TRIVIAL EXTENSIONS OF MONOMIAL ALGEBRAS

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Abstract. We describe the ideal of relations for the trivial extension \( T(\Lambda) \) of a finite-dimensional monomial algebra \( \Lambda \). When \( \Lambda \) is, moreover, a gentle algebra, we solve the converse problem: given an algebra \( B \), determine whether \( B \) is the trivial extension of a gentle algebra. We characterize such algebras \( B \) through properties of the cycles of their quiver, and show how to obtain all gentle algebras \( \Lambda \) such that \( T(\Lambda) \cong B \). We prove that indecomposable trivial extensions of gentle algebras coincide with Brauer graph algebras with multiplicity one in all vertices in the associated Brauer graph, result proven by S. Schroll.

1. Introduction

Let \( \Lambda \) be a finite-dimensional \( k \)-algebra (associative, with identity) over an algebraically closed field \( k \). Consider the trivial extension algebra \( T(\Lambda) = \Lambda \rtimes D(\Lambda) \) of \( \Lambda \) by the \( \Lambda \)-bimodule \( D(\Lambda) = \text{Hom}_k(\Lambda, k) \), that is, \( T(\Lambda) = \Lambda \oplus D(\Lambda) \) as \( k \)-vector space and the multiplication in \( T(\Lambda) \) is given by \( (a, f)(b, g) = (ab, ag + fb) \) for \( a, b \in \Lambda \) and \( f, g \in D(\Lambda) \).

The ordinary quiver of the trivial extension \( T(\Lambda) \) of a finite-dimensional algebra \( \Lambda = kQ_\Lambda/I_\Lambda \) was described by Fernández and Platzeck [3], where also the relations of such trivial extension are given under the assumption that any oriented cycle in the ordinary quiver of \( \Lambda \) is zero in \( \Lambda \).

In this work we will describe the relations for \( T(\Lambda) \) when \( \Lambda \) is monomial. These algebras form a broad family containing, among others, string algebras and gentle algebras.

If \( \Lambda \) is monomial, the ordinary quiver of \( T(\Lambda) \) is obtained from \( Q_\Lambda \) by adding \( t \) arrows, where \( t \) is the number of paths of \( kQ_\Lambda \) which are maximal nonzero in \( \Lambda \). For each of these maximal paths we add an arrow in the opposite direction. In this way, we obtain an oriented cycle, which we call elementary. We prove that each

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nonzero path $\gamma$ of the quiver of $T(\Lambda)$ is contained in an elementary cycle $C$ such that $C = \gamma \mu$ for some path $\mu$, and we say that $\mu$ is a supplement of $\gamma$. The ideal of relations for $T(\Lambda)$ is generated by

(i) the paths not contained in an elementary cycle, and

(ii) the elements $\mu - \mu'$, where $\mu, \mu'$ are different paths from $kQ_{T(\Lambda)}$ with a common supplement $\gamma$ in elementary cycles $C$ and $C'$, respectively.

When $\Lambda$ is gentle, elementary cycles of $kQ_{T(\Lambda)}$ do not overlap (that is, have no common arrows). Thus the description of $I_{T(\Lambda)}$ is easier, because the generators described in (ii) can be replaced by the elements $C - C'$, where $C$ and $C'$ are elementary cycles starting at the same vertex of $Q_{T(\Lambda)}$.

Let $\Lambda$ be a gentle algebra. Then the bound quiver of $B = T(\Lambda)$ satisfies the following properties:

(T$_1$) Any permutation of a maximal cycle is a maximal cycle.
(T$_2$) Any path $u$ of $kQ_B$ nonzero in $B$ is contained in a maximal cycle of $kQ_B$, which is unique up to permutations if $u$ is nontrivial.
(T$_3$) There are at most two different cycles from $j$ to $j$ of $kQ_B$ maximal nonzero in $B$ for any vertex $j$ of $(Q_B)_0$. If there are two such cycles, they are equal in $B$.
(T$_4$) If $\alpha_s \cdots \alpha_1 : j \to j$ is a nonzero cycle of $kQ_B$ which is not maximal, then $\alpha_1 \alpha_s = 0$ in $B$ and the dimension of the endomorphism ring of the projective associated to the vertex $j$ is four.

We prove that these properties characterize trivial extensions of gentle algebras. Moreover, we show how to find all gentle algebras $\Lambda$ such that $T(\Lambda) \simeq B$, as stated in the following result.

**Theorem.** Let $B = kQ_B/I_B$ be a finite-dimensional algebra satisfying (T$_1$), (T$_2$), (T$_3$) and (T$_4$).

(i) Let $Q$ be the quiver obtained from $Q_B$ by eliminating exactly one arrow of each cycle of $Q_B$ maximal in $B$, and let $I = kQ \cap I_B$. Then $\Lambda = kQ/I$ is a gentle algebra and $T(\Lambda) \cong B$.

(ii) If $\Lambda$ is a gentle algebra such that $T(\Lambda) \cong B$, then $\Lambda = kQ/I$, with $Q$ and $I$ as in (i).

Finally, we prove that trivial extensions of indecomposable gentle algebras coincide with Brauer graph algebras with multiplicity one in all vertices in the associated Brauer graph, result proven by S. Schroll in [4] with a different approach. To prove this result we show that an indecomposable algebra $B$ is a Brauer graph algebra with multiplicity one in all vertices in the associated Brauer graph if and only if its cycles satisfy the properties (T$_1$), (T$_2$), (T$_3$) and (T$_4$) which characterize trivial extensions of gentle algebras.
2. Preliminaries

Throughout this paper $k$ will denote an algebraically closed field. The algebras considered are finite-dimensional $k$-algebras which we will also assume to be basic and indecomposable. Thus, for an algebra $\Lambda$, we have that $\Lambda \simeq kQ_{\Lambda}/I_{\Lambda}$, where $Q_{\Lambda}$ is a finite connected quiver and the ideal $I_{\Lambda}$ is admissible. Given an element $x$ of $kQ_{\Lambda}$, we will denote by $\overline{x}$ the corresponding element of $kQ_{\Lambda}/I_{\Lambda}$.

If $Q$ is a quiver, we will denote by $Q_0$ the set of vertices, and by $Q_1$ the set of arrows between vertices. For each arrow $\alpha$, $s(\alpha)$ and $e(\alpha)$ will denote the start and end vertices of $\alpha$, respectively. For each $i \in Q_0$, $S_i$ will be the simple $\Lambda$-module associated to $i$, and $P_i$ and $I_i$ will denote the projective cover and injective envelope of $S_i$, respectively. Thus, if $e_i$ is the trivial path of $kQ_{\Lambda}$ corresponding to the vertex $i$ of $Q_{\Lambda}$, then $P_i = \Lambda e_i$.

We recall now the description of the ordinary quiver for $T(\Lambda)$ given in [3]. Let $\Lambda$ be an algebra with ordinary quiver $Q_{\Lambda}$, and let $p_1, \ldots, p_t$ be elements in $kQ_{\Lambda}$ such that $M = \{p_1, \ldots, p_t\}$ is a $k$-basis for the socle $\text{soc}_{k}\Lambda$ of $\Lambda$ considered as a module over the enveloping algebra $\Lambda^e$, where each $p_i$ is a linear combination of paths with the same origin $s(p_i)$ and the same endpoint $e(p_i)$. Then the ordinary quiver of $T(\Lambda)$ is given by

(i) $(Q_{T(\Lambda)})_0 = (Q_{\Lambda})_0$
(ii) $(Q_{T(\Lambda)})_1 = (Q_{\Lambda})_1 \cup \{\beta_{p_1}, \ldots, \beta_{p_t}\}$, where $\beta_{p_i}$ is an arrow from $e(p_i)$ to $s(p_i)$ for each $i = 1, \ldots, t$.

The notion of elementary cycle, given in [3], is essential in the description of the relations for $\Lambda$. We recall the definition now. Let $p_{t+1}, \ldots, p_r$ be paths of $Q_{T(\Lambda)}$ such that $B = \{p_1, \ldots, p_t, p_{t+1}, \ldots, p_r\}$ is a $k$-basis of $\Lambda$, and let $B^* = \{p_1^*, \ldots, p_r^*\}$ denote the dual basis. Following [3], we say that an oriented cycle $C$ of $kQ_{T(\Lambda)}$ is elementary if it is of the form $C = \alpha_j \cdots \alpha_1 \beta_p \alpha_m \cdots \alpha_{j+1}$, where $\alpha_1, \ldots, \alpha_m \in (Q_{\Lambda})_1$, $p \in M$ and $p^* (\overline{\alpha_m} \cdots \overline{\alpha_1}) \neq 0$.

It follows from the definition that cyclic permutations $\sigma_j \cdots \sigma_i \sigma_m \cdots \sigma_{j+1}$ of elementary cycles $\sigma_m \cdots \sigma_1$ are elementary cycles. When we refer to a permutation of a cycle we will always mean a cyclic permutation of it.

We will say that a path $\gamma$ in $kQ_{\Lambda}$ is maximal in $\Lambda$ if $0 \neq \overline{\gamma} \in \Lambda$, and $\overline{\gamma} \overline{\alpha} = 0$ for any arrow $\alpha$ of $Q_{\Lambda}$.

We will also need the notion of supplement given in [3]. Let $q$ be a path in an elementary cycle $C$. If $q = C$ we say that the supplement of $q$ in $C$ is the trivial path $e_{s(q)}$. Otherwise the supplement of $q$ in $C$ is the path consisting of the remaining arrows of $C$. More precisely, if $C = \mu_s \cdots \mu_1$ is an elementary cycle, with $\mu_1, \ldots, \mu_s \in (Q_{\Lambda})_1$, and $q = \mu_{j+r} \cdots \mu_j$ is a subpath of $C$, then the supplement of $q$ in $C$ is $\mu_{j-1} \cdots \mu_1 \mu_s \cdots \mu_{j+r+1}$.

We consider, as in [3], the morphism of $k$-algebras $\Phi : kQ_{T(\Lambda)} \to T(\Lambda)$ defined by

$$
\Phi(e_i) = (\overline{\gamma_i}, 0) \quad \text{for } i = 1, \ldots, n,
\Phi(\alpha) = (\overline{\alpha}, 0), \quad \Phi(\beta_p) = (0, \overline{p}^*) \quad \text{for every } \alpha \in (Q_{\Lambda})_1, p \in M.
$$
Then $\Phi$ is surjective and so we can identify $T(\Lambda)$ with $kQ_{T(\Lambda)}/\ker \Phi$. Thus, the class $\bar{x}$ of an element $x \in kQ_{T(\Lambda)}$ is nonzero in $T(\Lambda)$ if and only if $\Phi(x) \neq 0$. Any path in $kQ_{T(\Lambda)}$ containing at least two arrows $\beta_p$ is zero in $T(\Lambda)$. Moreover, a path in $kQ_{\Lambda}$ which is not zero in $\Lambda$ is always contained in an elementary cycle (see [3, Remark 3.3]) and is therefore not maximal in $T(\Lambda)$. Thus maximal paths in $T(\Lambda)$ contain exactly one arrow $\beta_p$.

Associated with $\Phi$ are the compositions $\varphi_1 = \pi_1 \Phi : kQ_{T(\Lambda)} \to \Lambda$ and $\varphi_2 = \pi_2 \Phi : kQ_{T(\Lambda)} \to D(\Lambda)$, where $\pi_1, \pi_2$ are the projections induced by the decomposition $T(\Lambda) = \Lambda \oplus D(\Lambda)$.

