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CLUSTER ALGEBRAS OF TYPE $\mathbb{A}_{n-1}$ THROUGH THE PERMUTATION GROUPS S_n

KODJO ESSONANA MAGNANI

Abstract. Flips of triangulations appear in the definition of cluster algebras by Fomin and Zelevinsky. In this article we give an interpretation of mutation in the sense of permutation using triangulations of a convex polygon. We thus establish a link between cluster variables and permutation mutations in the case of cluster algebras of type $\mathbb{A}$.

1. Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in [8, 9]. They are a class of commutative algebras which was shown to be connected to various areas of mathematics like combinatorics, Lie theory, Poisson geometry, Teichmüller theory, mathematical physics, and representation theory of algebras. A cluster algebra is generated by a set of variables, called *cluster variables*, obtained recursively by a combinatorial process known as *mutation* starting from a set of initial cluster variables.

Triangulating a convex polygon plays a central role in the theory of cluster algebras. Consider a triangulation T of a convex polygon which is the partition of its interior into triangles by non-intersecting diagonals [2, 3, 10]. Each diagonal d in the triangulation T is the diagonal of some quadrilateral. A new triangulation T' is obtained by replacing the diagonal d with the other diagonal of that quadrilateral. This well-known process is called a *flip*. The cluster variables are in natural bijection with the diagonals of a convex polygon [1] and the flip of diagonals corresponds to a mutation of cluster variables [6].

The present work is motivated by the correspondence between triangulations and permutations in [4]. Our objective here is to show how one can mutate a permutation in the permutation groups S_n. We establish a link between the mutation of cluster variables and the mutation of a permutation in the permutation groups S_n. For this, taking a permutation, we show how to enumerate all diagonals composing the corresponding triangulation. This correspondence allows us to prove that the cluster algebra associated with a permutation in S_n is of type $\mathbb{A}_{n-1}$.

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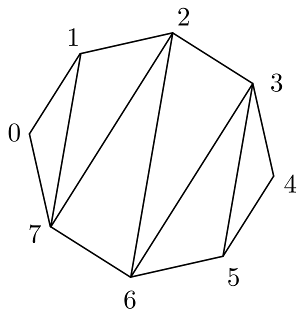
The article is organized as follows. In Section 2, we recall some basic notions on triangulations and permutations, and set some results. In Section 3, we establish a link between cluster algebras of type $\mathbb{A}_{n-1}$ and the permutation groups S_n.

2. Triangulations and permutations

Let $n \geq 1$ be an integer. Let P_{n+2} be a convex polygon with $n+2$ vertices labelled $0, 1, \dots, n+1$ in clockwise order. The partition of the interior of P_{n+2} into triangles by non-crossing diagonals is called a *triangulation* of P_{n+2}. The partition uses $n-1$ diagonals. The set of triangulations of P_{n+2} will be denoted by T_n, and its cardinality by t_n. It is well known that t_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, $n \geq 1$ (see [10]).

Example 2.1. Let $n = 6$, and let P_8 be a convex octagon with vertices labelled 0, 1, 2, 3, 4, 5, 6, 7. We can have the following triangulation:



We denote by S_n the group of permutations of $\{1, 2, \dots, n\}$ and write the elementary transpositions as $(i, i+1)$ for $1 \leqslant i \leqslant n-1$. Let w be the set of words on $\{1, 2, \dots, n\}$. A word in w is said to be *standard* if its letters are pairwise distinct. The set of standard words of length n in w will be identified with S_n. In this way a permutation σ in S_n will be represented by the word such that $\sigma = a_1 a_2 \dots a_n \in w$, where $a_i = \sigma(i)$ for all $1 \leqslant i \leqslant n$. Note that the left multiplication of the word σ by the elementary transposition $\tau = (i, i+1)$ results in the word where the i-th and $(i+1)$-th letters of σ are permuted. Indeed, if $\sigma = u a_i a_{i+1} v$, where $u = a_1 a_2 \dots a_{i-1}$ and $v = a_{i+2} \dots a_n$ (with the convention that u or v is empty if $i = 1$ or $i = n-1$, respectively), then $\tau\sigma = u a_{i+1} a_i v$.

