x_1,x_2,\dots,x_n)=x_1\circledast x_2\circledast\cdots\circledast x_n$$

for all $x_i\in X_i$ $(i=1,2,\dots,n)$.

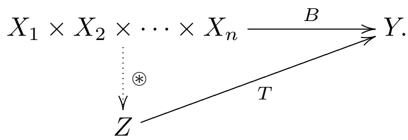
We will call this particular map a *norming product* if the inclusion $B_Z\subseteq \circledast(B_{X_1}\times B_{X_2}\times\cdots\times B_{X_n})$ holds.

Here are some examples of norming product:

- The algebraic multiplication is a norming product for all unital Banach algebras.
- The convolution operation $*$ is a norming product from $\mathcal{L}^2(\mathbb{T})\times\mathcal{L}^2(\mathbb{T})$ to $\mathcal{W}(\mathbb{T})$.
- Let (Ω,Σ,μ) be a σ-finite measure space and let $\sum_{i=1}^n \frac{1}{p_i}=\frac{1}{r}$ and $p_i, r\geq 1$. Then the pointwise product $\odot$ defined on $\mathcal{L}^{p_1}(\mu)\times\cdots\times\mathcal{L}^{p_n}(\mu)$ to $\mathcal{L}^r(\mu)$ is a norming product (see [14, Section 4]). The pointwise product $\odot$: $\ell^{p_1}\times\cdots\times\ell^{p_n}\to\ell^r$ is a norming product for sequence spaces also.

Recall that a multilinear operator $B : X_1 \times \cdots \times X_n \to Y$ is called $\circledast$-*factorable* for the norming product $\circledast$ if it can be factored through the product $\circledast : X_1 \times \cdots \times X_n \to Z$ and a linear operator $T : Z \to Y$ such that $B(x_1, x_2, \ldots, x_n) = T \circ \circledast(x_1, x_2, \ldots, x_n) = T(x_1 \circledast x_2 \circledast \cdots \circledast x_n)$ for all $x_i \in X_i$ $(i = 1, \ldots, n)$ (see [14, Definition 2.1]).

Thus, the following triangular diagram commutes for a certain continuous linear operator $T : Z \to Y$:



The author showed [14, Lemma 2.3] that a multilinear operator $B : X_1 \times X_2 \times \cdots \times X_n \to Y$ is $\circledast$-factorable for the norming product $\circledast : X_1 \times X_2 \times \cdots \times X_n \to Z$ if and only if there is a constant $k > 0$ such that for every finite set of vectors $(x_i^j)_{i=1}^m \in X_j$ $(j = 1, 2, \ldots, n)$, the following inequality holds:

$$\left\| \sum_{i=1}^{m} B(x_i^1, x_i^2, \ldots, x_i^n) \right\|_Y \leq k \left\| \sum_{i=1}^{m} x_i^1 \circledast x_i^2 \circledast \cdots \circledast x_i^n \right\|_Z .$$

In addition to this necessary and sufficient condition for the $\circledast$-factorability, it has been proved that, for specified domains, $\circledast$-factorable multilinear maps satisfy another property, called zero product preservation.

A multilinear map $B : X_1 \times X_2 \times \cdots \times X_n \to Y$ is called *zero product preserving* (*zpp* for short) if

$$B(x_1, x_2, \ldots, x_n) = 0 \quad \text{whenever } x_1 \circledast x_2 \circledast \cdots \circledast x_n = 0 \text{ for some } x_i \in X_i,$$

where $i \in \{1, 2, \ldots, n\}$. The class of zpp multilinear operators is a Banach space endowed with the usual multilinear operator norm. The Banach space of n-linear zpp operators defined on $X_1 \times X_2 \times \cdots \times X_n$ to Y will be denoted by $\mathfrak{L}_0^n(X_1 \times X_2 \times \cdots \times X_n, Y)$.

3. Convolution-factorability of multilinear operators

To obtain a class of $*$-factorable multilinear operators we will use some well-known results of harmonic analysis. Let us recall them. $\mathcal{I}(\mathbb{T})$ consists of all the trigonometric polynomials on $\mathbb{T}$ in the ordinary sense, that is, it is the set of all functions $f(t) = \sum_{k=-n}^{n} a_k \exp(ikt)$.

For the compact group $\mathbb{T}$, $\mathcal{L}^1(\mathbb{T})$ is a Banach algebra under convolution, where the normalized Haar measure is the normalized Lebesgue measure (see [17, 28.46] and [19, p. 202]). The non-unital Banach algebra $\mathcal{L}^1(\mathbb{T})$ has a bounded left approximate identity $\{e_\tau\}_{\tau \in \Lambda} \in \mathcal{L}^1(\mathbb{T})$ consisting of positive trigonometric polynomials such that $\|e_\tau\|_{\mathcal{L}^1} \leq 1$ for each τ and $\lim_\tau \|e_\tau * f - f\|_{\mathcal{L}^1} = 0$ for all $f \in \mathcal{L}^1(\mathbb{T})$ (see [17, Theorem 28.53]).