Notice that an elementary cycle $C = \alpha_j \cdots \alpha_1 \beta_p \alpha_m \cdots \alpha_{j+1} : e \to e$ is nonzero in $T(\Lambda)$. In fact, $\Phi(C) = (0, \varphi_2(C))$, and using the structure of $D(\Lambda)$ as a $\Lambda$-bimodule we obtain

$$
\varphi_2(C)(e) = \varphi_2(\alpha_j \cdots \alpha_1 \beta_p \alpha_m \cdots \alpha_{j+1})(\bar{e}) = \overline{\alpha_m \cdots \alpha_{j+1} e \alpha_j \cdots \alpha_1} = \overline{\alpha_m \cdots \alpha_1} \neq 0.
$$

3. Trivial extensions of monomial algebras

From now on we will assume that $\Lambda$ is a monomial algebra. That is, $\Lambda \simeq kQ_{\Lambda}/I_{\Lambda}$, where $I_{\Lambda}$ is an admissible ideal generated by paths. In this case, the set of classes of maximal paths is a basis for soc $\Lambda$. We will always assume that $p_1, \ldots, p_t$ in the chosen basis $\mathcal{M} = \{\overline{p_1}, \ldots, \overline{p_t}\}$ of soc $\Lambda$ are maximal paths. The extension $\mathcal{B}$ of $\mathcal{M}$ to a basis of $\Lambda$ considered above is then the set of classes of paths in $kQ_{\Lambda}$ that are nonzero in $\Lambda$. Notice that this set is a basis for $\Lambda$ because $\Lambda$ is a monomial algebra.

In this section we will find generators for the ideal of relations of the trivial extension of a monomial algebra. We do it by adapting the approach followed in [3] to our case.

In order to describe the relations for $T(\Lambda)$ we have to find generators for $\ker \Phi$. The next proposition gives a description of the elementary cycles of $T(\Lambda)$.

**Proposition 3.1.** Let $\Lambda = kQ_{\Lambda}/I_{\Lambda}$ be a monomial algebra and $C$ an oriented cycle of $kQ_{T(\Lambda)}$. Then the following conditions are equivalent:

(i) $C$ is an elementary cycle.

(ii) $C$ is a cyclic permutation of the cycle $p\beta_p$ for some $p \in \mathcal{M}$.

(iii) $C$ is maximal in $T(\Lambda)$.

**Proof.** (i) $\rightarrow$ (ii) Let $C$ be an elementary cycle of $kQ_{T(\Lambda)}$, $C = \alpha_j \cdots \alpha_1 \beta_p \alpha_m \cdots \alpha_{j+1}$, with $\alpha_1, \ldots, \alpha_n \in (Q_{\Lambda})_1, p \in \mathcal{M}$. Since $\mathcal{B}$ is generated by all the paths that are nonzero in $\Lambda$ and $\overline{p}^{*}(\overline{\alpha_n \cdots \alpha_1}) \neq 0$, it follows that $\alpha_n \cdots \alpha_1 = p$. Thus $C$ is a permutation of the cycle $p\beta_p$.

(ii) $\rightarrow$ (i) Let $p \in \mathcal{M}$. Since $\overline{p}^{*}(\overline{p}) = 1$ the cycle $p\beta_p$ is elementary by definition, and so is any permutation $C$ of $p\beta_p$.

(i) $\rightarrow$ (iii) Let $C = \alpha_j \cdots \alpha_1 \beta_p \alpha_m \cdots \alpha_{j+1}$ be an elementary cycle, with $\alpha_n \cdots \alpha_1 = p \in \mathcal{M}$. We know that $\overline{C} \neq 0$ in $T(\Lambda)$. Suppose $C$ is not maximal. Then there is an arrow $\alpha \in Q_{T(\Lambda)}$ such that $\alpha \overline{C} \neq 0$ or $\overline{C} \alpha \neq 0$. We assume that $\alpha \overline{C} \neq 0$. 

Thus \( \Phi(\alpha \alpha_j \cdots \alpha_1 \beta_p \alpha_n \cdots \alpha_{j+1}) \neq (0, 0) \). Therefore,
\[
\Phi(\alpha \alpha_j \cdots \alpha_1) \Phi(\beta_p) \Phi(\alpha_n \cdots \alpha_{j+1}) = (\alpha \alpha_j \cdots \alpha_1, 0) (0, p^*) (\alpha_n \cdots \alpha_{j+1}, 0) \neq (0, 0).
\]
That is, \( (0, \alpha \alpha_j \cdots \alpha_1 p^* \alpha_n \cdots \alpha_{j+1}) \neq (0, 0) \). Thus in \( D(\Lambda) \) we have that \( \alpha \alpha_j \cdots \alpha_1 p^* \alpha_n \cdots \alpha_{j+1} \neq 0 \). So there is a path \( q \) of \( kQ_\Lambda \) such that \( \alpha \alpha_j \cdots \alpha_1 p^* \alpha_n \cdots \alpha_{j+1}(\eta) \neq 0 \). By definition of the structure of \( D(\Lambda) \) as a \( \Lambda \)-bimodule we get \( \alpha \alpha_j \cdots \alpha_1 p^* \alpha_n \cdots \alpha_{j+1}(\eta) = p^*(\alpha_n \cdots \alpha_{j+1} \eta) \alpha \alpha_j \cdots \alpha_1 \neq 0 \), which contradicts that \( p^*(p) = 0 \) for any path \( p \neq p \) of \( kQ_\Lambda \). In a similar way we prove that \( \overline{C \alpha} \neq 0 \) in \( T(\Lambda) \) leads to a contradiction. Therefore \( C \) is maximal.

(iii) \( \Rightarrow \) (i) Suppose that \( C \) is a cycle of \( Q_{T(\Lambda)} \) maximal in \( T(\Lambda) \), \( C : e \rightarrow e \). As observed above, \( C \) contains exactly one arrow \( \beta_p \). Then \( C = \alpha_j \cdots \alpha_1 \beta_p \alpha_n \cdots \alpha_{j+1} \) and \( \Phi(C) = (0, \varphi_2(C)) \neq 0 \). So, there is a path \( \gamma \) in \( Q_\Lambda \) such that \( \varphi_2(C)(\gamma) \neq 0 \). Then \( \varphi_2(C \gamma)(\overline{\gamma}) = \varphi_2(C)(\overline{\gamma}) \neq 0 \), so \( \varphi_2(C \gamma) \neq 0 \). That is, \( 0 \neq \varphi_2(C \gamma) = \overline{C \gamma} \) in \( T(\Lambda) \). Since \( C \) is maximal in \( T(\Lambda) \) it follows that \( \gamma \) is a trivial path, that is, \( \gamma = e \). Thus \( \varphi_2(C)(\overline{\gamma}) \neq 0 \). Since \( \varphi_2(C)(\overline{\gamma}) = p^*(\alpha_n \cdots \alpha_{j+1} \overline{\gamma}) \) it follows that \( p^*(\alpha_n \cdots \alpha_1) \neq 0 \). The paths \( p, \alpha_n \cdots \alpha_1 \) belong to the chosen basis \( B \) of \( \Lambda \), thus \( \alpha_n \cdots \alpha_1 = p \), and this proves that the cycle \( C \) is elementary. 

We will briefly say that “a path has a supplement” to mean that it has a supplement in some elementary cycle. If \( \{x_i\}_{i \in I} \) is a family of elements in an algebra, we will denote by \( (x_i)_{i \in I} \) the two-sided ideal generated by them.

As a consequence of the preceding proposition we obtain the following result.

**Corollary 3.2.** Let \( \Lambda = kQ_\Lambda / I_\Lambda \) be a monomial algebra. Then:

(i) Every arrow \( \beta_p \) of \( Q_{T(\Lambda)} \) is contained in a single elementary cycle, up to permutations for any \( p \in M \).

(ii) If a path \( \gamma \in (\beta_p)_{p \in M} \) has supplements \( \mu, \mu' \) then \( \mu = \mu' \).

**Proof.** (i) The only elementary cycles containing \( \beta_p \) are the permutations of \( C = p \beta_p \).

(ii) Let \( \gamma \in (\beta_p)_{p \in M} \) be a path of \( kQ_{T(\Lambda)} \) with supplements \( \mu, \mu' \). Then \( \gamma = \delta \beta_p \rho \), with \( \delta, \rho \) paths of \( kQ_\Lambda \), and let \( C \) be the elementary cycle containing \( \beta_p \), which is unique up to permutations. Then any supplement of \( \gamma \) is a supplement in \( C \), and is the path consisting of the remaining arrows in \( C \). So \( \mu = \mu' \).

The next technical lemma will be used in what follows.

**Lemma 3.3.** Let \( \Lambda = kQ_\Lambda / I_\Lambda \) be a monomial algebra.

(i) Let \( q \) and \( u \) be paths in \( kQ_{T(\Lambda)} \). If \( \varphi_2(q)(\overline{\gamma}) \neq 0 \), then \( \varphi_2(q)(\overline{\gamma}) = 1 \), and \( u \) is a supplement of \( q \).

(ii) If \( C : e \rightarrow e \) is an elementary cycle, then \( \varphi_2(C)(e) \neq 0 \).

**Proof.** (i) Suppose that \( \varphi_2(q)(\overline{\gamma}) \neq 0 \). Then \( q = \gamma \beta_p \delta \), with \( \gamma \) and \( \delta \) paths of \( kQ_\Lambda \). So \( 0 \neq \varphi_2(q)(\overline{\gamma}) = \varphi_2(\gamma \beta_p \delta)(\overline{\gamma}) = (\overline{\gamma}, 0)(0, p^*)(\overline{\delta}, 0)(\overline{\gamma}) = ((0, \overline{\gamma}) p^*(\overline{\delta}, 0))(\overline{\gamma}) = (0, \overline{\gamma} p^*(\overline{\delta u \gamma})) \). Thus \( p^*(\overline{\delta u \gamma}) \neq 0 \). Then \( u \) is a supplement of \( q \) in the elementary cycle \( \gamma \beta_p \delta u \).

This proves (i), and (ii) follows directly from the definition of \( \varphi_2 \).
The next proposition is a first step to describe the ideal of relations of the trivial extension of a monomial algebra.

**Proposition 3.4.** Let $\Lambda = kQ_\Lambda / I_\Lambda$ be a monomial algebra. Let $\Phi$ be the morphism defined above. For each $j \in (Q_{T(\Lambda)})_0$, let $Y_j$ be the ideal of $kQ_{T(\Lambda)}$ generated by

(i) oriented cycles from $j$ to $j$ which are not contained in an elementary cycle,
(ii) all the elements $C-C'$, where $C$ and $C'$ are elementary cycles with origin $j$.

Then $Y_j \subseteq \text{Ker } \Phi \cap e_jkQ_{T(\Lambda)}e_j$.

**Proof.** It suffices to prove that $\Phi$ vanishes in the generators of $Y_j$.

Suppose $v$ is a generator of $Y_j$ as considered in (i), that is, a path from $j$ to $j$ not contained in an elementary cycle. As we observed after the definition of $\Phi$, paths of $kQ_\Lambda$ are contained in elementary cycles, so $v$ contains one arrow $\beta_p$, with $p \in \mathcal{M}$. Thus $\Phi(v) = (0, \varphi_2(v))$.

If $v$ has two or more arrows $\beta_p$ then $\varphi_2(v) = 0$, so $v \in \text{Ker } \Phi \cap e_jkQ_{T(\Lambda)}e_j$.

Otherwise, $v = \gamma\beta_p\delta$, with $\gamma$ and $\delta$ paths of $kQ_\Lambda$, and $\varphi_2(v) = \pi^*p^*\delta$. Suppose $\varphi_2(v) \neq 0$. Then there is a path $u$ of $kQ_\Lambda$ such that $\varphi_2(v)(\overline{\pi}) \neq 0$. That is, $\overline{p}^*(\delta u) \neq 0$ and therefore $\gamma\beta_p\delta u$ is an elementary cycle containing $v = \gamma\beta_p\delta$. This contradicts the hypothesis that $v$ is not contained in an elementary cycle. Therefore $\varphi_2(v) = 0$. So $v \in \text{Ker } \Phi \cap e_jkQ_{T(\Lambda)}e_j$.