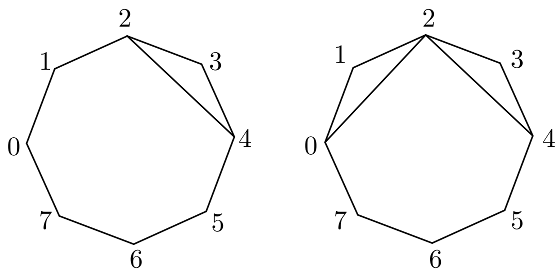
Example 2.2. Let σ be in S_6 such that $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{smallmatrix}\right)$. The word associated with the permutation σ is $\sigma = 314265$.

The triangulations of a fixed convex plane $(n+2)$-gon P_{n+2} will be now associated with permutations in S_n. The way to associate a permutation with a triangulation is described in the following example.

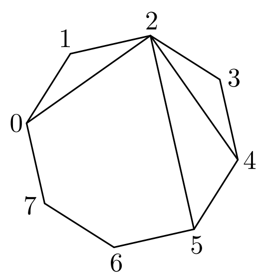
Example 2.3. Let $n = 6$ and let P_8 be a convex octagon with vertices labelled in clockwise order from 0 to 7.

Let $\sigma = 314265 \in \mathrm{S}_6$. We associate with σ the triangulation T of P_8 constructed by the following procedure. We read the word σ from left to right.

- The first letter being 3 gives rise to the diagonal joining its two neighbors in P_8, namely 2 and 4. Cutting vertex 3 from P_8, the newly added diagonal $\{2, 4\}$ gives rise to a 7-gon P_8' on the vertices $0, 1, 2, 4, 5, 6, 7$.
- The next letter in σ is 1 and gives rise to the diagonal $\{0, 2\}$ joining its two neighbours in P_8'. Now we cut vertex 1 from P_8' and denote by P_8'' the resulting hexagon on vertices $0, 2, 4, 5, 6, 7$.

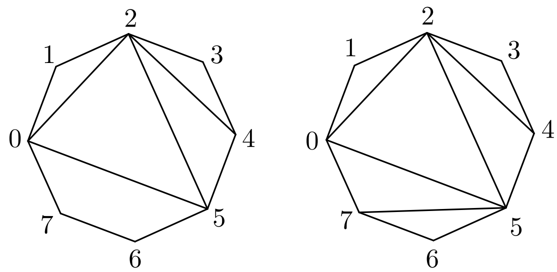


- The third letter of σ is 4 and gives rise to the diagonal $\{2, 5\}$ between its neighbours in P_8''.



- The fourth letter of σ is 2 and gives rise to the diagonal $\{0, 5\}$ between its two neighbours in the vertex sequence $0, 2, 5, 6, 7$.
- Finally, the fifth letter of σ is 6 and gives rise to the diagonal $\{5, 7\}$ joining its neighbours in the square on $0, 5, 6, 7$.

The resulting triangulation T of P_8 has inner diagonals $\{2, 4\}$, $\{0, 2\}$, $\{2, 5\}$, $\{0, 5\}$, $\{5, 7\}$.



The *degree* of a vertex i in P_{n+2} is the number of edges of T which are incident to i. A vertex of degree exactly 2 is called an *ear* in T.

The words in S_n obtained with the cutting procedure will be called the *readings* of the triangulation T.

According to [4, Lemma 13], the map $t\colon S_n \to T_n$ is surjective. Then each word will be associated with a triangulation. The following example gives the readings of such triangulation T in T_n.

Example 2.4. The readings of the triangulation $T = t(\sigma = 314265)$ obtained in Example 2.3 are

$$\begin{array}{rlrlrl}
\sigma &= 314265 & \sigma_5 &= 134265 & \sigma_{10} &= 613425\\
\sigma_1 &= 341625 & \sigma_6 &= 136425 & \sigma_{11} &= 341265\\
\sigma_2 &= 346125 & \sigma_7 &= 163425 & \sigma_{12} &= 134625\\
\sigma_3 &= 316425 & \sigma_8 &= 634125 & &\\
\sigma_4 &= 314625 & \sigma_9 &= 631425, & &
\end{array}$$

all in S_6.

Definition 2.5. Let $T \in T_n$, and let $t\colon S_n \to T_n$ be the surjective map. The *canonical reading* of T is the lexicographically smallest word in the fiber $t^{-1}(T)$.