It is known that if $\mathcal{U}(\mathbb{T})$ is a left Banach $\mathcal{L}^1(\mathbb{T})$-module under convolution such that the set $\mathcal{I}(\mathbb{T})$ of trigonometric polynomials defined on $\mathbb{T}$ is contained as a dense set, then any bounded left approximate identity $\{e_\tau\}_{\tau\in\Lambda}$ of $\mathcal{L}^1(\mathbb{T})$ is a bounded left approximate identity for the subalgebra $\mathcal{U}(\mathbb{T})$, i.e., $\lim_\tau \|e_\tau * g - g\|_{\mathcal{U}} = 0$ for all $g \in \mathcal{U}(\mathbb{T})$. Besides, this implies a factorization for $\mathcal{U}(\mathbb{T})$ such that $\mathcal{L}^1(\mathbb{T}) * \mathcal{U}(\mathbb{T}) = \mathcal{U}(\mathbb{T})$ (see [17, Remark 38.6] and also [7] for more information about Banach algebras and modules). There are some specific subalgebras of the Banach algebra $\mathcal{L}^1(\mathbb{T})$ that have the properties attributed to $\mathcal{U}(\mathbb{T})$ above: the spaces $\mathcal{L}^p(\mathbb{T})$ $(1 \le p < \infty)$, $\mathcal{C}(\mathbb{T})$, and $\mathcal{W}(\mathbb{T})$ are left Banach $\mathcal{L}^1(\mathbb{T})$-modules with respect to the convolution that include the Banach algebra $\mathcal{I}(\mathbb{T})$ as a dense set.

In this main section we will be interested in the multilinear operators

$$B : \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}) \to Y$$

and the n-convolution map

$$* : \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}) \to \mathcal{U}(\mathbb{T}),$$

where $\mathcal{U}(\mathbb{T}) \in \{\mathcal{L}^p(\mathbb{T})\,(1 \le p < \infty), \mathcal{C}(\mathbb{T}), \mathcal{W}(\mathbb{T})\}$. By using the associativity of the convolution we can write that

$$\begin{aligned} f_1 * f_2 * \cdots * f_{n-1} * f_n &= f_1 * (f_2 * (\cdots * (f_{n-1} * f_n) \ldots)) \\ &= ((\ldots((f_1 * f_2) * f_3) \cdots * f_{n-1}) * f_n), \end{aligned}$$

where $f_1, \ldots, f_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $f_n \in \mathcal{U}(\mathbb{T})$.

Remark 3.1. The convolution product $*$ is a norming product from $\mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T})$ to $\mathcal{U}(\mathbb{T})$, where $\mathcal{U}(\mathbb{T}) \in \{\mathcal{L}^p(\mathbb{T})\,(1 \le p < \infty), \mathcal{W}(\mathbb{T}), \mathcal{C}(\mathbb{T})\}$. Indeed, this is seen from the Cohen factorization theorem and the factorization $\mathcal{L}^1(\mathbb{T}) * \mathcal{U}(\mathbb{T}) = \mathcal{U}(\mathbb{T})$ for the mentioned function spaces (see [11] and [17, Remark 38.6]). For $f \in \mathcal{U}(\mathbb{T})$, it is possible to write a factorization $f = h * g$ for some $h \in \mathcal{L}^1(\mathbb{T})$ and $g \in \mathcal{U}(\mathbb{T})$ by the factorization $\mathcal{L}^1(\mathbb{T}) * \mathcal{U}(\mathbb{T}) = \mathcal{U}(\mathbb{T})$. Besides, the function f can be written as $f = f_1 * \cdots * f_{n-1}$ by Cohen's factorization theorem, where $f_1, \ldots, f_{n-1} \in \mathcal{L}^1(\mathbb{T})$.

Theorem 3.2. *Consider a Banach-space-valued n-linear operator $B : \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}) \to Y$, where $\mathcal{U}(\mathbb{T}) \in \{\mathcal{L}^p(\mathbb{T})\,(1 \le p < \infty), C(\mathbb{T}), W(\mathbb{T})\}$. The following statements are equivalent:*

(i) *The map B is zero product preserving.*

(ii) *The operator B is $*$-factorable for the norming product*

$$* : \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}) \to \mathcal{U}(\mathbb{T}).$$

(iii) *There exists a constant $k > 0$ such that, for every finite set of functions $(f_i^j)_{i=1}^m \in \mathcal{L}^1(\mathbb{T})\,(1 \le j \le n-1)$ and $(f_i^n)_{i=1}^m \in \mathcal{U}(\mathbb{T})$,*

$$\Big\| \sum_{i=1}^m B(f_i^1, f_i^2, \ldots, f_i^n) \Big\|_Y \le k \Big\| \sum_{i=1}^m f_i^1 * f_i^2 * \cdots * f_i^n \Big\|_{\mathcal{U}}.$$

Thus, the following diagram commutes for a certain linear operator $T : \mathcal{U}(\mathbb{T}) \to Y$:

$$\begin{array}{ccc} \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}) & \xrightarrow{\quad B \quad} & Y. \\ \Big\downarrow{\scriptstyle *} & \nearrow_{T} & \\ \mathcal{U}(\mathbb{T}) & & \end{array}$$

Proof. Let us consider a zero product preserving multilinear operator $B : \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}) \to Y$ and denote by $\{e_\tau\}_{\tau\in\Lambda}$ the left approximate identity of $\mathcal{L}^1(\mathbb{T})$ with bound 1. As stated before, this is an approximate identity for $\mathcal{U}(\mathbb{T})$ as well, and, in addition, for all τ, e_τ is a positive trigonometric polynomial that can be represented by the form $e_\tau(t) = \sum_{k=-N_\tau}^{N_\tau} \alpha_k^\tau \exp(ikt)$.