Suppose now that $v$ is a generator of $Y_j$ of the type (ii), that is $v = C-C'$, where $C$ and $C'$ are elementary cycles with origin $j$. Then $\Phi(v) = (0, \varphi_2(v))$ and $\varphi_2(v) = \varphi_2(C) - \varphi_2(C')$.

Let $u \in kQ_\Lambda$. We will prove that $\varphi_2(v)(\overline{\pi}) = 0$.

If $u \neq e_j$, then $u$ is not a supplement for $C$, and thus $\varphi_2(C)(\overline{\pi}) = \varphi_2(C')(\overline{\pi}) = 0$, by Lemma 3.3(i), so $\varphi_2(v)(\overline{\pi}) = 0$. On the other hand, $\varphi_2(C)(\overline{\tau_j}) = 1$ and $\varphi_2(C')(\overline{\tau_j}) = 1$, by Lemma 3.3(ii). Thus $\varphi_2(v)(\overline{\tau_j}) = 1-1 = 0$.

Thus, $\varphi_2(v) = 0$, that is, $v \in \text{Ker } \Phi \cap e_jkQ_{T(\Lambda)}e_j$. \qed

Now we state the main result of this section.

**Theorem 3.5.** Let $\Lambda = kQ_\Lambda / I_\Lambda$ be a monomial algebra. Let $I'$ be the ideal in $kQ_{T(\Lambda)}$ generated by

(i) the paths not contained in an elementary cycle, and
(ii) the elements $\mu - \mu'$, where $\mu$, $\mu'$ are different paths from $kQ_{T(\Lambda)}$ with a common supplement $\gamma$ in elementary cycles $C$ and $C'$, respectively.

Then $I'$ is admissible and $I' = \text{Ker } \Phi$. That is, $T(\Lambda) \simeq kQ_{T(\Lambda)}/I'$.

Before proving this theorem, we will make some useful observations about the ideal $I'$ defined in its statement.

**Remark 3.6.**

(a) $I_\Lambda \subseteq I'$, because if $q$ is a path of $I_\Lambda$, then $\overline{q} = 0$ in $\Lambda$ and so $q$ is not contained in an elementary cycle, because elementary cycles are nonzero in $T(\Lambda)$. Thus $I_\Lambda \subseteq I' \cap kQ_\Lambda$. Conversely, a path in $kQ_\Lambda$ which is not zero in $\Lambda$ is contained in an elementary cycle, as observed above. Thus $I_\Lambda = I' \cap kQ_\Lambda$.
(b) If \( \mu - \mu' \) is a generator of \( I' \) of type (ii) in Theorem 3.5, then \( \mu, \mu' \in (\beta_p)_{p \in \mathcal{M}} \).

In fact, since they are different paths with a common supplement \( \gamma \) it follows from Corollary 3.2(ii) that \( \gamma \in kQ_\Lambda \). Thus any supplement of \( \gamma \) has an arrow \( \beta_p \).

(c) Suppose \( q \) in \( kQ_{T(\Lambda)} \) is not in \( I' \). Then \( q \) has a supplement in some elementary cycle. In fact, \( q \) is contained in an elementary cycle \( C \), by the definition of \( I' \). Thus \( q \) has a supplement in \( C \).

Now we will prove Theorem 3.5

**Proof.** We must prove that \( I' = \ker \Phi \), where \( \Phi \) is the morphism defined above.

We will prove first that \( I' \subseteq \ker \Phi \). For this, we prove that the generators given in \( I' \) belong to \( \ker \Phi \).

**Case 1.** The generator is a path \( q \) in \( I' \) not contained in an elementary cycle. Then:

- If \( q \in kQ_\Lambda \), then \( q \in kQ_\Lambda \cap I' = I_\Lambda \), by Remark 3.6(a). Then \( \overline{q} = 0 \) and thus \( \Phi(q) = (0, 0) \).

- If \( q \in (\beta_p)_{p \in \mathcal{M}}, \) then \( q = \gamma \beta_p \delta \), with \( p \in \mathcal{M} \), where \( \gamma \) and \( \delta \) are paths of \( kQ_\Lambda \). Thus \( \Phi(q) = (0, \varphi_2(q)) \).

Suppose that \( \varphi_2(q) \neq 0 \). Then there is a path \( u \) of \( kQ_\Lambda \) such that \( \varphi_2(q)(\overline{u}) \neq 0 \). Thus \( \overline{p'}(\delta u \gamma) \neq 0 \). So \( u \) is a supplement of \( q \) in the elementary cycle \( \gamma \beta_p \delta u \). This is a contradiction because we are assuming that \( q \) is not contained in an elementary cycle. Thus \( \varphi_2(q) = 0 \) and then \( q \in \ker \Phi \).

**Case 2.** Suppose \( v \) is a generator of \( I' \) of the form \( v = \mu - \mu' \), where \( \mu, \mu' \) are different paths of \( kQ_{T(\Lambda)} \) from \( i \) to \( j \) with a common supplement \( \gamma \) in elementary cycles \( C \) and \( C' \), respectively. Then \( C = \gamma \mu \) and \( C' = \gamma \mu' \), and we have that \( \gamma \in kQ_\Lambda \), because \( \mu, \mu' \in (\beta_p)_{p \in \mathcal{M}} \) by Remark 3.6(b). Since \( v \in (\beta_p)_{p \in \mathcal{M}} \) we know that \( \Phi(v) = (0, \varphi_2(v)) \). We will prove that \( \varphi_2(v) = 0 \).

The product \( \gamma v = \gamma \mu - \gamma \mu' = C - C' \) is in the ideal \( Y_j \) defined in Proposition 3.4. We proved in the same proposition that \( Y_j \subseteq \ker \Phi \), so \( \gamma v \in \ker \Phi \) and therefore \( \varphi_2(\gamma v) = 0 \). Then \( 0 = \varphi_2(\gamma v)(\overline{e_7}) = \varphi_2(v)(\overline{e_7}) = \varphi_2(v)(\overline{7}). \) Thus \( \varphi_2(v)(\overline{7}) = 0 \).

Suppose that there is a path \( q \in kQ_\Lambda \) such that \( \varphi_2(v)(\overline{q}) \neq 0 \). Then \( \varphi_2(v)(\overline{q}) = \varphi_2(\mu - \mu')(\overline{q}) = \varphi_2(\mu)(\overline{q}) - \varphi_2(\mu')(\overline{q}) \neq 0 \). So either \( \varphi_2(\mu)(\overline{q}) \) or \( \varphi_2(\mu')(\overline{q}) \) is not zero.

Without loss of generality we may assume that \( \varphi_2(\mu)(\overline{q}) \neq 0 \). Then \( q \) is a supplement of \( \mu \) by Lemma 3.3 and therefore \( q = \gamma \), because paths in \( (\beta_p)_{p \in \mathcal{M}} \) have a unique supplement, by Corollary 3.2(ii). So \( \varphi_2(v)(\overline{q}) = \varphi_2(\mu)(\overline{q}) \neq 0 \), which contradicts that \( \varphi_2(v)(\overline{7}) = 0 \). This proves that \( \varphi_2(v) = 0 \), as desired.

The preceding case-by-case analysis proves that \( I' \subseteq \ker \Phi \).

Let \( \pi : kQ_{T(\Lambda)} \rightarrow kQ_{T(\Lambda)}/I' \) be the canonical epimorphism and denote \( \pi(y) = \bar{y} \).

Since \( I' \subseteq \ker \Phi \), the epimorphism \( \Phi : kQ_{T(\Lambda)} \rightarrow kQ_{T(\Lambda)}/\ker \Phi = T(\Lambda) \) induces an epimorphism \( \overline{\Phi} : kQ_{T(\Lambda)}/I' \rightarrow kQ_{T(\Lambda)}/\ker \Phi = T(\Lambda) \) such that \( \overline{\Phi} \circ \pi = \Phi \). To prove the equality \( I' = \ker \Phi \) it is enough to prove that \( \dim_k kQ_{T(\Lambda)}/I' = \dim_k T(\Lambda) = 2 \dim_k \Lambda \).
The inclusion of \( \Lambda \) in \( T(\Lambda) \) factors through \( kQ_{T(\Lambda)}/I' \) because \( I_{\Lambda} \subseteq I' \). So, the morphism \( \iota : \Lambda \to kQ_{T(\Lambda)}/I' \) induced by the inclusion of \( kQ_{\Lambda} \) en \( kQ_{T(\Lambda)} \) is a monomorphism.

Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
  kQ_{\Lambda} & \xrightarrow{\iota} & kQ_{T(\Lambda)} \\
  \Lambda = kQ_{\Lambda}/I_{\Lambda} & \xrightarrow{\pi} & kQ_{T(\Lambda)}/I' \\
  & \xrightarrow{\Phi} & T(\Lambda) = kQ_{T(\Lambda)}/\text{Ker}\Phi \\
\end{array}
\]

with \( \Phi \circ \pi = \Phi \).

We know that \( kQ_{T(\Lambda)} = kQ_{\Lambda} + (\beta_p)_{p \in M} \). Therefore \( e_j kQ_{T(\Lambda)} e_i = e_j kQ_{\Lambda} e_i + e_j (\beta_p)_{p \in M} e_i \) for each \( i, j \in (Q_{T(\Lambda)})_0 \). We consider in \( kQ_{T(\Lambda)}/I' \) the subspaces \( P_{ij} = \pi(e_j kQ_{\Lambda} e_i) \) and \( F_{ij} = \pi(e_j (\beta_p)_{p \in M} e_i) \). Then \( P_{ij} = \iota(e_j \Lambda e_i) \simeq e_j \Lambda e_i \). So \( \sum_{i,j} \dim_k P_{ij} = \dim_k \Lambda \). We will show that \( \dim_k (P_{ij}) \geq \dim_k (F_{ji}) \).

We start by proving that \( F_{ji} \neq 0 \) if and only if \( P_{ij} \neq 0 \). In fact, if \( F_{ji} \neq 0 \), there is a path \( q \) of \( e_i (\beta_p)_{p \in M} e_j \) which is not in \( I' \). So \( q \) has a supplement \( \gamma \) by Remark 3.6(c). Then \( \gamma \) is a path of \( kQ_{\Lambda} \), because \( q \) is in \( (\beta_p)_{p \in M} \), and \( \gamma \) does not belong to \( I' \). Then \( 0 \neq \gamma \in P_{ij} \) and so \( P_{ij} \neq 0 \). Conversely, if \( P_{ij} \neq 0 \) there is a path \( q \) in \( kQ_{\Lambda} \) such that \( q \) does not belong to \( I' \). Then \( q \) has a supplement \( \gamma \), again by Remark 3.6(c), and \( \gamma \) is a path of \( e_i (\beta_p)_{p \in M} e_j \) which does not belong to \( I' \). Then \( 0 \neq \gamma \in F_{ji} \) and so \( F_{ji} \neq 0 \).

Therefore we assume that both, \( F_{ji} \) and \( P_{ij} \) are nonzero and we choose paths \( \mu_1, \ldots, \mu_f \in (\beta_p)_{p \in M} \) such that \( \{\tilde{\mu}_1, \ldots, \tilde{\mu}_f\} \) is a basis of \( F_{ji} \). Then \( \mu_t \notin I' \) if \( t \in \{1, \ldots, f\} \). So, \( \mu_t \) has a supplement in an elementary cycle \( C_t = \mu_t \gamma_t \) for all \( t \in \{1, \ldots, f\} \) by Remark 3.6(c). The paths \( \gamma_1, \ldots, \gamma_f \) belong to \( kQ_{\Lambda} \). We will prove that that \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_f \) are linearly independent in \( kQ_{T(\Lambda)}/I' \). Assume, on the contrary, that \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_f \) are linearly dependent in \( kQ_{T(\Lambda)}/I' \). Then \( \{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_f\} \) are linearly dependent in \( \Lambda \), because \( \iota \) is a monomorphism. Since \( \tilde{\gamma}_t \) is not zero in \( \Lambda \) for \( t \in \{1, \ldots, f\} \), we have that \( \gamma_1, \ldots, \gamma_f \) are not pairwise different, because \( \Lambda \) is monomial, and therefore classes in \( \Lambda \) of pairwise different nonzero paths are linearly independent. Therefore two of the paths \( \gamma_1, \ldots, \gamma_f \) are equal, let’s say \( \gamma_1 = \gamma_2 \). Then \( C_1 = \mu_1 \gamma_1 \) and \( C_2 = \mu_2 \gamma_1 \), so the elementary cycles \( C_1 \) and \( C_2 \) contain the common path \( \gamma_1 \), and this proves that \( \mu_2 - \mu_1 \) is an element of \( I' \). Then \( \mu_2 = \mu_1 \), which contradicts the fact that \( \tilde{\mu}_1, \tilde{\mu}_2 \) are elements of a basis of \( F_{ji} \). This contradiction shows that \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_f \) are linearly independent in \( kQ_{T(\Lambda)}/I' \). So \( \dim_k (P_{ij}) \geq \dim_k (F_{ji}) \).