Example 2.6. In Example 2.4 the canonical reading of T is $\sigma_5 = 134265$.

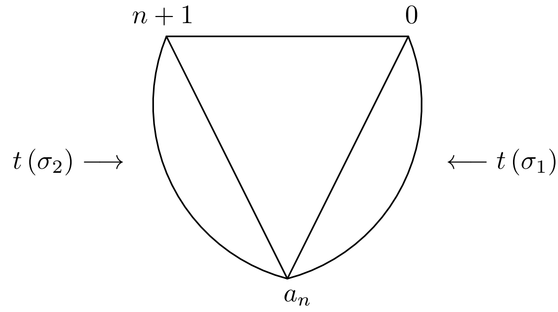
Remark 2.7. It was shown in [4, Lemma 15] that the set of readings of T is exactly the fiber $t^{-1}(T)$. It is clear that the canonical reading is unique in $t^{-1}(T)$.

Now we are going to define the separation of a permutation, as in [4].

Definition 2.8. Let $\sigma = a_1 a_2 \ldots a_n \in S_n$. A *separation* of σ is a factorization $\sigma = \sigma_1\sigma_2 a_n$ where the subwords σ_1 and σ_2 have the property that $a_i < a_n$ for every letter a_i in σ_1, and $a_j > a_n$ for every letter a_j in σ_2.

Example 2.9. Let $\sigma = 134265 \in \mathrm{S}_6$. Then σ has a separation given by $\sigma_1 = 1342$, $\sigma_2 = 6$, and $a_n = a_6 = 5$.

Remark 2.10. The separation of a permutation is unique if it exists. Let $\sigma = \sigma_1\sigma_2 a_n$ be a separation; the triangulation $T = t(\sigma)$ associated with σ is of the form



Note that a canonical reading is a separation.

We know that $a_n \in \{1, \ldots, n\}$; then we can rewrite a separation as follows: $\sigma = \sigma_1\sigma_2 i$ with $i \in \{1, \ldots, n\}$.

For a fixed i in $\{1, \ldots, n\}$, the following theorem gives the number of canonical readings ended by i.

Theorem 2.11. *Let $n \geqslant 1$ be an integer, $i \in \{1, \ldots, n\}$, and S_n the permutation group. Let $\sigma = \sigma_1\sigma_2 i$ be a canonical reading.*

(1) *The number of canonical readings in S_n ended by i is Δ_i such that we have*

$$\Delta_i = \frac{1}{i}\frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}.$$

(2) $\displaystyle\sum_{i=1}^{n} \Delta_i = \frac{1}{n+1}\binom{2n}{n}$.

Before giving the proof of the above theorem, let us state the following lemma.

Lemma 2.12. *Let $n \geqslant 1$ be an integer. We have*

$$\sum_{i=1}^{n} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i} = \frac{1}{2}\binom{2n}{n}.$$

Proof. The assertion holds for $n = 1$ and $n = 2$, as shown by

$$\frac{1}{1}\binom{0}{0}\binom{0}{0} = 1 = \frac{1}{2}C_2^1$$

and

$$\begin{aligned}\sum_{i=1}^{2} \frac{1}{i}\binom{2\,(i-1)}{i-1}\binom{2\,(2-i)}{2-i} &= \frac{1}{1}\binom{0}{0}\binom{2}{1} + \frac{1}{2}\binom{2}{1}\binom{0}{0} = 2+1 = 3\\ &= \frac{1}{2}\binom{4}{2} = \frac{1}{2}\frac{4\cdot 3}{2\cdot 1} = 3.\end{aligned}$$

We state a pair of identities that will be used below:

