ON THE PLANARITY, GENUS, AND CROSSECT OF THE
WEAKLY ZERO-DIVISOR GRAPH OF COMMUTATIVE RINGS

NADEEM UR REHMAN, MOHD NAZIM, AND SHABIR AHMAD MIR

Abstract. Let $R$ be a commutative ring and $Z(R)$ its zero-divisors set. The
weakly zero-divisor graph of $R$, denoted by $W^\Gamma(R)$, is an undirected graph
with the nonzero zero-divisors $Z(R)^*$ as vertex set and two distinct vertices
$x$ and $y$ are adjacent if and only if there exist $a \in \text{Ann}(x)$ and $b \in \text{Ann}(y)$
such that $ab = 0$. In this paper, we characterize finite rings $R$ for which
the weakly zero-divisor graph $W^\Gamma(R)$ belongs to some well-known families of
graphs. Further, we classify the finite rings $R$ for which $W^\Gamma(R)$ is planar,
toroidal or double toroidal. Finally, we classify the finite rings $R$ for which
the graph $W^\Gamma(R)$ has crossec at most two.

1. Introduction

All rings $R$ considered in this paper will be commutative with unit element
$1 \neq 0$. For $x \in R$, the set $\text{Ann}(x) = \{y \in R^* : xy = 0\}$ is the annihilator of $x$. The
set of all zero-divisors, nilpotent elements, minimal prime ideals and unit elements
of a ring $R$ are denoted by $Z(R)$, $\text{Nil}(R)$, $\text{Min}(R)$ and $U(R)$, respectively. We write
$S^* = S \setminus \{0\}$ for any subset $S$ of $R$. We refer the reader to [6] for any ambiguous
notation or vocabulary in ring theory.

Algebraic combinatorics is an area of mathematics which employs methods of
abstract algebra in various combinatorial contexts and vice versa. Lately, linking
a graph to the algebraic structure has received a lot of attention.

A variety of graphs attached to rings or other algebraic structures can be found
in the literature. In [7], Beck introduced for the first time a graph associated
to a commutative ring $R$ with the elements of $R$ as its vertices, and was mainly
interested in the coloring of commutative rings. In [3], Anderson and Livingston
introduced the zero-divisor graph of $R$, denoted by $\Gamma(R)$, with vertex set $Z(R)^*$
(the set of nonzero zero-divisors of $R$), and where two vertices $x \neq y \in Z(R)^*$

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are adjacent if and only if $xy = 0$. See [11, 2, 4, 8, 9, 16, 19] for more details. Several authors have looked at the zero-divisor graphs of commutative rings. Later, Redmond [15] established the zero-divisor graph of a noncommutative ring which corresponds to the concept introduced by Demeyer et al. in [10] for semigroups.

In [14], Nikmehr et al. introduced and studied the weakly zero-divisor graph of a commutative ring $R$, denoted by $W\Gamma(R)$. It is an undirected graph with vertex set $Z(R)^*$, where two distinct vertices $x$ and $y$ are adjacent if and only if there exist $a \in \text{Ann}(x)$ and $b \in \text{Ann}(y)$ such that $ab = 0$. The authors in [14] discussed some basic properties of the weakly zero-divisor graph and studied the similarities between $W\Gamma(R)$ and $\Gamma(R)$.

In this paper, we characterize the finite rings $R$ for which $W\Gamma(R)$ is a tree, a unicycle or a split graph. Then we classify the finite rings $R$ for which $W\Gamma(R)$ is a planar, ring, outerplanar, toroidal or double toroidal graph. Finally, we classify the finite rings $R$ for which the graph $W\Gamma(R)$ has crosscap at most two.

2. Preliminaries

Let $G$ be a graph with vertex set $V(G)$. The distance between two vertices $u$ and $v$ of $G$, denoted by $d(u, v)$, is the smallest path from $u$ to $v$. If there is no such path, then $d(u, v) = \infty$. The diameter of $G$ is defined as $\text{diam}(G) = \sup\{d(u, v) : u, v \in V(G)\}$. A cycle is a closed path in $G$. The girth of $G$, denoted by $\text{gr}(G)$, is the length of a shortest cycle in $G$. Note that $\text{gr}(G) = \infty$ whenever $G$ contains no cycle. A graph is said to be a complete graph if all its vertices are adjacent to each other. A complete graph with $n$ vertices is denoted by $K_n$. A bipartite graph is a graph $G$ whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge in $G$ has one end in $V_1$ and the other end in $V_2$. Further, if each vertex of $V_1$ is adjacent to every vertex of $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with partition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. We write $K_{m,\infty}$ (respectively, $K_{\infty,\infty}$) if one (respectively, both) of the disjoint vertex sets is infinite. A complete bipartite graph of the form $K_{1,n}$ is called a star graph. A connected graph is said to be a tree if it does not contain cycles. A graph is said to be a unicycle whenever it contains a unique cycle. A graph is said to be a split graph if its vertex set can be partitioned into a clique and an independent set. We say that a graph is planar whenever it can be drawn in the plane in such a way that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski’s Theorem says that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$. An undirected graph is said to be outerplanar if it can be embedded in the plane in such a way that all the vertices lie on the unbounded face of the drawing. For more details on graph theory, we refer the reader to [17, 18].

The following observation proved by Nikmehr et al. [14] is used frequently in this article and hence given below.
Lemma 2.1 ([14] Lemma 2.1). If $R$ is a commutative ring, then the following statements hold:

1. If $p - q$ is an edge of $\Gamma(R)$ for some distinct elements $x, y \in Z(R)^*$, then $p - q$ is an edge of $WT(R)$.
2. If $p \in \text{Nil}(R)^*$, then $p$ is adjacent to all other vertices.
3. $\text{Nil}(R)^*$ is a complete subgraph of $WT(R)$.

Theorem 2.2. If $R$ is a local ring, then $WT(R)$ is a complete graph.

Proof. It is clear from Lemma 2.1(3).

Theorem 2.3 ([14] Theorem 3.1]). If $R$ is a reduced ring which is not an integral domain, then $WT(R) = \Gamma(R)$ if and only if $|\text{Min}(R)| = 2$.

In the following examples, we calculate the weakly zero-divisor graph of some rings.

Example 2.4. If $R = \mathbb{Z}_8$, then $Z(R) = \{0, 2, 4, 6\}$. Also, $\text{Ann}(2) = \{4\}$, $\text{Ann}(4) = \{2, 4, 6\}$ and $\text{Ann}(6) = \{4\}$. Since $2 \cdot 4 = 0$, $4 \cdot 6 = 0$ and $4 \in \text{Ann}(2) \cap \text{Ann}(6)$ such that $4 \cdot 4 = 0$, we have that the graph $WT(R)$ is $K_3$.