Therefore

\[
\dim_k kQ_{T(\Lambda)}/I' \leq \sum_{i,j} (\dim_k P_{ij} + \dim_k F_{ij}) \leq \sum_{i,j} (\dim_k P_{ij} + \dim_k P_{ji}) = 2 \dim_k \Lambda = \dim_k T(\Lambda).
\]
Thus \( \dim_k kQ_{T(\Lambda)}/I' \leq \dim_k T(\Lambda) \), and this proves that the surjective morphism \( \Phi : kQ_{T(\Lambda)}/I' \to T(\Lambda) \) is an isomorphism, which ends the proof of the theorem. \( \square \)

**Example 3.7.** Let \( \Lambda = kQ_{\Lambda}/I_{\Lambda} \), where

\[
\begin{align*}
Q_{\Lambda}: & \quad \alpha_2 & \quad \alpha_1 \\
& \quad \alpha_3 & \quad \beta_{p_2} & \quad \beta_{p_1}
\end{align*}
\]
with relation \((\alpha_1)^2 = 0\).

The maximal paths are \( p_1 = \alpha_2\alpha_1 \) and \( p_2 = \alpha_3\alpha_1 \), which induce in \( Q_{T(\Lambda)} \) elementary cycles \( C_1 = \beta_{p_1}\alpha_2\alpha_1 \) and its permutations \( C_2 = \alpha_1\beta_{p_1}\alpha_2, C_3 = \alpha_2\alpha_1\beta_{p_1}, \) and \( C_4 = \beta_{p_2}\alpha_3\alpha_1 \) and its permutations \( C_5 = \alpha_1\beta_{p_2}\alpha_3, C_6 = \alpha_3\alpha_1\beta_{p_2}. \) Then \( T(\Lambda) \) is given by

\[
\begin{align*}
Q_{T(\Lambda)}: & \quad \alpha_2 & \quad \beta_{p_1} & \quad \alpha_3 & \quad \alpha_1 \\
& \quad \beta_{p_2} & \quad \alpha_1
\end{align*}
\]
with relations

\[
(\alpha_1)^2, \alpha_3\beta_{p_1}, \alpha_2\beta_{p_2},
\alpha_3\alpha_1\beta_{p_1}, \alpha_2\alpha_1\beta_{p_2},
\alpha_1C_1, \alpha_2C_2, \beta_{p_1}C_3,
\alpha_1C_4, \alpha_3C_5, \beta_{p_2}C_6,
\beta_{p_1}\alpha_2 - \beta_{p_2}\alpha_3.
\]

**Example 3.8.** Let \( \Lambda = kQ_{\Lambda}/I_{\Lambda} \), where

\[
\begin{align*}
Q_{\Lambda}: & \quad \alpha_1 & \quad \alpha_2 & \quad \alpha_3 \\
& \quad \gamma_1 & \quad \gamma_2 & \quad \gamma_3 & \quad \gamma_{i+1}
\end{align*}
\]
with relations \( \alpha_{i+1}\alpha_i, \gamma_{i+1}\gamma_i \) for \( i = 1, 2 \).

The maximal paths are \( p_1 = \alpha_3\gamma_2\alpha_1 \) and \( p_2 = \gamma_3\alpha_2\gamma_1 \), which induce in \( Q_{T(\Lambda)} \) elementary cycles \( C_1 = \beta_{p_1}\alpha_3\gamma_2\alpha_1 \) and its permutations, \( C_2 = \alpha_1\beta_{p_1}\alpha_3\gamma_2, C_3 = \gamma_2\alpha_1\beta_{p_1}\alpha_3, C_4 = \alpha_3\gamma_2\alpha_1\beta_{p_1} \) and \( C_5 = \beta_{p_2}\gamma_3\alpha_2\gamma_1 \) and its permutations, \( C_6 = \gamma_1\beta_{p_2}\gamma_3\alpha_2, C_7 = \alpha_2\gamma_1\beta_{p_2}\gamma_3, \) and \( C_8 = \gamma_3\alpha_2\gamma_1\beta_{p_2}. \)
Then $T(\Lambda)$ is given by

$$Q_{T(\Lambda)}:$$

with relations

$\alpha_{i+1} \alpha_i$, $\gamma_{i+1} \gamma_i$ for $i = 1, 2$,
$\alpha_1 C_1$, $\gamma_2 C_2$, $\alpha_3 C_3$, $\beta_{p_1} C_4$,
$\gamma_1 C_5$, $\alpha_2 C_6$, $\gamma_3 C_7$, $\beta_{p_2} C_8$,
$\alpha_1 \beta_{p_2}$, $\beta_{p_2} \alpha_3$, $\beta_{p_1} \gamma_3$, $\gamma_1 \beta_{p_1}$,
$C_1 - C_5$, $C_3 - C_7$, $C_2 - C_6$, $C_4 - C_8$.

**Example 3.9.** Let $\Lambda = kQ_\Lambda/I_\Lambda$, where

$Q_\Lambda:$

with relations $\alpha_2 \alpha_1$, $\alpha_1 \alpha_3$.

The maximal paths are $p_1 = \alpha_1$ and $p_2 = \alpha_3 \alpha_2$, which induce in $Q_{T(\Lambda)}$ the elementary cycles $C_1 = \beta_{p_1} \alpha_1$ and its permutation, $C_2 = \alpha_1 \beta_{p_1}$, $C_3 = \beta_{p_2} \alpha_3 \alpha_2$ and its permutations, $C_4 = \alpha_2 \beta_{p_2} \alpha_3$ and $C_5 = \alpha_3 \alpha_2 \beta_{p_2}$.

Then $T(\Lambda)$ is given by

$Q_{T(\Lambda)}:$

with relations

$\alpha_2 \alpha_1$, $\alpha_1 \alpha_3$, $\beta_{p_2} \beta_{p_1}$, $\beta_{p_1} \beta_{p_2}$,
$\alpha_1 C_1$, $\beta_{p_1} C_2$, $\alpha_2 C_3$, $\beta_{p_2} C_5$, $\alpha_3 C_4$,
$C_1 - C_5$, $C_2 - C_3$.

4. The gentle case

In this section we study the trivial extension $T(\Lambda)$ in the particular case when $\Lambda$ is a gentle algebra. We will prove that the given description of the relations of the trivial extension of a monomial algebra can be formulated in a simple way when the monomial algebra is gentle. Also, we will see that it is possible to determine when an algebra $B = kQ_B/I_B$, given by its quiver and relations, is the trivial extension.
We recall from [1] that an algebra $\Lambda$ is called gentle if it is Morita equivalent to $kQ/I$, where:

1. $I$ is generated by paths of length two.
2. Each vertex of $Q$ is the beginning and the target of at most two arrows.
3. For each arrow $\alpha$ of $Q$ there exists at most one arrow $\beta$ such that $\alpha \beta \in I$, and there exists at most one arrow $\gamma$ such that $\gamma \alpha \in I$.
4. For each arrow $\alpha$ of $Q$ there exists at most one arrow $\delta$ such that $\alpha \delta \notin I$ and there exists at most one arrow $\epsilon$ such that $\epsilon \alpha \notin I$.

It follows from the definition of a gentle algebra that every arrow is contained in a unique maximal path, at most two maximal paths begin at the same given vertex, and at most two maximal paths end at a given vertex. Thus, different maximal paths can not have common arrows.

In the next proposition we prove properties of the maximal cycles of the trivial extension of a gentle algebra, which will be very useful in what follows.

**Proposition 4.1.** Let $\Lambda = kQ_{\Lambda}/I_{\Lambda}$ be a gentle algebra. Then the trivial extension $T(\Lambda)$ satisfies the following properties:

(i) Any nontrivial path of $kQ_{T(\Lambda)}$ which is nonzero in $T(\Lambda)$ is contained in a maximal cycle, unique up to permutations.

(ii) If $C_1$ and $C_2$ are two maximal cycles from $j$ to $j$, then either $C_1$ is a permutation of $C_2$, or $C_1$ and $C_2$ do not overlap (i.e., they do not have common arrows) and $\tilde{C_1} = \tilde{C_2}$ in $T(\Lambda)$.

(iii) In $kQ_{T(\Lambda)}$ there are at most two different cycles from $j$ to $j$ which are maximal in $T(\Lambda)$ for all $j$ in $\{1, \ldots, n\}$.

**Proof.**

(i) Gentle algebras are monomial, so we know from Theorem 3.5 that every nontrivial path $\gamma$ of $kQ_{T(\Lambda)}$ which is nonzero in $T(\Lambda)$ is contained in an elementary cycle $C$. On the other hand, we know from Proposition 3.1 that elementary cycles coincide with maximal cycles in $T(\Lambda)$, and are permutations of cycles of the form $p_h \beta p_h$, with $p_h \in M$. Then the uniqueness up to permutations of the maximal cycle $C$ containing $\gamma$ is clear when $\gamma$ contains an arrow $\beta_p$. Otherwise such uniqueness follows from the fact that every nontrivial path of $kQ_{\Lambda}$ is contained in a unique maximal path, because $\Lambda$ is a gentle algebra.

(ii) If $C_1$ and $C_2$ are two maximal cycles from $j$ to $j$, then they have the common supplement $e_j$. So the path $C_1 - C_2 \in I'$ because it satisfies condition (ii) in Proposition 3.5. Thus $\tilde{C_1} = \tilde{C_2}$ in $T(\Lambda)$. The fact that $C_1$ and $C_2$ do not overlap is a direct consequence of (i).

(iii) Suppose that there are three cycles $C_1, C_2, C_3$ from $j$ to $j$ in $kQ_{T(\Lambda)}$ maximal in $T(\Lambda)$. Then each $C_i$ has exactly one arrow $\beta_{p_i}$, with $p_i \in M$, and the length of $C_i$ is at least 2. So we write $C_1 = \delta_1 q_1 \gamma_1, C_2 = \delta_2 q_2 \gamma_2$ and $C_3 = \delta_3 q_3 \gamma_3$, where $\gamma_i, \delta_i$ are arrows of $Q_{T(\Lambda)}$ and $q_i$ is a path of $kQ_{T(\Lambda)}$ for $i \in \{1, 2, 3\}$. 
By (ii) we have that the cycles $C_1$, $C_2$, $C_3$ have no common arrow. Moreover, $\gamma_1$, $\gamma_2$, $\gamma_3$ are not all arrows of $Q_\Lambda$ because $\Lambda$ is gentle. If none of them is an arrow of $Q_\Lambda$ then $\delta_1$, $\delta_2$, $\delta_3$ lie in $Q_\Lambda$, and this can not happen because $\Lambda$ is gentle. Assume two of the arrows $\gamma_i$, say $\gamma_1$ and $\gamma_2$, are arrows of $Q_\Lambda$. Then $\gamma_3 = \beta_{p_3}$, and therefore $\delta_3$ is an arrow of $Q_\Lambda$. Thus one of the paths $\gamma_1 \delta_3$, $\gamma_2 \delta_3$ determines a nonzero class in $\Lambda$, because $\Lambda$ is a gentle algebra. Then such a path is contained in a maximal cycle $C'$. The maximal cycles $C' = C_3$, because maximal cycles do not overlap. This is a contradiction because $\gamma_1$, $\gamma_2$ are not arrows of $C_3$. This proves that only one arrow $\gamma_i$ is an arrow of $Q_\Lambda$. In the same way one proves that only one arrow $\delta_i$ is an arrow of $Q_\Lambda$. We may assume that the arrow $\gamma_i$ in $Q_\Lambda$ is $\gamma_1$. Then $\gamma_2 = \beta_{p_2}$ and $\gamma_3 = \beta_{p_3}$. So $\delta_2$, $\delta_3$ are arrows of $Q_\Lambda$, and this contradiction shows that there are at most two maximal cycles from $j$ to $j$.