Consider the trigonometric polynomials $f_1, \ldots, f_n \in \mathcal{I}(\mathbb{T})$ and represent them as $f_j = \sum_{l=-N_j}^{N_j} f_l^j \exp(ilt)$ for all $j \in \{1, 2, \ldots, n\}$. Since the convolution of $\exp(ikt)$ and $\exp(ilt)$ is zero valued unless k and l are the same, i.e., $\exp(ikt) * \exp(ilt) = 0$ ($k \neq l$), it is seen that

$$e_\tau * f_j(t) = \sum_{k=-N_{\tau j}}^{N_{\tau j}} \alpha_k^\tau f_k^j \exp(ikt)$$

for all $j \in \{1, 2, \ldots, n\}$ and $\tau \in \Lambda$, where $N_{\tau j} = \min\{N_\tau, N_j\}$.

Let us fix $\tau \in \Lambda$ and define the multilinear operator $B_\tau(h_1, \ldots, h_n) = B(e_\tau * h_1, \ldots, e_\tau * h_n)$ for all $h_1, \ldots, h_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $h_n \in \mathcal{U}(\mathbb{T})$. Clearly, the net $\{B_\tau\}_{\tau\in\Lambda}$ of operators consists of well-defined, multilinear operators. Due to the assumption of zero product preservation of the map B, we get that

$$\begin{aligned} &B_\tau(f_1, \ldots, f_n) \\ &\quad = B(e_\tau * f_1, \ldots, e_\tau * f_n) \\ &\quad = B\left(\sum_{k_1=-N_{\tau 1}}^{N_{\tau 1}} \alpha_{k_1}^\tau f_{k_1}^1 \exp(ik_1 t), \ldots, \sum_{k_n=-N_{\tau n}}^{N_{\tau n}} \alpha_{k_n}^\tau f_{k_n}^n \exp(ik_n t) \right) \\ &\quad = \sum_{k=-N}^{N} (\alpha_k^\tau)^n f_k^1 \ldots f_k^n B(\exp(ikt), \ldots, \exp(ikt)) \quad (N = \min\{N_{\tau 1}, \ldots, N_{\tau n}\}) \\ &\quad = B\left(\sum_{k=-N}^{N} \alpha_k^\tau \exp(ikt), \sum_{k=-N}^{N} \alpha_k^\tau \exp(ikt), \ldots, \sum_{k=-N}^{N} \alpha_k^\tau f_k^1 \ldots f_k^n \exp(ikt) \right) \\ &\quad = B(e_\tau, e_\tau, \ldots, e_\tau, e_\tau * f_1 * \cdots * f_n) \end{aligned}$$

for the trigonometric polynomials $f_1, \ldots, f_n$. Since the algebra $\mathcal{I}(\mathbb{T})$ is dense in both $\mathcal{L}^1(\mathbb{T})$ and $\mathcal{U}(\mathbb{T})$, the equality above holds for arbitrary functions in $\mathcal{L}^1(\mathbb{T})$ and $\mathcal{U}(\mathbb{T})$ by the continuity of B and the convolution product $*$. That is, for every $h_1, \ldots, h_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $h_n \in \mathcal{U}(\mathbb{T})$,

$$B_\tau(h_1, \ldots, h_n) = B(e_\tau, \ldots, e_\tau, e_\tau * h_1 * \cdots * h_n).$$

Secondly, define the map $T_\tau : \mathcal{U}(\mathbb{T}) \to Y$ given by $T_\tau(h) = T_\tau(h_1 * \cdots * h_n) = B_\tau(h_1, \ldots, h_n)$ for all $h = h_1 * \cdots * h_n$ and each $\tau \in \Lambda$, where $h_1, \ldots, h_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $h_n \in \mathcal{U}(\mathbb{T})$. The class $\{T_\tau\}_{\tau\in\Lambda}$ of maps is a net of well-defined, linear and continuous operators. Indeed, for all τ, it is easily seen that T_τ is well defined and linear by linearity of the multilinear operator B_τ in each variable. The boundedness of T_τ is obtained by the bounedness of B and algebraic properties as follows:

$$\begin{aligned}\|T_\tau(h_1 * \cdots * h_n)\|_Y &= \|B_\tau(h_1, \ldots, h_n)\|_Y \\ &= \|B(e_\tau, \ldots, e_\tau, e_\tau * h_1 * \cdots * h_n)\|_Y \\ &\le \|B\| \|e_\tau\|_{\mathcal{L}^1}^{n-1} \|e_\tau * h_1 * \cdots * h_n\|_{\mathcal{U}} \\ &\le \|B\| \|e_\tau * h_1 * \cdots * h_n\|_{\mathcal{U}} \\ &\le \|B\| \|e_\tau\|_{\mathcal{L}^1} \|h_1 * \cdots * h_n\|_{\mathcal{U}} = K\|h_1 * \cdots * h_n\|_{\mathcal{U}}.\end{aligned}$$

Besides, the operator T_τ is independent of the representation of the function h. To see this, assume $h = h_1 * \cdots * h_n = h'_1 * \cdots * h'_n$. Then we get

$$\begin{aligned}T_\tau(h_1 * \cdots * h_n) &= B(e_\tau, \ldots, e_\tau, e_\tau * h_1 * \cdots * h_n) \\ &= B(e_\tau, \ldots, e_\tau, e_\tau * h'_1 * \cdots * h'_n) = T_\tau(h'_1 * \cdots * h'_n).\end{aligned}$$