$$\binom{n}{p} = \frac{n}{p}\binom{n-1}{p-1}, \tag{2.1}$$

$$\binom{n}{p} = \binom{n}{n-p}. \tag{2.2}$$

Now fix $n \in \mathbb{N}$, and assume that the assertion is valid for every $i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
&\sum_{i=1}^{n+1} \frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n+1-i)}{n+1-i} \\
&\quad = \sum_{i=1}^{n}\left[\frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n-i+1)}{n-i+1}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&\quad = \sum_{i=1}^{n}\left[\frac{2}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)+1}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&\quad \overset{\text{by (2.1)}}{=} \sum_{i=1}^{n}\left[\frac{2}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)+1}{n-i+1}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&\quad \overset{\text{by (2.2)}}{=} \sum_{i=1}^{n}\left[\frac{2}{i}\binom{2(i-1)}{i-1}\frac{2(n-i)+1}{n-i+1}\binom{2(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&\quad \overset{\text{by (2.1)}}{=} \sum_{i=1}^{n}\left[\frac{2}{i}\left(2 - \frac{1}{n-i+1}\right)\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&\quad = \sum_{i=1}^{n}\left[\left(\frac{4}{i} - \frac{2}{i(n-i+1)}\right)\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&\quad = \sum_{i=1}^{n}\left[\left(\frac{4}{i} - \frac{2}{n+1}\left(\frac{1}{i} + \frac{1}{n-i+1}\right)\right)\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

Due to the fact that

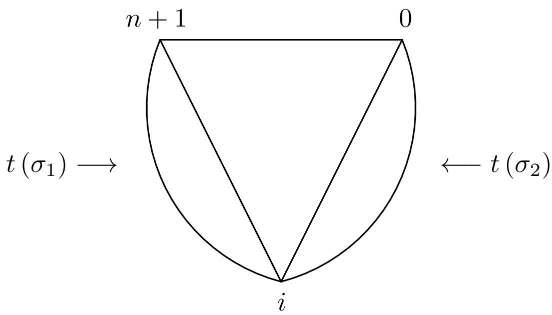
$$\sum_{i=1}^{n} \frac{1}{i}\binom{2\,(i-1)}{i-1}\binom{2\,(n-i)}{n-i} = \sum_{i=1}^{n}\frac{1}{n-i+1}\binom{2\,(i-1)}{i-1}\binom{2\,(n-i)}{n-i},$$

we obtain

$$\begin{aligned}
&\sum_{i=1}^{n+1} \frac{1}{i}\binom{2\,(i-1)}{i-1}\binom{2\,(n+1-i)}{n+1-i} \\
&\quad = \sum_{i=1}^{n}\left[\left[\frac{4}{i} - \frac{2}{n+1}\frac{2}{i}\right]\binom{2\,(i-1)}{i-1}\binom{2\,(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n} \\
&\quad = \sum_{i=1}^{n}\left[\left(\frac{2n}{n+1}\frac{2}{i}\right)\binom{2\,(i-1)}{i-1}\binom{2\,(n-i)}{n-i}\right] + \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2n}{n+1}\binom{2n}{n} + \frac{1}{n+1}\binom{2n}{n} \\
&= \frac{2n+1}{n+1}\binom{2n}{n} \\
&\overset{\text{by (2.1)}}{=} \binom{2n+1}{n+1} \\
&\overset{\text{by (2.2)}}{=} \binom{2n+1}{n} \\
&\overset{\text{by (2.1)}}{=} \frac{n+1}{2(n+1)}\binom{2\,(n+1)}{n+1} \\
&= \frac{1}{2}\binom{2\,(n+1)}{n+1}. \qquad \square
\end{aligned}
$$

Proof of Theorem 2.11. (1) Let $\sigma = \sigma_1\sigma_2 i$ be a separation, $i \in \{2, \ldots, (n-1)\}$. The triangulation $T = t\,(\sigma)$ associated with σ is of the form



Note that $t\,(\sigma_1)$ is a triangulation of a $(i+1)$-gon. The number of triangulations of a $(i+1)$-gon is the Catalan number $t_{i-1} = \frac{1}{i}\binom{2(i-1)}{i-1}$.

Considering $t(\sigma_2)$, we will see that it is a triangulation of a $(n-i+2)$-gon, whose number of triangulations is

$$t_{n-i} = \frac{1}{n-i+1}\binom{2\,(n-i)}{n-i}.$$

Due to the fact that the triangle with vertices 0, i, $n+1$ is fixed in $T = t(\sigma)$, all other separations ended by i are obtained by making flips in $t(\sigma_1)$ and $t\,(\sigma_2)$. Then, combining this, we obtain that the number Δ_i of separations in S_n ended by i is $\Delta_i = t_{i-1}t_{n-i}$. Thus

$$\Delta_i = \frac{1}{i}\frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}.$$

In the cases $i = 1$ and $i = n$, the triangulation is associated with an $(n+1)$-gon. In these cases the number of triangulations is the same and coincides with t_{n-1}. Therefore, the formula of Δ_i above can apply for $i = 1$ and $i = n$. Indeed, i running through $\{1, 2, \ldots, n\}$.