The above result shows that the description of the ideal of relations for $T(\Lambda)$ given in Theorem 3.5 can be simplified in the gentle case, as we state in the following theorem.

**Theorem 4.2.** Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a gentle algebra. Let $I'$ be the ideal in $kQ_{T(\Lambda)}$ generated by

(i) the paths not contained in elementary cycles, and

(ii) the elements $C - C'$, where $C$ and $C'$ are elementary cycles starting at the same vertex of $Q_{T(\Lambda)}$.

Then $I'$ is admissible and $I' = \ker \Phi$. That is, $T(\Lambda) \simeq kQ_{T(\Lambda)}/I'$.

We illustrate the previous theorem with an example.

**Example 4.3.** Let $\Lambda$ be given the quiver

![Quiver](image)

with the relation $\alpha_1 \alpha_2 = 0$. Then the unique maximal path is $p_1 = \alpha_2 \alpha_1$. So $Q_{T(\Lambda)}$ is the quiver

![Quiver](image)

The elementary cycles are $C_1 = \alpha_2 \alpha_1 \beta_{p_1}$ and its permutations, $C_2 = \beta_{p_1} \alpha_2 \alpha_1$ and $C_3 = \alpha_1 \beta_{p_1} \alpha_2$. The relations are

$\alpha_1 \alpha_2 = 0, \quad \beta_{p_1}^2 = 0, \quad \beta_{p_1} \alpha_2 \alpha_1 \beta_{p_1} = 0,$

$\alpha_1 \beta_{p_1} \alpha_2 \alpha_1 = 0, \quad \alpha_2 \alpha_1 \beta_{p_1} \alpha_2 = 0, \quad \alpha_2 \alpha_1 \beta_{p_1} = \beta_{p_1} \alpha_2 \alpha_1.$
Our next goal is to give a characterization of the trivial extension of a gentle algebra through the description of its cycles. We will see that maximal cycles play a fundamental role. This will require some preliminary lemmas.

**Lemma 4.4.** If $\Lambda = kQ_\Lambda / I_\Lambda$ is an algebra such that $I_\Lambda$ is generated by paths of length two and $C = \alpha_s \cdots \alpha_1$ is a cycle with origin $j$ and nonzero in $\Lambda$, then $\bar{\alpha_1} \bar{\alpha_s} = 0$.

**Proof.** We have that $\bar{\alpha_s} \cdots \bar{\alpha_1} \neq 0$ because $C$ is not zero. Suppose $\bar{\alpha_1} \bar{\alpha_s} \neq 0$. Then $\bar{\alpha_1} C = \bar{\alpha_1} \bar{\alpha_s} \cdots \bar{\alpha_1} \neq 0$ since $I_\Lambda$ is generated by paths of length two. For the same reason $\alpha_1 C^k \neq 0$ for all $k$ greater than one. This contradicts that $\Lambda$ is of finite dimension. Thus $\bar{\alpha_1} \bar{\alpha_s} = 0$. □

Observe that the previous lemma holds for gentle algebras.

**Lemma 4.5.** Let $\Lambda = kQ_\Lambda / I_\Lambda$ be a gentle algebra. Then

(i) There is at most one cycle of $kQ_\Lambda$ with origin $j$ and nonzero in $\Lambda$ for any vertex $j$ of $Q_\Lambda$.

(ii) There are at most two elementary cycles of $kQ_{T(\Lambda)}$ with origin $j$ in the same permutation class.

**Proof.** (i) Assume that there are two nonzero cycles $C_1, C_2$ of $kQ_\Lambda$ with origin $j$ and nonzero in $\Lambda$. Let $C_1 = \alpha_r \cdots \alpha_1$, $C_2 = \beta_r \cdots \beta_1$, with $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ arrows of $Q_\Lambda$. Since $\Lambda$ is gentle and $C_1, C_2$ are nonzero in $\Lambda$ we have that $\alpha_1 \neq \beta_1$ and $\alpha_r \neq \beta_s$. By Lemma 4.4 we know that $\bar{\alpha_1} \bar{\alpha_r} = 0$ and $\bar{\beta_1} \bar{\beta_s} = 0$. Then, $\bar{\beta_1} \bar{\alpha_r} \neq 0$ and $\bar{\alpha_1} \bar{\beta_s} \neq 0$ because $\Lambda$ is gentle. Therefore $(C_1 C_2)^t \neq 0$ for all $t$ greater than zero, because $\Lambda$ is generated by paths of length two. This contradicts that $\Lambda$ is a finite-dimensional algebra.

Thus (i) holds, and (ii) is a direct consequence of Proposition 4.1 □

**Lemma 4.6.** Let $\Lambda = kQ_\Lambda / I_\Lambda$ be a gentle algebra. If $\dim_k \text{End}_{T(\Lambda)}(P_j) > 2$ for some indecomposable projective $T(\Lambda)$-module $P_j$, then there is a cycle of $kQ_\Lambda$ with origin $j$ and nonzero in $\Lambda$.

**Proof.** We know from Proposition 4.1(ii) that elementary cycles with the same origin are equal in $T(\Lambda)$. Thus the hypothesis $\dim_k \text{End}_{T(\Lambda)}(P_j) > 2$ implies that there is a cycle $\gamma$ from $j$ to $j$ which is not elementary and is nonzero in $T(\Lambda)$. Then there is an elementary cycle $C$ with origin $j$ containing $\gamma$, so $C = \delta \gamma$, with $\delta$ not trivial. Then $\delta$ is a cycle with origin $j$ and is nonzero in $T(\Lambda)$ because elementary cycles do not vanish in $T(\Lambda)$. On the other hand, a permutation of $C$ has the form

\begin{center}
\begin{tikzpicture}
\node (j) at (0,0) {$j$};
\node (a1) at (-2,1.5) {$\alpha_1$};
\node (a2) at (2,1.5) {$\alpha_r$};
\node (b1) at (-2,-1.5) {$\beta_1$};
\node (b2) at (2,-1.5) {$\beta_r$};
\draw[-stealth] (j) to[bend right] (a1);
\draw[-stealth] (j) to[bend right] (a2);
\draw[-stealth] (j) to[bend right] (b1);
\draw[-stealth] (j) to[bend right] (b2);
\end{tikzpicture}
\end{center}
\( p \beta_p \) with \( p \in \mathcal{M} \), by Proposition 3.1. The arrow \( \beta_p \) is either in \( \delta \) or in \( \gamma \). In the first case the cycle \( \gamma \) is in \( kQ_\Lambda \), and otherwise the cycle \( \delta \) is in \( kQ_\Lambda \). This ends the proof of the lemma because both cycles are nonzero in \( T(\Lambda) \).

The previous lemmas are helpful to determine \( \dim_k \operatorname{End}_{T(\Lambda)}(P_j) \) when \( \Lambda \) is a gentle algebra with a cycle in \( j \) nonzero in \( \Lambda \), as we show in the next proposition.

**Proposition 4.7.** Let \( \Lambda = kQ_\Lambda / I_\Lambda \) be a gentle algebra with a cycle \( \gamma \) with origin \( j \) nonzero in \( \Lambda \). Then there is a cycle \( \delta \) with origin \( j \) in \( kQ_{\Gamma(\Lambda)} \) nonzero in \( T(\Lambda) \), \( \delta \neq \gamma \), such that \( \bar{e}_j, \bar{\gamma}, \bar{\delta}, \bar{\gamma} \delta = \bar{\delta} \gamma \) form a basis of \( \operatorname{End}_{T(\Lambda)} P_j \).

**Proof.** Suppose \( \gamma \) is a cycle with origin \( j \) of \( kQ_\Lambda \) nonzero in \( \Lambda \), \( \gamma = \alpha_s \cdots \alpha_1 \), where \( \alpha_i \) are arrows of \( Q_\Lambda \) for all \( i \in \{1, \ldots, s\} \). We know that there is an elementary cycle \( C \) of \( kQ_{\Gamma(\Lambda)} \) with origin \( j \) containing \( \gamma \) which is unique up to permutations, by Proposition 4.1(i). Let \( \delta = \epsilon_1 \epsilon_2 \cdots \epsilon_1 \) be the supplement of \( \gamma \) in \( C \), where \( \epsilon_1, \ldots, \epsilon_r \) are arrows of \( Q_{\Gamma(\Lambda)} \). Then \( \delta \) is a cycle with origin \( j \) and is nonzero in \( T(\Lambda) \), \( \delta \neq \gamma \), and \( C \) is a permutation of \( \delta \gamma \). From the fact, proven in Lemma 4.5 that there is at most one cycle of \( Q_\Lambda \) starting at \( j \), we can conclude that either \( C = \delta \gamma \) or \( C = \gamma \delta \). We may assume \( C = \gamma \delta \).

We have that \( \alpha_1 \alpha_s = 0 \) from Lemma 4.4. Moreover, \( \epsilon_1 \epsilon_r = 0 \) because the path \( \epsilon_1 \epsilon_r \) is not contained in any elementary cycle. Moreover, the paths \( \bar{\epsilon}_1 \bar{\alpha}_s \) and \( \bar{\alpha}_1 \bar{\epsilon}_r \) are nonzero in \( kQ_{\Gamma(\Lambda)} \) because any permutation of a maximal cycle is a maximal cycle.

We claim that the only cycles with origin \( j \) and nonzero in \( T(\Lambda) \) are \( \gamma, \delta, C = \gamma \delta \) and \( C'' = \delta \gamma \). In fact, we know that \( C = \alpha_s \cdots \alpha_1 \epsilon_r \cdots \epsilon_1 \) and \( \bar{\alpha}_1 \bar{\alpha}_s = 0 \). Then \( \epsilon_1 \neq \alpha_s \) and \( \epsilon_r \neq \alpha_1 \). Suppose that there is another cycle of \( kQ_{\Gamma(\Lambda)} \) with origin \( j \) and nonzero in \( T(\Lambda) \). Such a cycle is contained in an elementary cycle \( C'' = \rho_t \cdots \rho_1 \) starting at \( j \). Then either \( C'' \) is a permutation of \( C \), or \( C \) and \( C'' \) do not have common arrows, by Proposition 4.1(ii). We also know that \( C'' \) has a permutation of the form \( p' \beta_p' \) with \( p' \in \mathcal{M} \). On the other hand, we know that the permutation class of \( C \) contains at most two cycles, by Lemma 4.5(ii), so \( C \) and \( C'' \) have no common arrow. We consider two cases:

**Case 1.** \( \gamma \) is maximal in \( \Lambda \). Then \( \delta = \beta_p \). Since \( C'' \) is an elementary cycle, it contains only one arrow \( \beta_p' \), so \( \rho_1 \neq \beta_p' \) or \( \rho_t \neq \beta_p' \). Suppose \( \rho_1 \neq \beta_p' \). Then \( \rho_1 \alpha_s \neq 0 \) because \( \Lambda \) is gentle. This contradicts the maximality of \( \gamma \). Analogously, if \( \rho_t \neq \beta_p' \) we have a contradiction.