Example 2.5. If $R = \mathbb{Z}_{25}$, then $Z(R) = \{0, 5, 10, 15, 20\}$. Also, since $5 \cdot 10 = 0$, $5 \cdot 15 = 0$, $5 \cdot 20 = 0$, $10 \cdot 15 = 0$, $10 \cdot 20 = 0$ and $15 \cdot 20 = 0$, we have that the graph $WT(R)$ is $K_4$.

Some finite local rings and their weakly zero-divisor graphs are given in Table [1].

3. Basic properties of $WT(R)$

In this section, we classify the finite rings for which the weakly zero-divisor graph is a unicycle, a tree or a split graph. The following results will play an important role in the characterization of commutative rings whose weakly zero-divisor graph is a unicycle, a tree or a split graph.

Lemma 3.1 ([12] Theorem VI-2]). Let $R$ be a finite commutative ring. Then $R$ decomposes uniquely (up to order of summands) as a direct sum of local rings.

Lemma 3.2. If $m \geq 3$ and $R = R_1 \times R_2 \times \cdots \times R_m$ for some commutative rings $R_i$, then $WT(R)$ contains $K_5$ as a subgraph.

Proof. Let $p_1 = e_1$, $p_2 = e_1 + e_2$, $p_3 = e_2$, $p_4 = e_2 + e_3$, $p_5 = e_3 \in Z(R)^*$, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$. Since $p_3 \in \text{Ann}(p_1)$ and $p_5 \in \text{Ann}(p_2)$ such that $p_3p_5 = 0$, $p_1p_3 = 0$, $p_1p_4 = 0$, $p_1p_5 = 0$, $p_5 \in \text{Ann}(p_2)$ and $p_1 \in \text{Ann}(p_3)$ such that $p_1p_5 = 0$, $p_2p_5 = 0$, $p_5 \in \text{Ann}(p_3)$ and $p_1 \in \text{Ann}(p_4)$ such that $p_1p_5 = 0$, $p_3p_5 = 0$, $p_1 \in \text{Ann}(p_4)$ and $p_3 \in \text{Ann}(p_5)$ such that $p_1p_3 = 0$, we see that the vertices $\{p_1, p_2, p_3, p_4, p_5\}$ induce a complete graph with five vertices. □
Table 1. Weakly zero-divisor graphs of some finite local commutative rings

<table>
<thead>
<tr>
<th></th>
<th>$Z(R)^*$</th>
<th>Local ring $R$</th>
<th>WT($R$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$</td>
<td></td>
<td>$K_1$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$</td>
<td></td>
<td>$K_2$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^4)}$, $\frac{\mathbb{Z}_4[x]}{(x^4)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,y^2)}$, $\frac{F_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$</td>
<td></td>
<td>$K_3$</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[y]}{(x^2)}$</td>
<td></td>
<td>$K_4$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{(x^2)}$</td>
<td></td>
<td>$K_6$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{(x^4)}$, $\frac{\mathbb{Z}_4[x]}{(x^2,x^2-2x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^2-2x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2,x^2-2x^2)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,x^2,y^2-x^2)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^2,x^2,y^2-x^2)}$, $\frac{F_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, $\frac{F_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$</td>
<td></td>
<td>$K_7$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{Z}_{27}$, $\frac{\mathbb{Z}_3[x]}{(3x^2-3)}$, $\frac{\mathbb{Z}_3[x]}{(3x^2-6)}$, $\frac{\mathbb{Z}_3[x]}{(5x^2)}$, $\frac{\mathbb{Z}_3[x]}{(x^2+y^2)}$, $\frac{\mathbb{Z}_3[x]}{(x^2+y^2)}$, $\frac{\mathbb{Z}_3[x]}{(x^2+y^2)}$, $\frac{\mathbb{Z}_3[x]}{(x^2+y^2)}$, $\frac{F_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, $\frac{F_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$</td>
<td></td>
<td>$K_8$</td>
</tr>
</tbody>
</table>

Lemma 3.3. Let $R_1$ and $R_2$ be local commutative rings. If either $R_1$ or $R_2$ is not a field, then $\text{WT}(R_1 \times R_2)$ contains $K_4$ as a subgraph.

Proof. Suppose without loss of generality that $R_1$ is not a field with nonzero maximal ideal $\mathfrak{M}_1$. Then there exists $\alpha \in \mathfrak{M}_1^*$ such that $\text{Ann}(\alpha) = \mathfrak{M}_1$. Let $q_1 = (1,0)$, $q_2 = (\alpha,0)$, $q_3 = (\alpha,1)$ and $q_4 = (0,1) \in Z(R)^*$. Since $q_4 \in \text{Ann}(q_1)$ and $q_2 \in \text{Ann}(q_2)$ with $q_2q_4 = 0$, $q_4 \in \text{Ann}(q_1)$ and $q_2 \in \text{Ann}(q_3)$ with $q_2q_4 = 0$, $q_1q_4 = 0$, $q_1q_4 = 0$, $q_2 \in \text{Ann}(q_3)$ and $q_2 \in \text{Ann}(q_4)$ with $q_2q_4 = 0$, we get that $\{q_1, q_2, q_3, q_4\}$ induces a $K_4$ in $\text{WT}(R_1 \times R_2)$. \hfill $\square$

Now we are ready to characterize the finite commutative rings such that their weakly zero-divisor graph is a unicycle, a tree or a split graph.

Theorem 3.4. If $R$ is a finite commutative ring, then $\text{WT}(R)$ is a unicycle if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^4)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,y^2)}$, $\frac{F_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$.

Proof. Assume that $\text{WT}(R)$ is a unicycle. Since $R$ is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \mathfrak{M}_i)$ is a local ring for each $i$ and $m \geq 1$. If $m \geq 3$, then by Lemma 3.2, $\text{WT}(R)$ contains two different cycles, which contradicts our assumption.

If $m = 2$ and either $R_1$ or $R_2$ is not a field, then by Lemma 3.3, $\text{WT}(R)$ contains two different cycles, a contradiction. Hence $R_1$ and $R_2$ are both fields.
This implies that $WT(R) \cong K_{|R_1^*|, |R_2^*|}$. Since we are assuming that $WT(R)$ is a unicycle, $|R_1^*| = 2$ and $|R_2^*| = 2$. Therefore, $R \cong Z_3 \times Z_3$.

Finally, if $m = 1$, then $R$ is a local ring. Thus by Theorem 2.2, $WT(R)$ is a complete graph, because $WT(R)$ contains a cycle, a contradiction. Hence $m \leq 2$.