Therefore, the net $\{T_\tau\}_{\tau\in\Lambda}$ is a net of continuous linear operators. On the other hand, this net is pointwise convergent for each $h = h_1 * \cdots * h_n \in \mathcal{U}(\mathbb{T})$. Indeed,

$$\begin{aligned}\lim_\tau T_\tau(h_1 * \cdots * h_n) &= \lim_\tau B_\tau(h_1, \ldots, h_n) = \lim_\tau B(e_\tau * h_1, \ldots, e_\tau * h_n) \\ &= B(\lim_\tau e_\tau * h_1, \ldots, \lim_\tau e_\tau * h_n) = B(h_1, \ldots, h_n).\end{aligned}$$

So, $\{T_\tau(h_1 * \cdots * h_n)\}_{\tau\in\Lambda}$ converges to $B(h_1, \ldots, h_n)$ for all $h = h_1 * \cdots * h_n$. Define the pointwise limit $T(h_1 * \cdots * h_n) = \lim_\tau T_\tau(h_1 * \cdots * h_n)$. It is clear that the limit map T is well defined and linear. Let us see the boundedness of the limit operator. By using the version of the Uniform Boundedness Principle (Banach–Steinhaus theorem) for nets, it is seen that T is continuous (see [21, p. 141]). Summing up, the linear operator $T(h_1 * \cdots * h_n) = B(h_1, \ldots, h_n)$ is the desired factorization operator.

The implication between the statements (ii) and (iii) is obtained by [14, Lemma 2.3].

Finally, let us show the inequality given in the statement (iii) implies zero product preservation. Assume that $f_1 * \cdots * f_n = 0$. Then $\|f_1 * \cdots * f_n\| = 0$. By the inequality given in (iii), we get $\|B(f_1, \ldots, f_n)\| = 0$. So $B(f_1, \ldots, f_n) = 0$ and B is zpp. This finishes the proof. □

As a consequence of the above theorem we get the following isomorphism between $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}), Y)$ and $\mathfrak{L}(\mathcal{U}(\mathbb{T}), Y)$ (see the last paragraph of Section 2, p. 106, for the definition of $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}), Y)$).

Corollary 3.3. *The correspondence $T \longleftrightarrow B$ is an isomorphism between the Banach spaces $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T}), Y)$ and $\mathfrak{L}(\mathcal{U}(\mathbb{T}), Y)$. In particular, for $Y = \mathbb{R}$, we get $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{U}(\mathbb{T})) \cong (\mathcal{U}(\mathbb{T}))'$.*

Corollary 3.4. *The above theorem gives the following isomorphisms for specified* $\mathcal{U}(\mathbb{T})$*:*

- $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{W}(\mathbb{T}))\cong(\mathcal{W}(\mathbb{T}))'\cong(\ell^1)'=\ell^\infty$.
- $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{C}(\mathbb{T}))\cong(\mathcal{C}(\mathbb{T}))'=\mathcal{M}(\mathbb{T})$.
- $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{L}^p(\mathbb{T}))\cong(\mathcal{L}^p(\mathbb{T}))'=\mathcal{L}^q(\mathbb{T})\quad(\frac{1}{p}+\frac{1}{q}=1)$.
- $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{L}^1(\mathbb{T}))\cong(\mathcal{L}^1(\mathbb{T}))'=\mathcal{L}^\infty(\mathbb{T})$.
- $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{L}^2(\mathbb{T}))\cong(\mathcal{L}^2(\mathbb{T}))'=\mathcal{L}^2(\mathbb{T})\cong\ell^2$.

4. Domination inequalities and integral representations of $*$-factorable multilinear operators

We start this section by giving some basic notions from measure theory and Banach function spaces. Recall that $\mathcal{L}^p$-spaces are Banach function spaces (or Köthe function spaces) (see [22, Definition 1.b.17] for the definition of Köthe function spaces). Besides, $\mathcal{L}^p(\mathbb{T})$ ($p\geq 1$) has order continuous norm, that is, for every downward directed set $\{h_i\}_{i\in\Gamma}$ in $\mathcal{L}^p(\mathbb{T})$ converging to zero in dt-a.e., we get $\lim_i\|h_i\|=0$ (see [22, Definition 1.a.6]).

For the Köthe function spaces, another duality space called Köthe dual space (also called associate space) appears that will be denoted X^* for a Banach function space X. For a Banach function space $X(\Omega,\Sigma,\mu)$, the Köthe dual space X^* is the set of all the functionals belonging to the topological dual X' that can be represented as integrals, i.e., if ψ is such a functional, there is a measurable function g_ψ such that $\psi(f)=\int_\Omega g_\psi f\,d\mu$ for all $f\in X$. It is known that $X^*=X'$ for an order continuous Banach function space (see [22, p. 29]).