**Case 2.** \( \gamma \) is not maximal in \( \Lambda \). So \( \epsilon_1 \neq \beta_p \) or \( \epsilon_r \neq \beta_p \). Suppose \( \epsilon_1 \neq \beta_p \). Then \( \rho_1 = \beta_p' \) because \( \Lambda \) is gentle. So \( \rho_1 \neq \beta_p' \) and \( \alpha_1 \rho_t \neq 0 \) again because \( \Lambda \) is gentle. Therefore \( \alpha_1 \rho_t \) is contained in a maximal cycle \( C'' \) and so \( \alpha_1 \) is contained in two different cycles, which is a contradiction.

Therefore the only cycles with origin \( j \) and nonzero in \( T(\Lambda) \) are \( \gamma, \delta, C = \gamma \delta \) and its permutation \( \delta \gamma \). So \( \operatorname{End}_{T(\Lambda)} P_j \) is generated by \( \bar{e}_j, \bar{\gamma}, \bar{\delta}, \bar{\gamma} \delta = \bar{\delta} \gamma \). \( \square \)
Summarizing the previous results we can state important properties of the cycles of the trivial extension of a gentle algebra, which we list in the following proposition.

**Proposition 4.8.** Let $\Lambda$ be a gentle algebra. Then the bound quiver of $B = T(\Lambda)$ satisfies the following properties:

1. Any permutation of a maximal cycle is a maximal cycle.
2. Any path $u$ of $kQ_B$ which is nonzero in $B$ is contained in a maximal cycle of $kQ_B$, which is unique up to permutations if $u$ is nontrivial.
3. There are at most two different cycles with origin $j$ in $kQ_B$ which are maximal and nonzero in $B$ for any vertex $j$ of $(Q_B)_0$. If there are two such cycles, they are equal in $B$.
4. If $\alpha_s \ldots \alpha_1 : j \rightarrow j$ is a cycle of $kQ_B$ which is nonzero and not maximal in $B$, then $\alpha_s \alpha_1 = 0$ in $B$ and $\dim_k \text{End}_K(P_j) = 4$.

**Proof.** Property (T1) holds because maximal cycles coincide with elementary cycles, by Proposition 3.1. Properties (T2) and (T3) follow directly from Proposition 4.1, and property (T4) is a consequence of Proposition 4.7. □

Notice that (T1) holds for any monomial algebra as proven in Proposition 3.1. However, (T2), (T3) and (T4) do not hold in general for monomial algebras. In fact, in Example 3.7 the arrow $\alpha_1$ belongs to the elementary cycles $C_1$ and $C_4$, so (T2) does not hold in this case.

On the other hand, let $\Lambda$ be the hereditary algebra

Then $T(\Lambda)$ has quiver

where the elementary cycles are $C_1 = \beta_{p_1} \alpha_1$, $C_2 = \beta_{p_2} \alpha_2$, $C_3 = \beta_{p_3} \alpha_3$ and their permutations. Then the vertex 1 is the origin of $C_1$, $C_2$ and $C_3$, so (T3) does not hold.

Finally, if $\Lambda$ is given by the quiver with relation $\alpha^4 = 0$, then $T(\Lambda)$ does not satisfy (T3).
Remark 4.9. Let $B = kQ_B/I_B$ be an algebra satisfying $(T_1), (T_2), (T_3)$ and $(T_4)$. Then $(T_2)$ implies that $B$ is not semisimple, and any arrow of $Q_B$ is contained in a nonzero cycle.

Let $j$ be a vertex of $Q_B$. Then:

(a) If there is only one cycle $C$ in $kQ_B$ with origin $j$ and nonzero in $B$ then $C$ is maximal by $(T_2)$, and there is a unique arrow starting at $j$.

(b) If there are exactly two different cycles $C_1$ and $C_2$ with origin $j$ nonzero in $B$ then these are maximal in $B$ from $(T_4)$, since otherwise we would have $\dim_k \text{End}_B(P) = 4$. This situation is illustrated in the following diagram:

```
\begin{tikzpicture}
  \node (j) at (0,0) {$\bullet$};
  \node (C1) at (-1,1) {$\bullet$};
  \node (C2) at (1,1) {$\bullet$};
  \node (q1) at (-1.5,-1) {$\bullet$};
  \node (q2) at (1.5,-1) {$\bullet$};

  \draw[->] (j) -- (C1);\node[above] at (0,1) {$\gamma = \alpha_1 \ q_1 \ \epsilon_1$};
  \draw[->] (j) -- (C2);\node[above] at (0,1) {$\delta = \alpha_2 \ q_2 \ \epsilon_2$};
  \draw[<->] (C1) -- (C2);
  \draw[->] (C1) .. controls (-2,-1) and (-1,-2) .. (q1);
  \draw[->] (C2) .. controls (2,-1) and (1,-2) .. (q2);
\end{tikzpicture}
```

where $C_1 = C_2$, and $\dim_k \text{End}_B(P) = 2$.

(c) If there are more than two different cycles with origin $j$ and nonzero in $B$, then we know from $(T_3)$ that one of them is not maximal. So $\dim_k \text{End}_B(P) = 4$ from $(T_4)$, and $kQ_B$ contains four cycles with origin $j$. Two of them are maximal and are in the same permutation class. The situation is locally as follows:

```
\begin{tikzpicture}
  \node (j) at (0,0) {$\bullet$};
  \node (q1) at (-1,-1) {$\bullet$};
  \node (q2) at (1,-1) {$\bullet$};

  \draw[->] (j) -- (q1);\node[below] at (0,-1) {$\gamma = \alpha_1 \ q_1 \ \epsilon_1$};
  \draw[->] (j) -- (q2);\node[below] at (0,-1) {$\delta = \alpha_2 \ q_2 \ \epsilon_2$};
  \draw[<->] (q1) -- (q2);
  \draw[->] (q1) .. controls (-1,-2) and (-2,-1) .. (j);
  \draw[->] (q2) .. controls (1,-2) and (2,-1) .. (j);
\end{tikzpicture}
```

where $\gamma$ and $\delta$ nonmaximal cycles. The cycles of $kQ_B$ maximal in $B$ are $C_1 = \gamma \delta$, $C_2 = \delta \gamma$, and they are equal in $B$.

In any case, at most two arrows start at $j$ and at most two arrows end at $j$.

Remark 4.10.

(a) An algebra $B = kQ_B/I_B$ that satisfies properties $(T_1), (T_2), (T_3)$ and $(T_4)$ is uniquely determined by its quiver $Q_B$ and by the cycles of $kQ_B$ maximal in $B$. That is, if the algebras $A, B$ satisfy $(T_1), (T_2), (T_3)$ and $(T_4)$, then they are isomorphic if and only if they have the same quiver and the same maximal cycles.
(b) Assume $B$ is an algebra satisfying $(T_1)$ and $(T_2)$. Let $C_1, \ldots, C_t$ be a complete set of representatives of the equivalence classes under permutations of nonzero cycles of $Q_B$ maximal in $B$, and let $\alpha_i$ be an arrow in $C_i$ for $i = 1, \ldots, t$. Then $\alpha_i$ is not an arrow of $C_j$ for $i \neq j$, because $(T_2)$ holds. Thus we can always consider a quiver obtained from $Q_B$ by eliminating exactly one arrow of each cycle of $Q_B$ maximal in $B$.

For an algebra $B$ satisfying properties $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$, it is possible to construct a gentle algebra whose trivial extension is isomorphic to $B$. The following theorem shows this fact and describes all such gentle algebras.

**Theorem 4.11.** Let $B = kQ_B/I_B$ be a finite-dimensional algebra satisfying $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$.

(i) Let $Q$ be the quiver obtained from $Q_B$ by eliminating exactly one arrow of each cycle of $Q_B$ maximal in $B$, and let $I = kQ \cap I_B$. Then $\Lambda = kQ/I$ is a gentle algebra and $T(\Lambda) \cong B$.

(ii) If $\Lambda$ is a gentle algebra such that $T(\Lambda) \cong B$, then $\Lambda = kQ/I$, with $Q$ and $I$ as in (i).

**Proof.** (i) Let $Q$ be the quiver obtained from $Q_B$ by eliminating exactly one arrow of each cycle of $Q_B$ maximal in $B$. We will prove all the conditions that must be satisfied so that $\Lambda = kQ/I$ is gentle.

(G1) $I$ is generated by paths of length two. Indeed:

Assume that there is a relation $\overline{k_1 q_1} + \overline{k_2 q_2} + \ldots + \overline{k_r q_r} = 0$, where $0 \neq k_i \in k$ for all $i \in \{1, \ldots, r\}$ and all the $q_i$ are different paths of $kQ_\Lambda$ nonzero in $\Lambda$. We choose such a sum with $r$ minimum. If one of the $q_i$’s, say $q_1$, is not maximal in $\Lambda$, then it can be extended to a maximal nonzero path $q’q_1q$ and we replace the original relation by $\overline{k_1 q’_1 q_1} + \overline{k_2 q’_2 q_2} + \ldots + \overline{k_r q’_r q_r} = 0$. So we may assume that all the $q_i$’s are maximal paths in $\Lambda$. From the construction of $Q$ we know that there is an arrow $\epsilon \in Q_B$ such that $q_1\epsilon$ is a maximal cycle of $B$, and $\epsilon \notin Q$. Then $\overline{k_1 q_1\epsilon} + \overline{k_2 q_2 \epsilon} + \ldots + \overline{k_r q_r \epsilon} = 0$ in $B$ with the first summand nonzero. Then there is another summand which is nonzero. Let’s say $\overline{q_2\epsilon} \neq 0$. Next we prove that the path $q_2\epsilon$ is maximal in $B$. Since property $(T_2)$ holds, this path can be completed to a maximal cycle $\epsilon$. On the other hand, by $(T_1)$ we know that permutations of maximal cycles are maximal cycles, so we may assume that $C = gq_2\epsilon$, where $g$ is a path in $kQ_B$. Since $\epsilon$ is not an arrow of $Q$, we have that $gq_2$ is in $kQ$. Since $gq_2$ is nonzero and $q_2$ is maximal in $\Lambda$, it follows that the path $g$ is trivial, so $C = q_2\epsilon$ is a maximal cycle of $B$. Thus $q_1\epsilon, q_2\epsilon$ are maximal cycles of $B$ and they contain $\epsilon$. Since $(T_2)$ holds, there is, up to permutations, a unique cycle containing $\epsilon$, so $q_1\epsilon = q_2\epsilon$. Thus $q_1 = q_2$, and this is a contradiction because all the $q_i$’s are different paths of $kQ$. This proves that $I$ is generated by paths.

Suppose now that $\alpha_r \cdots \alpha_2 \alpha_1$ is an element of $I$ with $r > 2$, where each $\alpha_i$ is an arrow of $Q$, and $\alpha_r \cdots \alpha_2, \alpha_{r-1} \cdots \alpha_1 \notin I$. Then $\alpha_r \cdots \alpha_2$ and $\alpha_{r-1} \cdots \alpha_1$ are nonzero paths in $\Lambda$ with a common arrow $\alpha_2$. From $(T_2)$ we know that $\alpha_2$ is contained in a cycle of $kQ_B$, maximal in $B$, and such a cycle is unique up to permutations. Thus $\alpha_{r-1} \cdots \alpha_1$ and $\alpha_r \cdots \alpha_2$ are both contained in the same

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maximal cycle $C$. We observe first that $\alpha_1 \neq \alpha_2, \ldots, \alpha_r$. In fact, if we assume, on the contrary, that $\alpha_1 = \alpha_k$, with $2 \leq k \leq r$, we get that $\alpha_1 \alpha_{k-1} \cdots \alpha_2$ is a cycle contained in $C$, and it is not maximal because any maximal cycle contains one arrow which is not an arrow of $Q$, and all $\alpha_i$’s are in $Q_1$. Since $(T_4)$ holds we have that $\overline{\alpha_2 \alpha_1} = 0$, contradicting our assumption that $\overline{\alpha_{r-1} \cdots \alpha_1} \neq 0$. Thus $\alpha_r \neq \alpha_2, \cdots, \alpha_1$, and therefore $C = \alpha_r \cdots \alpha_2 \beta_s \cdots \beta_1$, where $\beta_1, \ldots, \beta_s$ are arrows of $Q_B$, and one of them is $\alpha_1$. Then we have $\beta_k = \alpha_1$, with $1 \leq k \leq s$, and $\alpha_1 \beta_{k-1} \cdots \beta_1 \alpha_r \cdots \alpha_2 = C_1$ is a cycle contained in $C$, which is not maximal unless $s = k$. Thus, if $s \neq k$, then $(T_4)$ implies that $\overline{\alpha_2 \alpha_1} = 0$, contradicting again that $\overline{\alpha_{r-1} \cdots \alpha_1} \neq 0$. So $s = k$ and $C_1 = C$ is a maximal cycle. Since $(T_1)$ holds we obtain that the permutation $\beta_{k-1} \cdots \beta_1 \alpha_r \cdots \alpha_2$ of $C_1$ is also a maximal cycle, contradicting that $\alpha_r \cdots \alpha_2 \alpha_1 \in I$. This contradiction proves that $I$ is generated by paths of length two.