Now, if $m = 2$ and either $R_1$ or $R_2$ is not a field, then by Lemma 3.3, $WT(R)$ contains a cycle, a contradiction. Hence $R_1$ and $R_2$ are both fields. Thus, $WT(R) \cong K_{|R_1^*|, |R_2^*|}$. Since we are assuming that $WT(R)$ is a tree if and only if it is isomorphic to one of the following rings: $Z_4$, $Z_2[x, y]/(x^2 + y^2)$, $Z_2[x]/(x^2)$ or $Z_2 \times F$.

**Theorem 3.5.** If $R$ is a finite commutative ring, then $WT(R)$ is a tree if and only if $R$ is isomorphic to one of the following rings: $Z_4$, $Z_2[x, y]/(x^2)$, $Z_2[x]/(x^2)$ or $Z_2 \times F$.

**Proof.** Assume that $WT(R)$ is a tree. Since $R$ is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \mathfrak{z}_i)$ is a local ring for each $i$ and $m \geq 1$. If $m = 3$, then by Lemma 3.2, $WT(R)$ contains a cycle, a contradiction. Hence $m \leq 2$.

Now, if $m = 2$ and either $R_1$ or $R_2$ is not a field, then by Lemma 3.3, $WT(R)$ contains a cycle, a contradiction. Hence $R_1$ and $R_2$ are both fields. Thus, $WT(R) \cong K_{|R_1^*|, |R_2^*|}$. Since we are assuming that $WT(R)$ is a tree, $|R_1^*| = 1$ or $|R_2^*| = 1$. Hence $R_1 \cong Z_2$ or $R_2 \cong Z_2$.

If $m = 1$, then $WT(R)$ is a complete graph by Theorem 2.2 because $R$ is a local ring. Also, we are assuming that $WT(R)$ is a tree, then $1 \leq |Z(R)^*| \leq 2$. Therefore by Table 1, $R \cong Z_4$, $Z_2[x, y]/(x^2)$, $Z_2[x]/(x^2)$ or $Z_2 \times F$. □

**Theorem 3.6 (17).** If $G$ is a connected graph, then $G$ is a split graph if and only if $G$ contains no induced subgraph isomorphic to $2K_2$, $C_4$ or $C_5$.

**Theorem 3.7.** If $R$ is a finite commutative ring with $|Z(R)^*| \geq 2$, then $WT(R)$ is a split graph if and only if $R$ is isomorphic to one of the following rings: $Z_9$, $Z_2[x, y]/(x^2)$, $Z_8$, $Z_2[x]/(x^2)$, $Z_4[x, y]/(x^2, y^2)$, $Z_4[x]/(x^2)$ or $Z_2 \times F$.

**Proof.** Assume that $WT(R)$ is a split graph. Since $R$ is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \mathfrak{z}_i)$ is a local ring for each $i$ and $m \geq 1$. If $m = 3$, then by Lemma 3.2, $WT(R)$ contains $C_4$, a contradiction by Theorem 3.6.

Now, if $m = 2$ and either $R_1$ or $R_2$ is a field, then by Lemma 3.3, $WT(R)$ contains $C_4$, a contradiction by Theorem 3.6. Hence $R_1$ and $R_2$ are both fields. Thus, $WT(R) \cong K_{|R_1^*|, |R_2^*|}$. Since we are assuming that $WT(R)$ is a split graph, $|R_1^*| = 1$ or $|R_2^*| = 1$. Hence $R_1 \cong Z_2$ or $R_2 \cong Z_2$.

Finally, if $m = 1$, then $WT(R)$ is a complete graph, because $R$ is local. Also, we are assuming that $WT(R)$ is a split graph, then $2 \leq |Z(R)^*| \leq 3$. Therefore by Table 1, $R \cong Z_9$, $Z_2[x, y]/(x^2)$, $Z_8$, $Z_2[x]/(x^2)$, $Z_4[x, y]/(x^2, y^2)$, $Z_4[x]/(x^2)$ or $Z_2 \times F$. □

4. Planar, outerplanar, and ring graph $WT(R)$

In this section, we characterize the finite commutative rings $R$ for which $WT(R)$ is a planar, a ring or an outerplanar graph. We recall the characterization of planar graphs given by Kuratowski, which will play an important role in the characterization of commutative rings whose weakly zero-divisor graph is planar.

**Theorem 4.1 (Kuratowski’s Theorem, 17).** A graph $G$ is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$.
Theorem 4.2. If $R$ is a finite commutative ring, then $WT(R)$ is a planar graph if and only if $R$ is isomorphic to one of the following: \( \mathbb{Z}_4 \), \( \mathbb{Z}_9 \), \( \mathbb{Z}_2[x]/(x^2) \), \( \mathbb{Z}_8 \), \( \mathbb{Z}_8[x]/(x^2) \), \( \mathbb{Z}_4 \), \( \mathbb{Z}_4[x]/(x^2) \), \( \mathbb{Z}_2 \times \mathbb{F}_2 \), \( \mathbb{Z}_3 \times \mathbb{F}_2 \), \( \mathbb{Z}_4 \times \mathbb{Z}_2 \), or \( \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2 \).

Proof. Assume that $WT(R)$ is planar. Since $R$ is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \mathfrak{S}_i)$ is a local ring for each $i$ and $m \geq 1$. If $m \geq 3$, then by Lemma 3.2 $WT(R)$ contains $K_5$ as a subgraph, a contradiction by Theorem 4.1.

If $m = 2$ and $\mathfrak{S}_i \neq (0)$ for each $i = 1, 2$, then by [14, Theorem 2.6], $WT(R)$ contains $K_5$ induced by the set \( \{(1,0), (\alpha_1,0), (0,\alpha_2), (\alpha_1,\alpha_2), (0,1)\} \), where $\alpha_i \in \mathfrak{S}_i^*$ for each $i$, a contradiction by Theorem 4.1. Hence one of the $R_i$ must be a field.

Consider the following cases:

**Case (i)** If $R_1$ and $R_2$ both are fields, then $WT(R) \cong K_{|R_1^*|,|R_2^*|}$. Since we are assuming that $WT(R)$ is planar, $|R_1^*| \leq 2$ or $|R_2^*| \leq 2$ by Theorem 4.1. Hence $R \cong \mathbb{Z}_2 \times \mathbb{F}$ or $\mathbb{Z}_3 \times \mathbb{F}$.