These notions will give some good integral representations for $*$-factorable multilinear operators. For example, since $\mathcal{L}^p(\mathbb{T})$ has order continuous norm for $1\leq p<\infty$ and $\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{L}^p(\mathbb{T}))\cong(\mathcal{L}^p(\mathbb{T}))'$ by Corollary 3.4, it is seen that a multilinear map $B\in\mathfrak{L}_0^n(\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{L}^p(\mathbb{T}))$ can be represented by the integral $B(f_1,\ldots,f_n)=\int_\mathbb{T}(f_1*\cdots*f_n)\,h\,dm$ for some $h\in\mathcal{L}^0(m)$.

Theorem 4.1. *For $1\leq p<\infty$ and any Banach-space-valued n-linear map $B:\mathcal{L}^1(\mathbb{T})\times\cdots\times\mathcal{L}^1(\mathbb{T})\times\mathcal{L}^p(\mathbb{T})\to Y$, the following are equivalent:*

(1) For any finite subsets $\{f_j^i\}_{j=1}^m\in\mathcal{L}^1(\mathbb{T})$ $(i=1,\ldots,n-1)$ and $\{f_j^n\}_{j=1}^m\in\mathcal{L}^p(\mathbb{T})$,

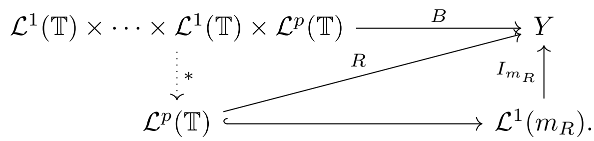
$$\sum_{j=1}^m\|B(f_j^1,\ldots,f_j^n)\|_Y^p\leq\sum_{j=1}^m\|f_j^1*\cdots*f_j^n\|_{\mathcal{L}^p}^p.$$

(2) There is a Y-valued vector measure μ on $\mathbb{T}$ such that $\mathcal{L}^p(\mathbb{T})\hookrightarrow\mathcal{L}^1(\mu)$ and

$$B(f_1,\ldots,f_n)=\int_\mathbb{T}(f_1*\cdots*f_n)\,d\mu$$

for all $f_1,\ldots,f_{n-1}\in\mathcal{L}^1(\mathbb{T})$ and $f_n\in\mathcal{L}^p(\mathbb{T})$.

Proof. (1)$\Rightarrow$(2) By the inequality given in (1) it is easily seen that the multilinear operator B is zpp and factors through a linear operator R and convolution $*$. By the order continuity of the space $\mathcal{L}^p(\mathbb{T})$, the linear map $R:\mathcal{L}^p(\mathbb{T})\to Y$ gives a countably additive vector measure, called the *vector measure associated to* R, defined by $m_R = R(\chi_A)$ for all $A \in \mathcal{B}(\mathbb{T})$. Using [23, Theorem 4.14], we get that the order continuous Banach function space $\mathcal{L}^1(m_R)$ is the largest space where $\mathcal{L}^p(\mathbb{T})$ is continuously embedded. Besides, there is a unique Y-valued, continuous linear extension of R and this extension is the integration operator $I_{m_R}(h) = \int_{\mathbb{T}} h\, dm_R$ for all $h \in \mathcal{L}^1(m_R)$. Thus the following diagram commutes:



By the continuity of the linear map R, the space $\mathcal{L}^1(m_R)$ is a Banach function space over a Rybakov measure η for m_R, and η is m-continuous (see [13, Sections I.2 and IX.2], [23, Theorem 3.7 (iv)]). Thus, the inclusion can be changed by the identification of classes $[f]_m \mapsto [f]_\eta$ that defines a continuous operator called an inclusion/quotient map that preserves the factorization. By using all of the information above, we obtain

$$B(f_1, \dots, f_n) = \int_{\mathbb{T}} (f_1 * \cdots * f_n)\, dm_R$$

for all $f_1, \dots, f_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $f_n \in \mathcal{L}^p(\mathbb{T})$. The measure m_R is the desired measure μ.

(2) $\Rightarrow$ (1) By direct computations and from the integral representation given in the statement (2), for every finite set of functions $\{f_i^j\}_{i=1}^m \in \mathcal{L}^1(\mathbb{T})$ $(1 \le j \le n-1)$ and $\{f_i^n\}_{i=1}^m \in \mathcal{L}^p(\mathbb{T})$, we get

$$\begin{aligned} \sum_{j=1}^m \left\| B(f_j^1, \dots, f_j^n) \right\|_Y^p &\le \sum_{j=1}^m \left\| (f_j^1 * \cdots * f_j^n) \right\|_{\mathcal{L}^1(m_R)}^p \\ &\le \sum_{j=1}^m \left\| (f_j^1 * \cdots * f_j^n) \right\|_{\mathcal{L}^p}^p. \end{aligned}$$ □

Theorem 4.2. *For an n-linear map $B : \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{C}(\mathbb{T}) \to Y$ and $1 \le p < \infty$, the following statements are equivalent:*

(1) *The following inequality holds for some constant $k > 0$ for every finite set $\{f_j^i\}_{j=1}^m \in \mathcal{L}^1(\mathbb{T})$ $(i = 1, \dots, n-1)$ and $\{f_j^n\}_{j=1}^m \in \mathcal{C}(\mathbb{T})$:*

$$\left(\sum_{j=1}^m \left\| B(f_j^1, \dots, f_j^n) \right\|_Y^p \right)^{1/p} \le k \left\| \left(\sum_{j=1}^m (f_j^1 * \cdots * f_j^n)^p \right)^{1/p} \right\|. \tag{4.1}$$

(2) *The multilinear map B admits a factorization of the form*

$$\mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{C}(\mathbb{T}) \xrightarrow{\ *\ } \mathcal{C}(\mathbb{T}) \xrightarrow{\ T\ } Y$$

such that T is a p-summing operator.