(2) Now let us compute $\sum_{i=1}^{n} \Delta_i$:

$$\begin{aligned}\sum_{i=1}^{n} \Delta_i &= \sum_{i=1}^{n} \frac{1}{i} \frac{1}{n-i+1} \binom{2\,(i-1)}{i-1} \binom{2\,(n-i)}{n-i} \\ &= \sum_{i=1}^{n} \frac{1}{n+1} \left[\frac{1}{i} + \frac{1}{n-i+1}\right] \binom{2\,(i-1)}{i-1} \binom{2\,(n-i)}{n-i}.\end{aligned}$$

Due to the fact that

$$\sum_{i=1}^{n} \frac{1}{n-i+1} \binom{2\,(i-1)}{i-1} \binom{2\,(n-i)}{n-i} = \sum_{i=1}^{n} \frac{1}{i} \binom{2\,(i-1)}{i-1} \binom{2\,(n-i)}{n-i},$$

we get

$$\sum_{i=1}^{n} \Delta_i = \frac{1}{n+1} \sum_{i=1}^{n} \frac{2}{i} \binom{2\,(i-1)}{i-1} \binom{2\,(n-i)}{n-i},$$

and by Lemma 2.12, we have

$$\sum_{i=1}^{n} \Delta_i = \frac{1}{n+1} \binom{2n}{n}. \qquad \square$$

3. Cluster algebras and permutations

In this section we establish the connection between cluster algebras of type $\mathbb{A}_{n-1}$ and the permutation groups S_n.

3.1. **Mutations in the permutation group S_n.** Consider a diagonal d in a triangulation T. This diagonal is the diagonal of some quadrilateral. Then there is a new triangulation T', which is obtained by replacing the diagonal d with the other diagonal of that quadrilateral. This process is called a *flip*. It is well known that flips are the mutations in triangulations [1, 6]. According to [2, Theorem 2.11], there exists a bijection between the set of triangulations and the set of canonical readings which are separations. Before stating how to mutate separations, we will define passivity classes.

Definition 3.1. Let $\sigma = ua_ia_{i+1}v$ and $\gamma = ua_{i+1}a_iv$ be two permutations of S_n. The word σ and γ are in the same *passivity class* if and only if the factor v contains a letter p in w which is between the letters a_i and a_{i+1}, that is, $a_i < p < a_{i+1}$ or $a_{i+1} < p < a_i$.

Example 3.2. Considering the readings of T in Example 2.4, we can say that $\sigma_3 = 316425$ and $\sigma_4 = 314625$ are in the same passivity class by taking $u = 31$, $v = 25$ and $p = 5$.

It results from [4, Proposition 16] that two permutations being in the same passivity class represent the same triangulation T in T_n. Then by Remark 2.7, the fiber $t^{-1}(T)$ represents a passivity class. It is now clear to see that each canonical reading represents a passivity class.

Now we are ready to define how to mutate a word σ in the permutation group S_n.

Definition 3.3. Let $\sigma = uxyv$ and $\gamma = uyxv$ be two permutations of S_n. The permutation γ will be called a *mutation* of σ, $\mu_{xy}(\sigma) = \gamma$, if and only if there is no letter p in v which is between the letters x and y.

Overall, to mutate a permutation is to exchange two of its consecutive letters under the above condition.

Example 3.4. Let $\sigma = 1234$ in S_4. We have $\mu_{23}(\sigma) = 1324$ and $\mu_{13}(\mu_{23}(\sigma)) = \mu_{13}(1324) = 3124$.

- $\mu_{13} \circ \mu_{23}(\sigma)$ is not defined because $\mu_{13} \circ \mu_{23}(\sigma)$ and $\mu_{23}(\sigma)$ are in the same passivity class. Then the words 1324 and 3124 represent the same triangulation.
- $\mu_{24}(1324) = 1342$ is defined but $\mu_{13}(1342)$ is not, because it stays in the same passivity class.