**Case (ii)** If $R_1$ is not a field with $\mathfrak{S}_1 \neq (0)$ and $R_2$ is a field, then there is $\alpha \in \mathfrak{S}_1^*$ such that $\text{Ann}(\alpha) = \mathfrak{S}_1$. Suppose $|\mathfrak{S}_1^*| \geq 2$. Let $q_1 = (0,1)$, $q_2 = (\alpha,0)$, $q_3 = (\beta,0)$, $r_1 = (1,0)$, $r_2 = (\gamma,0)$, $r_3 = (\delta,0)$, where $\alpha \neq \beta \in \mathfrak{S}_1^*$ and $1 \neq \gamma, \delta \in U(R_1)$.

Since $q_1r_i = 0$, $q_2 \in \text{Ann}(q_2)$ and $q_1 \in \text{Ann}(r_i)$ such that $q_1q_2 = 0$, $q_2 \in \text{Ann}(q_3)$ and $q_1 \in \text{Ann}(r_i)$ such that $q_1q_2 = 0$ for each $i = 1, 2, 3$. we get that $\{q_1, q_2, q_3, r_1, r_2, r_3\}$ induces $K_{3,3}$ in $WT(R)$, a contradiction by Theorem 4.1. Hence $|\mathfrak{S}_1^*| = 1$, which shows that $R_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

Suppose $|R_2^*| \geq 2$ and let $\alpha \in \mathfrak{S}_1^*$ such that $\alpha^2 = 0$. Let $s_1 = (1,0)$, $s_2 = (\alpha,0)$, $s_3 = (\alpha_1,0)$, $t_1 = (0,1)$, $t_2 = (0,\alpha_2)$, $t_3 = (\alpha,1) \in Z(R)^*$, where $1 \neq \alpha \in U(R_1)$ and $1 \neq \alpha \in U(R_2)$. Since $s_it_j = 0$ for each $j = 1, 2$, $t_1 \in \text{Ann}(s_i)$ and $s_2 \in \text{Ann}(t_2)$ such that $s_2t_1 = 0$ for each $i = 1, 2, 3$, we get that $\{s_1, s_2, s_3, t_1, t_2, t_3\}$ induces $K_{3,3}$ in $WT(R)$, a contradiction by Theorem 4.1. Hence $|R_2^*| = 1$, which shows that $R_2 \cong \mathbb{Z}_2$.

Finally, if $m = 1$, then $WT(R)$ is a complete graph by Theorem 2.2 because $R$ is a local ring. Also, we are assuming that $WT(R)$ is a planar graph, then $1 \leq |Z(R)^*| \leq 4$ by Theorem 4.1. Therefore by Table 1 $R \cong \mathbb{Z}_4$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_8[x]/(x^2)$, $\mathbb{Z}_4$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_{25}$ or $\mathbb{Z}_2[x]/(x^2)$.

Conversely, if $R \cong \mathbb{Z}_4 \times \mathbb{F}$ or $\mathbb{Z}_9 \times \mathbb{F}$, then $WT(R) \cong K_{1,n}$ or $K_{2,n}$, where $n \geq 1$ is a positive integer. Hence $WT(R)$ is planar by Theorem 4.1. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$, the planar embedding of $WT(R)$ is shown in Figure 1. Also, if $R \cong \mathbb{Z}_4$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_8[x]/(x^2)$, $\mathbb{Z}_4$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_{25}$ or $\mathbb{Z}_2[x]/(x^2)$, then the result follows from Table 1 and Theorem 4.1.

Let $C$ be a cycle of $G$. Any edge in $G$ that connects two nonadjacent vertices in $C$ is called a chord. A primitive cycle is one that has no chords. Furthermore, we claim that $G$ has the primitive cycle property (PCP) if any two primitive cycles
intersect in at most one edge. The \textit{frank} of $G$, denoted by \text{frank}(G), equals the number of primitive cycles of $G$. Also, \text{rank}(G) = q - n + r$, where $q$, $n$ and $r$ denote the number of edges, vertices and connected components of $G$, respectively.

Section 2 of [11] contains a detailed definition of a ring graph. The authors in [11] also demonstrated the following equivalence.

\textbf{Theorem 4.3 ([11])}. If $G$ is a connected graph, then following are equivalent:

1. $G$ is a ring graph,
2. \text{rank}(G) = \text{frank}(G),
3. $G$ satisfies PCP and $G$ does not contain a subdivision of $K_4$ as a subgraph.

As a result, each ring graph is planar. In the following theorem, we characterize all finite commutative rings $R$ for which $WT(R)$ is a ring graph.

\textbf{Theorem 4.4}. If $R$ is a finite commutative ring, then $WT(R)$ is a ring graph if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^2+2)$, $\mathbb{Z}_2[x,y]/(2x,2y)$, $\mathbb{Z}_2[x]/(x^2+y^2)$, $\mathbb{Z}_2[x]/(x^2+y^2)$, $\mathbb{Z}_2 \times \mathbb{F}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

\textit{Proof}. Since every ring graph is a planar graph, it is enough to deal with rings whose weakly zero-divisor graphs are planar. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, then $WT(R)$ contains $K_4$ induced by the set $\{(1,0), (0,1), (0,2), (2,1)\}$ as shown in Figure [1]. Hence by Theorem 4.3, $WT(R)$ is not a ring graph. Also, if $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$, then $WT(\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2) \cong WT(\mathbb{Z}_4 \times \mathbb{Z}_2)$, which implies that $WT(\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2)$ is not a ring graph.

If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, then $WT(R) \cong K_{1,n}$, where $n \geq 1$ is a positive integer. Thus, by Theorem 4.3, $WT(R)$ is a ring graph. Also, if $R \cong \mathbb{Z}_3 \times \mathbb{F}$, then $WT(R) \cong K_{2,n-1}$, where $n = |\mathbb{F}|$. Thus, \text{rank}(WT(R)) = n - 2 and \text{frank}(WT(R)) = \frac{(n-1)(n-2)}{2}.

Hence $WT(R)$ is a ring graph if and only if $n - 2 = \frac{(n-1)(n-2)}{2}$, which implies that $n = 2$ or $n = 3$. Hence $\mathbb{F} \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

If $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then $WT(R) \cong K_1$ by Table I, which is a ring graph. If $R \cong \mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^2)$, or $\mathbb{Z}_4[x]/(x^2, x^2+2)$,
\[
\frac{Z_4[x]}{(x^2, xy^2, y^4)}, \frac{Z_2[x, y]}{(x^2, xy^2)}, \frac{F_4[x]}{(x^2)} \quad \text{or} \quad \frac{Z_4[x]}{(x^2+y^2)}, \text{then } WT(R) \cong K_3 \text{ by Table 4.} \]
this is also a ring graph. If \( R \cong \mathbb{Z}_{25} \) or \( \frac{Z_5[x]}{(x^2)} \), then \( WT(R) \cong K_4 \), which is not a ring graph by Theorem 4.3.