(3) *There are a Borel–Radon measure η on $\mathbb{T}$ and a linear operator $S : \mathcal{L}^p(\eta) \to Y$ such that B has a factorization of the form*

$$\begin{array}{ccc} \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{C}(\mathbb{T}) & \xrightarrow{\ B\ } & Y \\ \Big\downarrow {\scriptstyle *} & & \Big\uparrow {\scriptstyle S} \\ \mathcal{C}(\mathbb{T}) & \xrightarrow[{[i]}]{} & \mathcal{L}^p(\eta), \end{array}$$

where $[i]$ is an inclusion/quotient map.

In particular, if B satisfies one of the previous statements, then it has an integral representation $B(f_1, \ldots, f_n) = \int_{\mathbb{T}} (f_1 * \cdots * f_n)\, d\eta$ by a countably additive vector measure η on the Borel sets $\mathcal{B}(\mathbb{T})$ to Y for all $f_1, \ldots, f_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $f_n \in \mathcal{C}(\mathbb{T})$.

Proof. (1) ⇒ (2) Inequality (4.1) implies the zero product preservation of B. Therefore, it factors through the linear map $T : \mathcal{C}(\mathbb{T}) \to Y$ given by $B(f_1, \ldots, f_n) = T(f_1 * \cdots * f_n)$ for all $f_1, \ldots, f_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $f_n \in \mathcal{C}(\mathbb{T})$. From the given inequality, we get

$$\left(\sum_{j=1}^{m} \left\| T(f_j^1 * \cdots * f_j^n) \right\|_Y^p \right)^{1/p} \leq \left\| \left(\sum_{j=1}^{m} (f_j^1 * \cdots * f_j^n)^p \right)^{1/p} \right\|.$$

This shows the p-summability of the linear map T.

(2) ⇒ (3) By using a consequence of Pietsch's Domination Theorem (see [12, Corollary 2.15]) it is seen that there is a regular Borel probability measure η on $\mathbb{T}$ such that every p-summing linear continuous operator on $\mathcal{C}(\mathbb{T})$ factors through $\mathcal{L}^p(\eta)$.

(3) ⇒ (1) It is known that the natural inclusion $[i] : \mathcal{C}(\mathbb{T}) \to \mathcal{L}^p(\eta)$ is p-summing, thus $S \circ [i]$ is a p-summing linear operator by the ideal property of p-summing operators. Therefore, we obtain the inequality (4.1) as follows:

$$\begin{aligned} \left(\sum_{j=1}^{m} \left\| B(f_j^1, \ldots, f_j^n) \right\|_Y^p \right)^{1/p} &= \left(\sum_{j=1}^{m} \left\| S \circ [i](f_j^1 * \cdots * f_j^n) \right\|_Y^p \right)^{1/p} \\ &\leq k \left\| \left(\sum_{j=1}^{m} (f_j^1 * \cdots * f_j^n)^p \right)^{1/p} \right\|. \end{aligned}$$

Let us show the map B has an integral representation if one of the statements (1), (2) or (3) is satisfied. The operator $T : \mathcal{C}(\mathbb{T}) \to Y$ appearing in the factorization of B is weakly compact since it is a p-summing linear operator. Theorem 3.2 in [5] implies that the linear operator T defined on $\mathcal{C}(\mathbb{T})$ is weakly compact if and only if it has an integral representation by a countably additive vector measure η on the Borel sets $\mathcal{B}(\mathbb{T})$ to Y for all $f \in \mathcal{C}(\mathbb{T})$. This gives the representation of B by factorization. □

5. Applications: polynomials and integral operators

We will finish the paper by giving some applications related with n-homogeneous polynomials and Hilbert–Schmidt operators.

Polynomials. Recall that an n-linear map $B : \times_n X \to Y$ is called symmetric if

$$B(x_1, \dots, x_n) = B(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad (x_1, \dots, x_n \in X)$$

for any permutation σ of the first n natural numbers. $\mathfrak{L}_s^n(\times_n X, Y)$ denotes the space of symmetric multilinear operators defined on X to Y.

Remark 5.1. The space $\mathfrak{L}_0^n(\times_n \mathcal{L}^1(\mathbb{T}), Y)$ of $*$-factorable n-linear operators is a proper subspace of the space $\mathfrak{L}_s^n(\times_n \mathcal{L}^1(\mathbb{T}), Y)$. Indeed, it is seen that any zpp multilinear map $B : \times_n \mathcal{L}^1(\mathbb{T}) \to Y$ is symmetric by the factorization. By the commutativity of the convolution $*$, the symmetry is seen as follows:

$$B(f_1, \dots, f_n) = T(f_1 * \dots * f_n) = T(f_{\sigma(1)} * \dots * f_{\sigma(n)}) = B(f_{\sigma(1)}, \dots, f_{\sigma(n)}).$$

To show the reverse inclusion is not true in general let us give a counterexample. Consider the bilinear symmetric functional $B : \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^1(\mathbb{T}) \to \mathbb{C}$ defined by $B(f, g) = (\int_{[0,\pi]} f\, dt) \cdot (\int_{[0,\pi]} g\, dt)$ for $f, g \in \mathcal{L}^1(\mathbb{T})$. For the functions $f(t) = \exp(it)$ and $g(t) = \exp(3it)$, it is known that $\exp(it) * \exp(3it) = 0$. However,

$$B(f, g) = B(\exp(it), \exp(3it)) = \left(\int_{[0,\pi]} \exp(it)\, dt \right) \left(\int_{[0,\pi]} \exp(3it)\, dt \right) = -\frac{4}{3} \neq 0.$$

Thus, we obtain that symmetry does not imply zero product preservation, i.e., $\mathfrak{L}_0^n(\times_n \mathcal{L}^1(\mathbb{T}), Y) \subset \mathfrak{L}_s^n(\times_n \mathcal{L}^1(\mathbb{T}), Y)$.