3.2. **Permutations and diagonals.** It is clear that each permutation corresponds to a triangulation. Then, taking a permutation, we need to recognize the diagonals composing the triangulation.

Let $\sigma = a_1 a_2 \cdots a_n$ be a permutation of S_n. We know that σ corresponds to a triangulation, so we need to enumerate all diagonals composing this triangulation through the permutation σ. To this end, we go step by step from left to right in reading σ. Recall that a diagonal joins two vertices of the convex polygon P_{n+2}. Then we denote a diagonal joining the vertices i and j by $\{i, j\}$. Each a_k in σ is an element of the set $\{1, 2, \ldots, n\}$. The vertices of the convex polygon P_{n+2} are labelled $0, 1, \ldots, n+1$ in clockwise order.

Now take a_k in σ and rewrite σ as $\sigma = U a_k V$, where $U = \{a_1, a_2, \ldots, a_{k-1}\}$ and $V = \{a_{k+1}, a_{k+2}, \ldots, a_n\}$. Considering a_k as an element of the ordered set $L = \{0, 1, \ldots, n+1\}$, we construct two subsets of L as follows: W_1 is the subset of L such that, for all $p \in W_1$, we have $p < a_k$; and W_2 is the subset of L such that, for all $q \in W_2$, we have $q > a_k$.

For a fixed a_k in σ, we can construct the vertices of the corresponding diagonal as follows: The label of the first vertex is equal to

$$\begin{cases} \max(W_1 \cap V) & \text{if } W_1 \cap V \text{ is not empty;} \\ 0 & \text{if } W_1 \cap V \text{ is empty.} \end{cases}$$

The label of the second vertex is equal to

$$\begin{cases} \min(W_2 \cap V) & \text{if } W_2 \cap V \text{ is not empty;} \\ n+1 & \text{if } W_2 \cap V \text{ is empty.} \end{cases}$$

Then we get the two vertices ending the diagonal. Indeed, taking the permutation $\sigma = a_1 a_2 \cdots a_n$ and starting the procedure with a_1 and ending with a_{n-1}, we obtain all diagonals composing the triangulation T associated with the permutation σ.

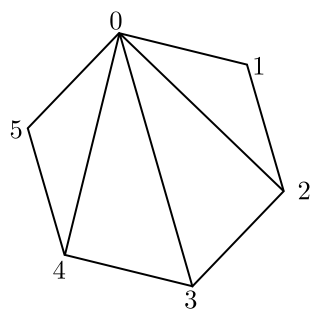
Example 3.5. Let σ_1 and σ_2 be in S_4 such that $\sigma_1 = 1234$ and $\sigma_2 = 4231$. First let us give all diagonals composing the triangulation T_1 associated with σ_1. Here the set $L = \{0, 1, 2, 3, 4, 5\}$ and $a_1 = 1$, $a_2 = 2$, $a_3 = 3$. For $a_1 = 1$, we have $U = \emptyset$, $V = \{2, 3, 4\}$, $W_1 = \{0\}$, $W_2 = \{2, 3, 4, 5\}$. Since $W_1 \cap V = \{0\} \cap \{2, 3, 4\} = \emptyset$, the first vertex of the diagonal is 0.

Next, we compute $W_2 \cap V = \{2, 3, 4, 5\} \cap \{2, 3, 4\} = \{2, 3, 4\}$.

Since $\min(W_2 \cap V) = \min\{2, 3, 4\} = 2$, the second vertex of the diagonal is 2. Thus the diagonal obtained is $\{0, 2\}$.

For $a_2 = 2$, we get the diagonal $\{0, 3\}$. For $a_3 = 3$, we get the diagonal $\{0, 4\}$. Then the diagonals composing the triangulation T_1 are $\{0, 2\}$, $\{0, 3\}$, $\{0, 4\}$.

T_1:



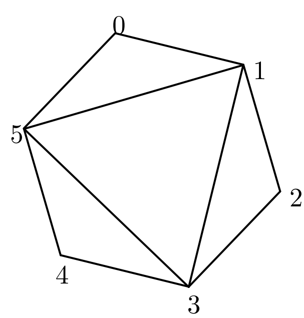
Now we give the diagonals composing the triangulation T_2 associated with σ_2. Here we have $L = \{0, 1, 2, 3, 4, 5\}$, $a_1 = 4$, $a_2 = 2$, $a_3 = 3$.