\section*{Theorem 4.5 (17).} A graph \( G \) is outerplanar if and only if it does not contain a subdivision of \( K_4 \) or \( K_{2,3} \).

In the next theorem, we determine all finite commutative rings with outerplanar weakly zero-divisor graphs.

\section*{Theorem 4.6.} If \( R \) is a finite commutative ring, then \( WT(R) \) is an outerplanar graph if and only if \( R \) is isomorphic to one of the following rings: \( \mathbb{Z}_4 \), \( \frac{Z_2[x]}{(x^2)} \), \( \mathbb{Z}_9 \), \( \frac{Z_4[x]}{(x^2+y^2)} \), \( \frac{Z_4[x]}{(x^2+y^2)} \), \( \frac{Z_2[x]}{(x^2)} \), \( \frac{Z_4[x]}{(x^2+y^2)} \), \( \frac{F_4[x]}{(x^2)} \), \( \frac{Z_4[x]}{(x^2+y^2)} \), \( \mathbb{Z}_2 \times \mathbb{F} \) or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \).

\section*{Proof.} In view of Theorems 4.3 and 4.5 one can say that every outerplanar graph is a ring graph. Thus it is enough to deal with the rings \( R \) for which \( WT(R) \) is a ring graph. Hence the result follows from Theorem 4.4.

\section*{5. Genus of WT(\( R \))}

In this section, we classify the finite commutative rings \( R \) for which \( WT(R) \) has genus at most two.

The minimal integer \( k \) such that a graph \( G \) can be drawn without crossing itself on a sphere with \( k \) handles (i.e. an oriented surface of genus \( k \)) is called the \textit{genus} of \( G \), denoted by \( \gamma(G) \). A planar graph has genus 0 because it can be drawn on a sphere without self-crossing. The following results deal with genus features of complete and complete bipartite graphs.

\section*{Lemma 5.1 (13).} \( \gamma(K_m) = \left\lceil \frac{(m-3)(m-4)}{12} \right\rceil \) if \( m \geq 3 \). In particular, \( \gamma(K_m) = 1 \) if \( m = 5, 6, 7 \).

\section*{Lemma 5.2 (13).} \( \gamma(K_{n,m}) = \left\lceil \frac{(n-2)(m-2)}{4} \right\rceil \) if \( n, m \geq 2 \). In particular, \( \gamma(K_{4,4}) = \gamma(K_{3,3}, m) = 1 \) if \( m = 3, 4, 5, 6 \). Also \( \gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{5,3}) = 2 \) if \( m = 7, 8, 9, 10 \).

\section*{Lemma 5.3 (13).} If \( G \) is a connected graph with \( q \) edges and \( m \geq 3 \) vertices, then
\[ \gamma(G) \geq \left\lceil \frac{q}{6} - \frac{m}{2} + 1 \right\rceil. \]

\section*{Lemma 5.4.} If \( R \cong R_1 \times R_2 \times \cdots \times R_m \) is a commutative ring, where \( (R_i, \mathcal{S}_i) \) is a commutative ring for each \( i \) and \( m \geq 4 \), then \( WT(R) \) contains \( K_9 \) as a subgraph.

\section*{Proof.} Let \( p_i = e_i \) for \( 1 \leq i \leq 4 \) and \( p_5 = e_3 + e_4, p_6 = e_2 + e_3, p_7 = e_2 + e_4, p_8 = e_2 + e_3 + e_4, p_9 = e_1 + e_2 \in Z(R)^*, \) where \( e_i = (0, 0, \ldots, 0, 0, \ldots, 0, 0) \). Since \( p_ip_j = 0 \) for each \( 1 \leq i, j \leq 4 \), the subgraph induced by the set \( \{p_1, p_2, p_3, p_4\} \) is \( K_4 \) in \( WT(R) \). Since \( p_1p_5 = 0, p_2p_5 = 0, p_1 \in Ann(p_i) \) and \( p_2 \in Ann(p_5) \) such that \( p_1p_2 = 0 \) for \( i = 1, 2 \), the subgraph induced by the set \( \{p_1, p_2, p_3, p_4, p_5\} \) is \( K_5 \) in \( WT(R) \). Since \( p_1p_6 = 0, p_1 \in Ann(p_i) \) and \( p_4 \in Ann(p_6) \) such that \( p_1p_4 = 0 \) for \( i = 2, 3, 5, p_4p_6 = 0 \), we have that \( \{p_1, p_2, \ldots, p_6\} \) induces \( K_6 \) in \( WT(R) \).
Lemma 5.5. Let $R \cong R_1 \times R_2 \times R_3$ be a commutative ring, where $R_i$ is a local ring for each $i = 1, 2, 3$. If $R_i$ is not a field for at least one $i = 1, 2, 3$, then $WT(R)$ contains $K_{4,7}$ as a subgraph.

Proof. Suppose without loss of generality that $R_1$ is not a field with nonzero maximal ideal $\mathfrak{m}_1$. Then there is $\alpha \in \mathfrak{m}_1^*$ such that $Ann(\alpha) = \mathfrak{m}_1$. Consider $q_1 = (1, 0, 0)$, $q_2 = (\alpha, 0, 0)$, $q_3 = (1, 1, 0)$, $q_4 = (0, 1, 0)$, $r_1 = (0, 0, 1)$, $r_2 = (1, 0, 1)$, $r_3 = (0, 1, 1)$, $r_4 = (\alpha, 1, 0)$, $r_5 = (\alpha, 0, 1)$, $r_6 = (\alpha, 1, 1)$, $r_7 = (u, 0, 0) \in Z(R)^*$, where $1 \neq u \in U(R_1)$. Since $q_1r_1 = 0$, $r_1 \in Ann(q_1)$ and $q_4 \in Ann(r_2)$ such that $q_4r_1 = 0$, $r_1 \in Ann(q_1)$ and $r_2 \in Ann(r_4)$ such that $q_4r_1 = 0$, $r_1 \in Ann(q_1)$ and $q_4 \in Ann(r_4)$ such that $q_4r_1 = 0$, $r_1 \in Ann(q_4)$ and $q_4 \in Ann(r_4)$ such that $q_4r_1 = 0$. We get that $\{q_1, q_2, q_3, q_4, r_1, r_2, \ldots, r_7\}$ induces $K_{4,7}$ in $WT(R)$. \qed

Lemma 5.6. Let $R \cong F_1 \times F_2 \times F_3$ be a commutative ring, where $F_i$ is a field for each $i = 1, 2, 3$. If $|F_i| \geq 3$ for some $i = 1, 2, 3$, then $WT(R)$ contains $K_9$ as a subgraph.