A map $P : X \to Y$ is called an *n-homogeneous polynomial* if it is associated with an n-linear symmetric map $B : \times_n X \to Y$ such that $P(x) = B(x, \dots, x)$ for all $x \in X$. An n-homogeneous polynomial defined on the Banach algebra X is called *orthogonally additive* if $P(x+y) = P(x) + P(y)$ whenever $xy = 0$ for $x, y \in X$. We denote by $\mathcal{P}(^n X, Y)$ (resp., $\mathcal{P}_0(^n X, Y)$) the spaces of n-homogeneous polynomials (resp., n-homogeneous orthogonally additive polynomials) from X to Y. We will write $\mathcal{P}(^n X)$ and $\mathcal{P}_0(^n X)$ for $Y = \mathbb{R}$.

The relation between n-homogeneous polynomials and zero product preserving n-linear operators has been studied by various authors in recent years for both Banach algebras and vector lattices (see [2, 6, 14, 18, 24, 25] and references therein). Now we will give a similar result for the orthogonally additive polynomials defined on the Banach algebra $\mathcal{L}^1(\mathbb{T})$.

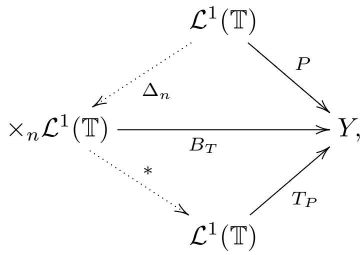
Theorem 5.2. *There is an isomorphism between the spaces $\mathfrak{L}_0^n(\times_n \mathcal{L}^1(\mathbb{T}), Y)$ and $\mathcal{P}_0(^n \mathcal{L}^1(\mathbb{T}), Y)$.*

Proof. Let us consider the zero product preserving map $B \in \mathfrak{L}_0^n(\times_n \mathcal{L}^1(\mathbb{T}), Y)$. Since B is symmetric and any symmetric multilinear map is associated to an n-homogeneous polynomial, it follows that the map B defines an n-homogeneous polynomial P_B such that it is orthogonally additive. Indeed, the orthogonal additivity of the polynomial P_B is seen as follows:

$$\begin{aligned} P_B(f+g) &= B(f+g,\dots,f+g) \\ &= \sum_{k=0}^{n} \binom{n}{k} B(f,\overset{k}{\ldots},f,g,\overset{n-k}{\ldots},g) \\ &= B(f,\overset{n}{\ldots},f) + B(g,\overset{n}{\ldots},g) \\ &= P_B(f) + P_B(g) \end{aligned}$$

by the zero product preservation of the map B, where $f, g \in \mathcal{L}^1(\mathbb{T})$ satisfy $f * g = 0$.

Now, consider an orthogonally additive n-homogeneous polynomial P. By [2, Theorem 3.1], the orthogonal additivity of the polynomial P implies the existence of a linear operator $T_P : \mathcal{L}^1(\mathbb{T}) \to Y$ defined by $P(f) = T_P(f * \overset{n}{\ldots} * f)$ for all $f \in \mathcal{L}^1(\mathbb{T})$. Since the linear operator T_P defines a $*$-factorable n-linear operator $B_T : \times_n \mathcal{L}^1(\mathbb{T}) \to Y$ by $T_P(f * \overset{n}{\ldots} * f) = B_T(f,\dots,f)$, it is seen that the orthogonally additive n-homogeneous polynomial P is associated with a $*$-factorable n-linear operator. This can be represented by the following diagram:



where Δ_n is the canonical embedding, called *diagonal mapping*, from $\mathcal{L}^1(\mathbb{T})$ to $\times_n \mathcal{L}^1(\mathbb{T})$. □

This theorem gives an integral representation for orthogonally additive polynomial forms defined on $\mathcal{L}^1(\mathbb{T})$.