(G2) Each vertex of $Q$ is the beginning and the target of at most two arrows. Indeed:

The vertices of $Q_B$ satisfy this property from Remark 4.9. Then the subquiver $Q$ of $Q_B$ inherits this property.

(G3) For each arrow $\alpha$ of $Q$ there exists at most one arrow $\beta$ such that $\alpha \beta \in I$, and there exists at most one arrow $\gamma$ such that $\gamma \alpha \in I$, and (G4) For each arrow $\alpha$ of $Q$ there exists at most one arrow $\delta$ such that $\alpha \delta \notin I$ and there exists at most one arrow $\epsilon$ such that $\epsilon \alpha \notin I$ is satisfied because any arrow $\alpha$ of $Q$ is an arrow of $Q_B$ and so it is a consequence of Remark 4.9.

Further, by the construction of $\Lambda$ we have that the quiver of the trivial extension $T(\Lambda)$ of $\Lambda$ is precisely the quiver $Q_B$ of the algebra $B$ and, moreover, the cycles of $kQ_B$ maximal in $B$ coincide with the cycles of $kQ_{T(\Lambda)}$ maximal in $T(\Lambda)$. On the other hand, we know from Proposition 4.8 that $T(\Lambda)$ satisfies the properties $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$. Thus $T(\Lambda) \cong B$ by Remark 4.10.

(ii) The quiver $Q_\Lambda$ is obtained from $Q_{T(\Lambda)}$ by deleting the arrow $\beta_p$ in each elementary cycle $p \beta_p$. Then (ii) holds because the maximal cycles of $B$ coincide with the elementary cycles of $T(\Lambda) \cong B$.

The previous theorem and Proposition 4.8 yield the following characterization of trivial extensions of gentle algebras.

**Theorem 4.12.** Let $B$ an indecomposable finite-dimensional $k$-algebra. Then $B$ satisfies $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$ if and only if there is a gentle algebra $\Lambda$ such that $T(\Lambda) \cong B$.

Given an algebra $\Lambda$, the quiver of its trivial extension is obtained from the quiver of $\Lambda$ by adding certain arrows $\beta_p$. When $\Lambda$ is a monomial algebra, we proved in Proposition 3.1 that elementary cycles coincide with maximal nonzero cycles in $T(\Lambda)$. Since $\Lambda$ is obtained from $T(\Lambda)$ by deleting the arrows $\beta_p$, and there is one arrow $\beta_p$ in each elementary cycle, it follows that $\Lambda$ is obtained from $T(\Lambda)$ by deleting exactly one arrow from each maximal cycle, and considering the induced relations. This shows that the construction in Theorem 4.11 of a gentle algebra $\Lambda$
whose trivial extension is isomorphic to \( B \) gives us in fact all gentle algebras with trivial extension isomorphic to \( B \).

5. Trivial extensions and Brauer graph algebras

In this section we relate Brauer graph algebras with the finite-dimensional algebras satisfying the properties \((T_1), (T_2), (T_3)\) and \((T_4)\) defined in the previous section.

We start with the necessary definitions, which generalize notions from the representation theory of finite groups, following the approach of Benson [2, section 4.18].

A Brauer graph is a finite connected graph (possibly with multiple edges and loops) where each vertex is assigned a cyclic ordering of the edges which are incident on it, and an integer greater than zero called the multiplicity of the vertex. We will always assume that the multiplicity at each vertex is one. If \( j_1, j_2, \ldots, j_r \) is the cyclic ordering of the edges around the vertex \( u \), then \( j_1, j_2, \ldots, j_r \) is called a sequence of successors of \( j_1 \) at the vertex \( u \).

If a loop \( j_k \) is incident on \( u \) it occurs twice in any sequence of successors at \( u \), and these occurrences are labeled \( j_k, \hat{j}_k \).

The sequence of successors of \( j_1 \) at the vertex \( u \) is unique if \( j_1 \) is not a loop; otherwise there are two: one starts at \( j_1 \), the other at \( \hat{j}_1 \).

We recall the definition of the Brauer graph algebra associated to a Brauer graph, which is a generalization of the classical Brauer tree algebra.

**Definition 5.1.** \( B_\Gamma \) is the Brauer graph algebra associated to the Brauer graph \( \Gamma \) if there is a one to one correspondence between the edges \( j \) of \( \Gamma \) and the simple modules \( S_j \) over \( B_\Gamma \), and \( P_j = P_0(S_j) \) is described by:

\[
P_j / \text{rad}(P_j) \cong \text{soc}(P_j) \cong S_j
\]

\[
\text{rad}(P_j) / \text{soc}(P_j) = U_j \oplus V_j,
\]

where \( U_j, V_j \) are the uniserial modules at the vertices \( u, v \) on which the edge \( j \) is incident, defined as follows: Let \( j = j_1, j_2, \ldots, j_r \) be the sequence of successors of \( j \) at the vertex \( u \). Then \( U_j \) is the uniserial module with composition factors (from the top) \( S_{j_2}, S_{j_3}, \ldots, S_{j_r} \) (\( U_j = 0 \) if \( r = 1 \)). If \( j \) is not a loop, \( V_j \) corresponds analogously to the sequence of successors of \( j \) at the vertex \( v \); otherwise, it corresponds to the sequence of successors at the vertex \( u \) starting at \( \hat{j}_1 \).

The quiver \( Q_\Gamma \) associated to the Brauer graph \( \Gamma \) is defined as follows. For each edge \( j_k \) of \( \Gamma \) there is a vertex \( v_{j_k} \) of \( Q_\Gamma \). If the edge \( j_{k+1} \) of \( \Gamma \) immediately follows the edge \( j_k \) in some sequence of successors, there is an arrow \( u_{j_k} \to u_{j_{k+1}} \) of \( Q_\Gamma \).

We observe that \( Q_\Gamma \) coincides with the quiver associated to the algebra \( B_\Gamma \). We describe next the maximal cycles in \( B_\Gamma \). For each sequence of successors \( j_1, j_2, \ldots, j_r \) of \( j_1 \) at the vertex \( u \) with \( r > 1 \), there is a cycle \( u = u_{j_1} \to u_{j_2} \to \cdots \to u_j \to u_{j_1} \) which is maximal in \( B_\Gamma \). If \( j_1 \) is not a loop, we will denote this cycle by \( C_{j_1,u} \). If \( j_1 \) is a loop, we will call \( C_{j_1,u}, C_{\hat{j}_1,u} \) the cycles which correspond to the two occurrences of \( j_1 \) in the cyclic ordering of the edges around the vertex \( u \).
Then the cycles described are all the maximal cycles in $B_\Gamma$, and every arrow is contained in one of them.

From the description of the indecomposable projective modules we can describe the relations for $B_\Gamma$, which are of the following three types:

- Relations of type one: $C_{j_1,u} - C_{j_1,v}$ if the edge $j_1$ of $\Gamma$ is incident on two different vertices $u, v \in \Gamma$; $C_{j_1,u} - C_{j_1,u}$ if the edge $j_1$ of $\Gamma$ is a loop in $u$.

- Relations of type two: $\alpha C_{j_1,u}$, where $u_{j_1} \xrightarrow{\alpha} u_{j_2}$.

- Relations of type three: paths of length two of $kQ_\Gamma$ which are not subpaths of a cycle $C_{j_1,u}$.

In the next theorem we characterize Brauer graph algebras given by their quiver and relations from the properties of their cycles, using for this the properties (T1), (T2), (T3) and (T4) given in Section 4.

**Theorem 5.2.** Let $B = kQ/I$ be an indecomposable finite-dimensional algebra. Then $B$ satisfies (T1), (T2), (T3) and (T4) if and only if there is a Brauer graph $\Gamma$ with multiplicity one in all the vertices such that the associated Brauer graph algebra $B_\Gamma$ is isomorphic to $B$.

**Proof.** Let $B_\Gamma$ be a Brauer graph algebra with associated Brauer graph $\Gamma$. We will prove that $B_\Gamma$ satisfies the properties (T1), (T2), (T3) and (T4). In fact:

(T1) Any permutation of a maximal cycle in $B_\Gamma$ is a maximal cycle in $B_\Gamma$.

This property is a direct consequence of the description of the maximal cycles of a Brauer graph algebra. In fact, a maximal cycle in $B_\Gamma$ is of the form $C_{j_1,u}$, and every permutation of the cycle $C_{j_1,u}$ is obtained by considering a cyclic permutation of the sequence of successors $j_1, j_2, \ldots, j_r$ of $j_1$ at the vertex $u$, which is also a sequence of successors at the same vertex, and therefore gives rise to another cycle of $kQ_\Gamma$ maximal in $B_\Gamma$.

(T2) Any path $u$ of $kQ_B$ which is nonzero in $B$ is contained in a maximal cycle of $kQ_B$, unique up to permutations if $u$ is nontrivial.

Since Brauer graph algebras are indecomposable and not semisimple, the trivial paths are not maximal. Suppose $u = u_{j_1} \rightarrow u_{j_2} \rightarrow \cdots \rightarrow u_{j_k}$ is a nontrivial path of $kQ_\Gamma$ which is nonzero in $B_\Gamma$. From the description of $Q_\Gamma$ we know that the edge $j_{t+1}$ immediately follows the edge $j_t$ in a sequence of successors of $j_1$ at some vertex $u$ of $\Gamma$ for $t = 1, \ldots, k-1$. Then the path considered is a subpath of the cycle $C_{j_1,u}$, which is maximal in $B_\Gamma$, and is the only maximal nonzero cycle containing the path $u$ up to permutations, by construction.

(T3) There are at most two different cycles with origin $j$ in $kQ_B$ which are maximal nonzero in $B_\Gamma$ for any vertex $j$ of $(Q_B)_0$. If there are two such cycles, they are equal in $B_\Gamma$. 

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The fact that this property holds in $B_\Gamma$ follows from the above description of the maximal cycles of $B_\Gamma$. In fact, if there are two maximal cycles starting at the same vertex, they are of the form $C_{j_1,u}$ and $C_{j_1,v}$ if $j_1$ is not a loop, or $C_{j_1,u}$ and $C_{\tilde{j}_1,u}$ if $j_1$ is a loop. In the first case $C_{j_1,u} - C_{j_1,v}$ is a relation of type one and in the second $C_{j_1,u} - C_{\tilde{j}_1,u}$ is also a relation of type one. Thus the two cycles are equal in $B_\Gamma$.

$(T_4)$ If $\alpha_1 \cdots \alpha_n : j \to j$ is a nonzero cycle of $kQ_{B_\Gamma}$ which is not maximal in $B_\Gamma$, then $\alpha_1 \alpha_n = 0$ in $B_\Gamma$ and $\dim_K \text{End}_K(P_j) = 4$.