Proof. Suppose without loss of generality that $|F_i| \geq 3$. Let $s_1 = (1, 0, 0)$, $s_2 = (\alpha, 0, 0)$, $s_3 = (0, 1, 0)$, $s_4 = (0, 0, 1)$, $s_5 = (1, 1, 0)$, $s_6 = (\alpha, 1, 0)$, $s_7 = (1, 0, 1)$, $s_8 = (\alpha, 0, 1)$, $s_9 = (0, 1, 1) \in Z(R)^*$, where $1 \neq \alpha \in F_i^*$. Since $s_i$ is adjacent with $s_j$ for each $i$ and $j$, $\{s_1, s_2, \ldots, s_9\}$ induces $K_9$ in $WT(R)$. \qed

Lemma 5.7. Let $R \cong R_1 \times F$ be a commutative ring, where $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq (0)$ and $F$ is a field. If $|\mathfrak{m}_1|^* = 2$, then $WT(R)$ contains $K_{6,5}$ as a subgraph.

Proof. Since $|\mathfrak{m}_1^*| = 2$, it follows that $R_1 \cong \mathbb{Z}_9$ or $\mathbb{Z}_9[\frac{x}{x^2}]$ and hence $|U(R_1)| = 6$. Let $\alpha, \beta \in \mathfrak{m}_1^*$ be such that $\alpha\beta = 0$ and $Ann(\alpha) = \mathfrak{m}_1$. Let $w_1 = (\delta_1, 0)$, $w_2 = (\delta_2, 0)$,
Proof. Suppose \(\alpha, \beta, \delta \in \mathcal{S}_1^+\) are such that \(\alpha \beta = \alpha \delta = 0\) and \(\text{Ann}(\alpha) = \mathcal{S}_1\). Let 
\(e_1 = (1,0), e_2 = (u,0), e_3 = (w,0), e_4 = (w,0), f_1 = (\alpha,0), f_2 = (\beta,0), f_3 = (0,1), f_4 = (\alpha,1), f_5 = (\beta,1), f_6 = (\delta,0), f_7 = (\delta,1) \in Z(R)^*,\)
where \(1 \neq u, v, w \in U(R_1)\). Since \(f_3 \in \text{Ann}(e_1)\) and \(f_1 \in \text{Ann}(f_2)\) such that \(e_1 f_3 = 0\) for \(1 \leq i \leq 4\) and \(1 \leq j \leq 7\), we have that \(\{e_1, e_2, e_3, e_4, f_1, f_2, ..., f_7\}\) induces \(K_{4,7}\) in \(WT(R)\). \(\square\)

Lemma 5.8. Let \(R \cong R_1 \times F\) be a commutative ring, where \((R_1, \mathcal{S}_1)\) is a local ring with \(\mathcal{S}_1 \neq (0)\) and \(F\) is a field. If \(|\mathcal{S}_1|^* \geq 3\), then \(WT(R)\) contains \(K_{4,7}\) as a subgraph.

Proof. Assume \(\gamma(\omega(\mathcal{S}_1))^* 1\). Since \(R\) is finite, by Lemma 3.1, \(R \cong R_1 \times R_2 \times \cdots \times R_m\), where \((R_i, \mathcal{S}_i)\) is a local ring for each \(i\) and \(m \geq 1\). If \(m \geq 4\), then by Lemma 5.4, \(WT(R)\) contains \(K_9\). Thus by Lemma 5.1, \(\gamma(\omega(\mathcal{S}_1))^* 1\). Consider the following cases:

Case (i) If \(m = 3\) and \(R_i\) is not a field for some \(i = 1, 2, 3\), then by Lemma 5.5, \(WT(R)\) contains \(K_{4,7}\) as a subgraph. Thus by Lemma 5.2, \(\gamma(\omega(\mathcal{S}_1))^* 1\). Hence \(R_i\) is a field for each \(i = 1, 2, 3\). If \(|R_i| \geq 3\) for some \(i = 1, 2, 3\), then by Lemma 5.6, \(WT(R)\) contains \(K_9\) as a subgraph. Thus by Lemma 5.1, \(\gamma(\omega(\mathcal{S}_1))^* 1\). Hence \(|R_i| = 2\) for each \(i = 1, 2, 3\). This implies that \(R \cong Z_2 \times Z_2 \times Z_2\).

Case (ii) If \(m = 2\) and \(\mathcal{S}_i \neq (0)\) for each \(i = 1, 2\), then by Theorem 2.6, \(WT(R)\) contains \(K_8\) induced by the set \(\{(1,0), (\alpha_1,0), (0,\alpha_2), (0,1), (\alpha_1,1), (1,\alpha_2), (\alpha_1,\alpha_2), (u,0)\}\), where \(\alpha_i \in \mathcal{S}_i\) for \(i = 1, 2\) and \(1 \neq u \in U(R_1)\). Thus, \(\gamma(\omega(\mathcal{S}_1))^* 1\) by
Lemma 5.1, a contradiction. Hence at least one of the $R_i$ is a field. Consider the following subcases:

**Subcase (a)** If $R_1$ and $R_2$ both are fields, then by Theorem 2.3 $WT(R) = \Gamma(R)$. Hence $R \cong F_4 \times F_4$, $F_4 \times Z_5$, $Z_5 \times Z_5$ or $F_4 \times Z_7$ by [19] Theorem 3.1.

**Subcase (b)** Suppose $R_1$ is not a field with $3_1 \neq (0)$ and $R_2$ is a field. If $|3_1| = 2$, then by Lemma 5.7 $WT(R)$ contains $K_9,5$. Thus $\gamma(WT(R)) \geq 3$ by Lemma 5.2, a contradiction. Also, if $|3_1| \geq 3$, then by Lemma 5.8 $WT(R)$ contains $K_{4,7}$, a contradiction by Lemma 5.2. Hence $|3_1| = 1$, which shows that $R_1 \cong Z_4$ or $Z_2[x]_{(x^2)}$. Finally, if $|R_2| \geq 4$, then by Lemma 5.9 $WT(R)$ contains $K_9 \setminus \{e\}$, a contradiction by Lemma 5.1. Hence $|R_2| \leq 3$. It is clear from Theorem 4.2 that $|R_1| \neq 2$. Hence $R_2 \cong Z_3$.