Corollary 5.3. *$\mathcal{P}_0(^n\mathcal{L}^1(\mathbb{T})) \cong \mathcal{L}^\infty(\mathbb{T})$ and every polynomial $P \in \mathcal{P}_0(^n\mathcal{L}^1(\mathbb{T}))$ has an integral representation $P(f) = \int_{\mathbb{T}} (f * \cdots * f) g \, dt$ for some $g \in \mathcal{L}^\infty(\mathbb{T})$.*

Proof. By Corollary 3.4 and Theorem 5.2, we get

$$\mathcal{P}_0(^n\mathcal{L}^1(\mathbb{T})) \cong \mathfrak{L}_0^n(\times_n \mathcal{L}^1(\mathbb{T})) \cong (\mathcal{L}^1(\mathbb{T}))' = \mathcal{L}^\infty(\mathbb{T}).$$

By the order continuity of the space $\mathcal{L}^1(\mathbb{T})$, any orthogonally additive polynomial form P is represented by an integral that is the factorization of a $*$-factorable multilinear form $B_P : \times_n \mathcal{L}^1(\mathbb{T}) \to \mathbb{R}$ as $P(f) = \int_{\mathbb{T}} (f * \cdots * f) g \, dt$ by a measurable function g. □

Integral operators. Our last applications are integral and series representations of $*$-factorable multilinear operators defined on $\mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^2(\mathbb{T})$. For these representations we will use the properties of Hilbert–Schmidt operators (see [20, 1.b.14] for the definition of Hilbert–Schmidt operators).

Corollary 5.4. *Let $B : \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^2(\mathbb{T}) \to \mathcal{L}^2(\mathbb{T})$ be an n-linear map. Then the following statements imply each other:*

1. *There exists a constant $k > 0$ for every finite set $\{f_j^i\}_{j=1}^m \in \mathcal{L}^1(\mathbb{T})$ ($i = 1, \dots, n-1$) and $\{f_j^n\}_{j=1}^m \in \mathcal{L}^2(\mathbb{T})$ such that*

$$\left(\sum_{j=1}^m \left\| B(f_j^1, \dots, f_j^n) \right\|_{\mathcal{L}^2}^2 \right)^{1/2} \leq k \sup_{\phi \in B_{\mathcal{L}^2(\mathbb{T})}} \left(\sum_{j=1}^m |\langle f_j^1 * f_j^2 * \cdots * f_j^{n-1} * f_j^n, \phi \rangle|^2 \right)^{1/2}.$$

2. *The multilinear map B admits a factorization of the form*

$$\mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^2(\mathbb{T}) \xrightarrow{\;*\;} \mathcal{L}^2(\mathbb{T}) \xrightarrow{\;T\;} Y$$

   such that T is a 2-summing operator.

3. *There is a kernel $K \in \mathcal{L}^2(\mathbb{T}^2, m^2)$ such that B has the following integral representation:*

$$B(f_1, \dots, f_n)(x) = \int_{\mathbb{T}} K(x,y)(f_1 * \cdots * f_n)\, dm(y) \quad x\, dm\text{-a.e.}$$

   for all $f_1, \dots, f_{n-1} \in \mathcal{L}^1(\mathbb{T})$, $f_n \in \mathcal{L}^2(\mathbb{T})$.

4. *There is a regular probability measure μ on a compact Hausdorff space S and a linear operator $\tilde{B} : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mathbb{T})$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{L}^1(\mathbb{T}) \times \cdots \times \mathcal{L}^1(\mathbb{T}) \times \mathcal{L}^2(\mathbb{T}) & & \xrightarrow{\quad B \quad} & & \mathcal{L}^2(\mathbb{T}) \\ \downarrow{\scriptstyle *} & & & & \uparrow{\scriptstyle \tilde{B}} \\ \mathcal{L}^2(\mathbb{T}) & \xrightarrow[I]{\qquad} & \mathcal{C}(S) & \xrightarrow[J]{} & \mathcal{L}^2(\mu), \end{array}$$

   where I is the formal inclusion map and J is the canonical map.

Proof. (1) $\Leftrightarrow$ (2) The inequality given in the first statement implies zero product preservation, hence factorization, of B such that the operator T appearing in the factorization is 2-summing by the definition of summing operators and the inequality. The converse is clear.

(2) $\Leftrightarrow$ (3) The map $T : \mathcal{L}^2(\mathbb{T}) \to \mathcal{L}^2(\mathbb{T})$ is a 2-summing operator if and only if it is a Hilbert–Schmidt operator and this implies an integral representation $T(f)(x) = \int_{\mathbb{T}} K(x,y) f(y)\, dm(y)\ x\, dm$-a.e. for a kernel $K \in \mathcal{L}^2(\mathbb{T}^2, m^2)$ and all $f \in \mathcal{L}^2(\mathbb{T})$ (see [12, Theorem 4.10] and [20, Prop. 1.b.15]). Since any $f \in \mathcal{L}^2(\mathbb{T})$ can be written as $f = f_1 * \cdots * f_n$ for $f_1, \dots, f_{n-1} \in \mathcal{L}^1(\mathbb{T})$ and $f_n \in \mathcal{L}^2(\mathbb{T})$, the map T defines the multilinear map B by the integral representation. For the converse, assume that B has the integral representation, then clearly it is zpp and admits the factorization such that T is a Hilbert–Schmidt operator due to the integral representation. Since Hilbert–Schmidt operators are 2-summing, so is T (see [12, Theorem 4.10] and [20, Prop. 1.b.15]).

(4) $\Leftrightarrow$ (1) B admits such a factorization if and only if the linear map $\tilde{B} \circ J \circ I$ is 2-summing (see [12, Corollary 2.16]). Thus, the inequality given in (1) follows by the 2-summability of $\tilde{B} \circ J \circ I$ and the factorization $B = \tilde{B} \circ J \circ I \circ *$. For the

converse, the first statement implies the $*$-factorability, and therefore we have the factorization $B = T \circ *$ with T 2-summing.

Therefore, the commutative diagram follows by [12, Corollary 2.16] again. □

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