Let $\gamma : u_{j_1} \to u_{j_2} \to \cdots \to u_{j_{s-1}} \to u_{j_1}$ be a cycle of $kQ_{B_\Gamma}$ nonzero and not maximal in $B_\Gamma$. From the above description of the maximal cycles of $B_\Gamma$ we conclude that the edge $j_1$ is a loop, there are two sequences of successors $j_1, j_2, \ldots, j_{s-1}, \tilde{j}_1, j_{s+1}, \ldots, j_r$ and $\tilde{j}_1, j_{s+1}, \ldots, j_r, j_1, j_2, \ldots, j_{s-1}$, and the maximal cycles with origin $u_{j_1}$ are $C_{j_1,u_{j_1}}, C_{\tilde{j}_1,u_{j_1}}$. Since the path $u_{j_{s-1}} \to u_{\alpha_1} \to u_{j_1} \to u_{j_2}$ is not a subpath of any of these two maximal cycles, it follows that $\alpha_1 \alpha_n$ is a relation of type three. Therefore $\alpha_1 \alpha_n = 0$ in $B_\Gamma$.

The cycle $\gamma' : u_{j_1} \to u_{j_{s+1}} \to \cdots \to u_{j_r} \to u_{j_1}$ with origin $u_{j_1}$ is also nonzero and not maximal in $B_\Gamma$, since it is properly contained in $C_{j_1,u_{j_1}}$. Then, the cycles starting at $u_j$ are $\gamma, \gamma'$ and the maximal cycles $\gamma' = C_{j_1,u_{j_1}}$ and $\gamma'' = C_{\tilde{j}_1,u_{j_1}}$. Since $C_{j_1,u_{j_1}} - C_{\tilde{j}_1,u_{j_1}}$ is a relation of type one, we have that $\text{End}_{B_\Gamma} P_{j_1}$ is generated by $\overline{C_{j_1}}, \overline{\gamma}, \overline{\gamma'}$ and $\overline{C_{\tilde{j}_1,u_{j_1}}}$. Thus $\dim_K \text{End}_{B_\Gamma} P_{j_1} = 4$, so $(T_4)$ holds for $B_\Gamma$. This ends the proof that Brauer graph algebras satisfy the properties $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$.

Now consider $B = kQ/I$ satisfying $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$. We will construct a Brauer graph $\Gamma$ such that $B \simeq B_\Gamma$. The set of maximal cycles of $B$ is not empty because $(T_2)$ holds. Consider the equivalence relation defined in this set by $C \sim C'$ if and only if $C$ is a permutation of $C'$, if $C, C'$ are maximal cycles. Let \{$C_1, \ldots, C_s$\} be the set of equivalence classes.

We associate to the quiver $Q$ a Brauer graph $\Gamma$ as follows. The set of vertices of $\Gamma$ is

$$\{u_{C_1}, u_{C_2}, \ldots, u_{C_s}\} \cup \{v_j : v_j \in Q_0 \text{ and } v_j \text{ is the origin of a unique maximal cycle}\}.$$ 

The edges $a_i$ of $\Gamma$ correspond to vertices $v_i$ of $Q$ in the following way: If $v_i$ is the beginning of two different maximal cycles $C_1, C_2$, the endpoints of $a_i$ are $u_{C_1}, u_{C_2}$. Notice that when $\overline{C_1} = \overline{C_2}$ we obtain a loop. If $v_i$ is the beginning of a unique maximal cycle $C'$ then the endpoints of $a_i$ are $u_{C'}, u_{v_i}$, and so $a_i$ is the only edge with $u_{v_i}$ as an endpoint. For each equivalence class $\overline{C_k}$, the edges that are incident on the vertex $u_{\overline{C_k}}$ correspond to the vertices of the maximal cycle $C_k$, and we define a cyclic ordering in this set of edges as follows. The edge $a_{j+1}$ is the immediate successor of the edge $a_j$ if there is an arrow $v_j \to v_{j+1}$ of $Q$ contained in a maximal cycle of $\overline{C_k}$. This ordering is well defined since any arrow is contained in a unique maximal cycle up to permutations, by $(T_2)$. Finally, we assign the multiplicity one
to each vertex of $\Gamma$, and $\Gamma$ is then a Brauer graph. From the Brauer graph $\Gamma$ we obtain the Brauer graph algebra $B_\Gamma = Q_\Gamma/I_\Gamma$. We know that $B_\Gamma$ satisfies ($T_1$), ($T_2$), ($T_3$) and ($T_4$), since we just proved that Brauer graph algebras do.

We will prove that $Q = Q_\Gamma$, and that the cycles of $Q$ maximal in $B$ coincide with the cycles of $Q = Q_\Gamma$ maximal in $B_\Gamma$.

We prove next that $Q = Q_\Gamma$. The Brauer graph $\Gamma$ was constructed in such a way that the vertices of $Q$ are in bijective correspondence with the edges of $\Gamma$. On the other hand, these edges are in bijective correspondence with the vertices of $Q_\Gamma$ maximal in $B_\Gamma$, as follows from the definition of Brauer graph algebra associated to a Brauer graph. Then the vertices of $Q$ are in bijective correspondence with those of $Q_\Gamma$.

We will denote by $u_i$ the vertex of $Q_\Gamma$ corresponding to the vertex $v_i$ of $Q$ under this bijection. Then $a_i$ is the edge of $\Gamma$ corresponding to $u_i$ in $Q_\Gamma$ and to $v_i$ in $Q$.

Let $v_j \to v_{j+1}$ be an arrow in $Q$. Then this arrow belongs to a unique maximal cycle $C_j$, by ($T_2$), and the edge $a_{j+1}$ of $\Gamma$ is an immediate predecessor of $a_j$ in the cyclic ordering of the edges of $\Gamma$ at the vertex $u_{C_j}$. This means that there is an arrow $u_j \to u_{j+1}$ in $Q_\Gamma$. Conversely, if there is an arrow $u_j \to u_{j+1}$ in $Q_\Gamma$ a similar argument shows that there is an arrow $v_j \to v_{j+1}$ in $Q$. Thus $Q$ and $Q_\Gamma$ have the same arrows, so we can identify the quivers $Q$ and $Q_\Gamma$.

Finally, we will prove that the cycles of $Q$ maximal in $B$ coincide with the cycles of $Q_\Gamma$ maximal in $B_\Gamma$. In fact, saying that $C_j = v_{j_1} \to v_{j_2} \to \cdots \to v_{j_t}$ is a cycle of $kQ$ maximal in $B$ is equivalent to saying that $a_{j_1}, a_{j_2}, \ldots, a_{j_t}$ are all the edges of $\Gamma$ with endpoint $u_{C_j}$, with the cyclic ordering $a_{j_1} \leq a_{j_2} \leq \cdots \leq a_{j_t}$. This means that $a_j, a_j, \ldots, a_j$ is a sequence of successors of $a_j$ at the vertex $u_{C_j}$, which amounts to say that $C_{j_1, u_{C_j}} = u_{j_1} \to u_{j_2} \to \cdots \to u_{j_t}$ is a cycle of $Q_\Gamma$ maximal in $B_\Gamma$.

Then cycles of $Q$ maximal in $B$ coincide with cycles of $Q_\Gamma$ maximal in $\Gamma$.

We know from Remark 4.10 that algebras that satisfy the properties ($T_1$), ($T_2$), ($T_3$) and ($T_4$) with the same quiver and the same maximal cycles are isomorphic, thus $B_\Gamma \cong B$. □

In Theorem 4.12 we proved that properties ($T_1$), ($T_2$), ($T_3$) and ($T_4$) characterize trivial extensions of gentle algebras. Combining this result with the previous theorem we get the following characterization of trivial extensions of gentle algebras, which was obtained by S. Schroll using a different approach.

**Corollary 5.3** (4, Theorem 1.2 and Corollary 1.4). Let $B$ be an indecomposable finite-dimensional algebra. Then $B$ is a Brauer graph algebra with multiplicity one in all the vertices of the associated Brauer graph if and only if $B$ is the trivial extension of a gentle algebra.

We summarize the results of the last two sections in the following theorem.
Theorem 5.4. Let $B = kQ_B/I_B$ be a finite-dimensional indecomposable $k$-algebra. Then the following conditions are equivalent:

(i) $B$ is isomorphic to the trivial extension of a gentle algebra.
(ii) $B$ satisfies the properties $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$.
(iii) $B$ is isomorphic to the Brauer graph algebra $B_\Gamma$ associated to a Brauer graph $\Gamma$ with multiplicity one in all the vertices.

Let us see an example to illustrate this result.

Example 5.5. Let $B = kQ_B/I_B$, where

\begin{align*}
Q_B: & \quad \bullet_1 \xrightarrow{\alpha_2} \bullet_2 \xrightarrow{\alpha_3} \bullet_3 \xrightarrow{\alpha_4} \\
& \quad \alpha_1 \quad \alpha_5 \\
I_B = & \quad (\alpha_1 \alpha_2 \alpha_1, \alpha_2 \alpha_1 \alpha_2, \alpha_3 \alpha_2, \\
& \quad \alpha_1 \alpha_5, \alpha_5 \alpha_3, \alpha_4 \alpha_4, \alpha_3 \alpha_5 \alpha_4, \alpha_3 \alpha_5 \alpha_3, \\
& \quad \alpha_4 \alpha_3 \alpha_5 \alpha_4, \alpha_5 \alpha_4 \alpha_3 \alpha_5, \\
& \quad \alpha_2 \alpha_1 - \alpha_5 \alpha_4 \alpha_3, \\
& \quad \alpha_3 \alpha_5 \alpha_4 - \alpha_4 \alpha_3 \alpha_5) 
\end{align*}

The cycles of $kQ_B$ maximal in $B$ are $C_1 = \alpha_1 \alpha_2$ and its permutation $\alpha_2 \alpha_1$, $C_2 = \alpha_5 \alpha_4 \alpha_3$ and its permutations $\alpha_3 \alpha_5 \alpha_4$ and $\alpha_4 \alpha_3 \alpha_5$. Then the algebra $B$ satisfies the properties $(T_1)$, $(T_2)$, $(T_3)$ and $(T_4)$. To describe the Brauer graph $\Gamma$ associated to $Q_B$ we observe that the vertex 1 is the beginning of a unique maximal cycle. The vertices of $\Gamma$ are $u_1$, $u_2 = u_\overline{C_1}$ and $u_3 = u_\overline{C_2}$. Since $Q_B$ has three vertices, $\Gamma$ has three edges: $a_1$ with endpoints $u_1$ and $u_2$, $a_2$ with endpoints $u_2$ and $u_3$, and $a_3$, which is a loop at $u_3$. Thus $\Gamma$ is the graph

\begin{center}
\begin{tikzpicture}
\node (u1) at (0,0) {$u_1$};
\node (u2) at (2,0) {$u_2$};
\node (u3) at (4,0) {$u_3$};
\node (s1) at (0,1) {$s_1$};
\node (s2) at (2,1) {$s_2$};
\draw (u1) -- (u2) -- (u3) -- (u1);
\draw (u2) -- (s1);
\draw (u3) -- (s2);
\end{tikzpicture}
\end{center}

where we choose the cyclic counterclockwise ordering of the edges around each vertex and we assign multiplicity one to each vertex.

On the other hand, if we delete in $Q_B$ one arrow of each maximal cycle we obtain a gentle algebra $\Lambda$ such that $T(\Lambda) \simeq B$. For example, when we choose the arrows $\alpha_2$ and $\alpha_5$ we obtain $\Lambda = kQ_\Lambda/I_\Lambda$, with

\begin{align*}
Q_\Lambda: & \quad \bullet_1 \xrightarrow{\alpha_1} \bullet_2 \xrightarrow{\alpha_3} \bullet_3 \xrightarrow{\alpha_4} \\
I_\Lambda = & \quad (\alpha_2^2)
\end{align*}

References


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