**Case (iii)** If $m = 1$, then $WT(R)$ is a complete graph, because $R$ is local. Also, we are assuming that $\gamma(WT(R)) = 1$, then $5 \leq |Z(R)^*| \leq 7$. Therefore by Table 1 $R \cong Z_{49}$, $Z_7[x]_{(x^2)}$, $Z_{49}$, $Z_2[x]_{(x)}$, $Z_{16}$, $Z_2[x]_{(x^2)}$, $Z_2[x]_{(x^2-2x)}$, $Z_2[x]_{(x^2-4x)}$, $Z_2[x]_{(x^2+2x-2x^2)}$, $Z_2[x]_{(x^2+2x-4x)}$, $Z_2[x]_{(x^2+2x+2x^2-2x^3)}$, $Z_2[x]_{(x^2+2x+2x^2-4x^3)}$, $Z_2[x]_{(x^2+2x+2x^2+y^2)}$, $Z_2[x]_{(x^2+2x+2x^2+y^2-x^2)}$, $Z_2[x]_{(x^2+2x+2x^2+y^2-2x^3)}$, $Z_2[x]_{(x^2+2x+2x^2+y^2-4x^3)}$, $Z_2[x]_{(x^2+2x+2x^2+y^2-2x^2)}$.

Conversely, if $R \cong Z_{49}$ or $Z_7[x]_{(x^2)}$, then $WT(R) \cong K_9$. Thus by Lemma 5.1 $\gamma(WT(R)) = 1$. If $R \cong Z_{16}$, $x^2-2x$, $x^2-4x$, $x^2+2x-2x^2$, $x^2+2x-4x$, $x^2+2x+2x^2-2x^3$, $x^2+2x+2x^2-4x^3$, $x^2+2x+2x^2+y^2$, $x^2+2x+2x^2+y^2-x^2$, $x^2+2x+2x^2+y^2-2x^3$, $x^2+2x+2x^2+y^2-4x^3$, $x^2+2x+2x^2+y^2-2x^2$, $x^2+2x+2x^2+y^2-4x^3$, $x^2+2x+2x^2+y^2-2x^2$, then $WT(R) \cong K_7$. Thus $\gamma(WT(R)) = 1$ again by Lemma 5.1. If $R \cong F_4 \times F_4$, $F_4 \times Z_5$, $Z_5 \times Z_5$ or $F_4 \times Z_7$, then $\gamma(WT(R)) = \gamma(\Gamma(R)) = 1$ by [19] Theorem 3.1. If $R \cong Z_4 \times Z_3$ or $Z_2[x]_{(x^2)} \times Z_3$, the toroidal embedding of $WT(R)$ is shown in Figure 2. If $R \cong Z_2 \times Z_2 \times Z_2$, then $WT(R) \cong K_6$ by [13] Theorem 2.6. Hence $\gamma(WT(R)) = 1$ by Lemma 5.1.

![Figure 2. Toroidal embedding of $WT(Z_4 \times Z_3) \cong WT(Z_2[x]_{(x^2)} \times Z_3)$](image-url)

We end this section with the classification of finite commutative rings $R$ with genus two $WT(R)$.

**Theorem 5.11.** If $R$ is a finite commutative ring, then $\gamma(WT(R)) = 2$ if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_{27}$, $\frac{\mathbb{Z}_9[x]}{(3x,x^2-3)}$, $\frac{\mathbb{Z}_9[x]}{(3x,x^2-6)}$, $\frac{\mathbb{Z}_9[x]}{(x^3)}$, $\frac{\mathbb{Z}_9[x]}{(x^2+1)}$, $\mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$ or $\mathbb{Z}_5 \times \mathbb{Z}_7$.

**Proof.** Assume that $\gamma(WT(R)) = 2$. Since $R$ is finite, by Lemma 3.1, $R \cong R_1 \times R_2 \times \cdots \times R_m$, where $(R_i, \mathbb{Z}_9)$ is a local ring for each $i$ and $m \geq 1$. If $m \geq 4$, then by Lemma 5.4, $WT(R)$ contains $K_9$. Thus by Lemma 5.1, $\gamma(WT(R)) \geq 3$, a contradiction. Hence $m \leq 3$. Consider the following cases:

**Case (i)** If $m = 3$ and $R_i$ is not a field for some $i = 1, 2, 3$, then by Lemma 5.5, $WT(R)$ contains $K_{1,7}$ as a subgraph. Thus by Lemma 5.2, $\gamma(WT(R)) \geq 3$, a contradiction. Hence $R_i$ is a field for each $i = 1, 2, 3$.

If $|R_i| \geq 3$ for some $i = 1, 2, 3$, then by Lemma 5.6, $WT(R)$ contains $K_9$ as a subgraph. Thus by Lemma 5.1, $\gamma(WT(R)) \geq 3$, a contradiction. Hence $|R_i| = 2$ for each $i = 1, 2, 3$. This implies that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $\gamma(WT(R)) = 1$ by Theorem 5.10, again a contradiction.

**Case (ii)** If $m = 2$ and $\mathbb{Z}_9 \neq \{0\}$ for each $i = 1, 2$, then by [14, Theorem 2.6], $WT(R)$ contains $K_9$ induced by the set $\{(0,0), (1,0), (0,1), (1,1), (0,0), (1,0), (0,0), (1,0), (0,0), (1,0)\}$, where $\mathbb{Z}_9$ contains $\mathbb{Z}_9$. Also, if $|R_1| \geq 3$, then by Lemma 5.8, $WT(R)$ contains $K_{4,7}$. Thus by Lemma 5.2, $\gamma(WT(R)) \geq 3$, a contradiction. Hence $|R_1| = 2$, then by Theorem 4.2, $\gamma(WT(R)) = 0$. Also, if $|R_2| = 3$, then by Theorem 5.10, $\gamma(WT(R)) = 1$. Hence in this case $\gamma(WT(R)) \neq 2$.

**Case (iii)** If $m = 1$, then $WT(R)$ is a complete graph, because $R$ is local. Also, we are assuming that $\gamma(WT(R)) = 2$, then $|Z(R)^*| = 8$. Therefore by Table 1, $R \cong \mathbb{Z}_{27}$, $\frac{\mathbb{Z}_9[x]}{(3x,x^2-3)}$, $\frac{\mathbb{Z}_9[x]}{(3x,x^2-6)}$, $\frac{\mathbb{Z}_9[x]}{(x^3)}$, $\frac{\mathbb{Z}_9[x]}{(x^2+1)}$, $\mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$ or $\mathbb{Z}_5 \times \mathbb{Z}_7$.

Conversely, if $R \cong \mathbb{Z}_{27}$, $\frac{\mathbb{Z}_9[x]}{(3x,x^2-3)}$, $\frac{\mathbb{Z}_9[x]}{(3x,x^2-6)}$, $\frac{\mathbb{Z}_9[x]}{(x^3)}$, $\frac{\mathbb{Z}_9[x]}{(x^2+1)}$, then $WT(R) \cong K_9$, which implies that $\gamma(WT(R)) = 2$ by Lemma 5.1. Also, if $R \cong \mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$ or $\mathbb{Z}_5 \times \mathbb{Z}_7$, then by Theorem 2.3 and [5, Theorem 4], $\gamma(WT(R)) = \gamma(\Gamma(R)) = 2$. □
6. Crosscap of WT(R)

In this section, we characterize the finite commutative rings $R$ for which $\gamma(WT(R))$ has crosscap at most two.

Let $N_k$ denote the sphere with $k$ crosscaps, where $k$ is a non-negative integer, that is, $N_k$ is a non-oriented surface with $k$ crosscaps. The crosscap number of a graph $G$, denoted by $\gamma(G)$, is the minimal integer $k$ such that $G$ can be embedded in $N_k$. Intuitively, $G$ is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. It is easy to see that $\gamma(H) \leq \gamma(G)$ for all subgraphs $H$ of $G$. The crosscap of various particular types of graphs are given in the following lemmas, which are useful for proving the results of this section.

Lemma 6.1 (13). If $m \geq 3$, then
\[
\gamma(K_m) = \begin{cases} 
\left\lceil \frac{(m-3)(m-4)}{6} \right\rceil & \text{if } m \geq 3 \text{ and } m \neq 7; \\
3 & \text{if } m = 7.
\end{cases}
\]

Lemma 6.2 (13). If $n, m \geq 2$, then
\[
\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.
\]

Lemma 6.3 (13). If $G$ is a connected graph with $q$ edges and $m \geq 3$ vertices, then
\[
\gamma(G) \geq \left\lceil \frac{q}{3} - m + 2 \right\rceil.
\]

Now, we can characterize the finite commutative rings $R$ with crosscap at most two $\gamma(WT(R))$.

Theorem 6.4. If $R$ is a finite commutative ring, then $\gamma(WT(R)) = 1$ if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{(x^2)}$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Since $\gamma(WT(R)) \leq \gamma(WT(R))$, it is enough to deal with the rings $R$ for which $\gamma(WT(R)) = 1$. If $R \cong \mathbb{Z}_{49}$ or $\frac{\mathbb{Z}_7[x]}{(x^2)}$, then $\gamma(WT(R)) \cong K_6$. Thus by Lemma 6.1, $\gamma(WT(R)) = 1$. If $R \cong \mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{(x^4)}$, $\frac{\mathbb{Z}_2[x]}{(x^2-2,x)}, \frac{\mathbb{Z}_2[x]}{(x^2-2,x,y)}, \frac{\mathbb{Z}_2[x]}{(x^2-2,x,y^2)}, \frac{\mathbb{Z}_2[x]}{(x^2-2,x,y,y^2)}, \frac{\mathbb{Z}_2[x]}{(x^2-2,x,y^2)}, \frac{\mathbb{Z}_2[x]}{(x^2-2,x,y,y^2)}, \frac{\mathbb{Z}_2[x]}{(x^2-2,x,y,y^2)}$, then $\gamma(WT(R)) \cong K_7$. Thus $\gamma(WT(R)) = 3$ by Lemma 6.1. If $R \cong \mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{F}_4 \times \mathbb{Z}_5$, then $\gamma(WT(R)) \cong K_{3,3}$ or $K_{3,4}$. Thus by Lemma 6.2, $\gamma(WT(R)) = 1$. If $R \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{Z}_7$, then $\gamma(WT(R)) \cong K_{4,4}$ or $K_{3,6}$. Thus by Lemma 6.2, $\gamma(WT(R)) = 2$. If $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_3$, then $\gamma(WT(R)) \cong K_7 \setminus \{e\}$ induced by the set $\{(1,0), (2,0), (3,0), (0,1), (0,2), (2,1), (2,2)\}$. Thus by Lemma 6.3, $\gamma(WT(R)) > 1$.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\gamma(WT(R)) \cong K_6$ by 13, Theorem 2.6. Hence $\gamma(WT(R)) = 1$ by Lemma 6.1.
Theorem 6.5. If $R$ is a finite commutative ring, then $\gamma(WT(R)) = 2$ if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_7$, $\mathbb{Z}_4 \times \mathbb{Z}_3$ or $\mathbb{Z}_2[x] / (x^2) \times \mathbb{Z}_3$.

Proof. Since $\gamma(WT(R)) \leq \gamma(WT(R))$, it is enough to deal with the rings $R$ for which $\gamma(WT(R))$ has genus at most two. It is clear from Theorem 6.4 that if $R \cong \mathbb{Z}_{49}$, $\mathbb{Z}_4[x] / (x^2)$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_7$, then $\gamma(WT(R)) = 1$. If $R \cong \mathbb{Z}_{16}$, $\langle x^2 \rangle$, $\langle x^2, y \rangle$, $\langle x, y \rangle$, $\mathbb{Z}_4[x] / (x^2, y^2, xy)$, $\langle x^2, y \rangle$, $\mathbb{Z}_4[x] / (x^2, y, x+y)$, $\mathbb{Z}_2[y] / (x^2, y)$, $\mathbb{Z}_2[x, y] / (x^2, y)$, then $\gamma(WT(R)) = 3$. If $R \cong \mathbb{Z}_9$, $\langle x^3 \rangle$, $\langle x^3, y \rangle$, $\mathbb{Z}_3[x] / (x^3, y)$, $\mathbb{Z}_3[x] / (x^3, y, x+y)$, $\mathbb{Z}_3[x] / (x^3, y, x+1)$, then $\gamma(WT(R)) = 3$. If $R \cong \mathbb{Z}_{27}$, $\langle 3x, x^2 \rangle$, $\langle 3x, x^2, 3y \rangle$, $\mathbb{Z}_3[x] / (3x, x^2)$, $\mathbb{Z}_3[x] / (3x, x^2, 3y)$, $\mathbb{Z}_3[x] / (3x, x^2, 3y, x+y)$, then $\gamma(WT(R)) = 3$. If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_4[x] / (x^2) \times \mathbb{Z}_3$, the embedding of $WT(R)$ onto $N_2$ is shown in Figure 3.

Figure 3. Embedding of $WT(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong WT(\mathbb{Z}_2[x] / (x^2) \times \mathbb{Z}_3)$ on $N_2$.

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References


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Nadeem ur Rehman
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
nu.rehman.mm@amu.ac.in

Mohd Nazim
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
mazim1882@gmail.com

Shabir Ahmad Mir
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
mirshabir967@gmail.com

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