Starting with $a_1 = 4$, we have $V = \{1, 2, 3\}$, $W_1 = \{0, 1, 2, 3\}$, $W_2 = \{5\}$.

Since $W_1 \cap V = \{1, 2, 3\} \neq \emptyset$, $\max(W_1 \cap V) = 3$, the first vertex of the diagonal is 3. Since $W_2 \cap V = \emptyset$, the second vertex of the diagonal is 5. Therefore, the diagonal obtained is $\{3, 5\}$.

For $a_2 = 2$, we get the diagonal $\{1, 3\}$. For $a_3 = 3$, we get the diagonal $\{1, 5\}$. Then the diagonals composing the triangulation T_2 are $\{1, 3\}$, $\{1, 5\}$ and $\{3, 5\}$.

T_2:



Let $\sigma = uxyv$ be a canonical reading in S_n. It is well known that (according to Definition 3.3) $\mu_{xy}(\sigma) = uyxv$. This action corresponds to a flip of a diagonal. To see this, we need some statements.

Note that a diagonal $\{i, j\}$ in a triangulation is a diagonal of some quadrilateral such that i and j are two of its four vertices. Assume that the diagonal $\{i, j\}$ is the corresponding diagonal of x in the word $\sigma = uxyv$. To get the other vertices we need the following definition.

Definition 3.6. Let α, β be two vertices of a polygon P_{n+2}. The vertices are said to be *adjacent* if they are related by a diagonal or a side of P_{n+2}.

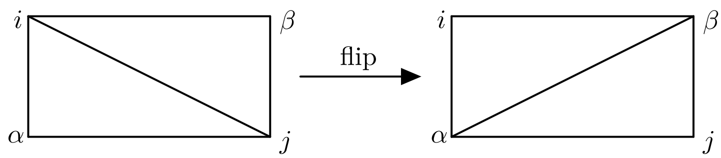
Denote by E_α the set of the vertices that are adjacent to vertex α:

$$E_\alpha = \{k \mid 0 \leqslant k \leqslant n+1,\ \{\alpha, k\} \in T\}.$$

It is clear that the remaining vertices of the quadrilateral that has $\{i, j\}$ as one of its diagonals compose the second diagonal and are obtained as $E_i \cap E_j$. The intersection $E_i \cap E_j$ gives two vertices because the triangulation T is of non-intersecting diagonals:

$$E_i \cap E_j = \{\alpha, \beta\}, \quad \alpha, \beta \in \{0, 1, \ldots, n+1\}.$$

Then the flip of the diagonal $\{i, j\}$ in the triangulation T gives the diagonal $\{\alpha, \beta\}$ with a new triangulation T':



Remark 3.7. According to the cutting procedure in Section 2, letter x in the word $\sigma = uxyv$ is a vertex of the quadrilateral having $\{i, j\}$ as one of its diagonals. Thus, x is one of the vertices of $E_i \cap E_j$.

This allows us to give the following theorem.

Theorem 3.8. *The mutation of permutations as defined in Definition* 3.3 *corresponds to the flip of diagonals in a triangulation of a polygon.* □

Example 3.9. (1) Let $\sigma_1 = 1234$ be a word in S_4 and $\mu_{23}(\sigma_1) = 1324$ (see Table 1). The diagonal corresponding to 2 in σ_1 is $\{0, 3\}$, $E_0 = \{1, 2, 3, 4, 5\}$, $E_3 = \{2, 4, 0\}$, and $E_3 \cap E_0 = \{2, 4\}$ Then the diagonal $\{2, 4\}$ replaces the diagonal $\{3, 0\}$ in the new triangulation T'_1.

(2) Let $\sigma_2 = 4231$ be a word in S_4. The corresponding diagonals are $\{3, 5\}$, $\{1, 3\}$, $\{1, 5\}$. The diagonal corresponding to 3 in σ_2 is $\{1, 5\}$, $E_1 = \{3, 5, 0, 2\}$, $E_5 = \{3, 1, 0, 4\}$, and $E_3 \cap E_5 = \{3, 0\}$.

In the triangulation T_2 associated with σ_2, we make a flip on the diagonal $\{1, 5\}$ and we get the new diagonal $\{0, 3\}$ in the new triangulation T'_2.

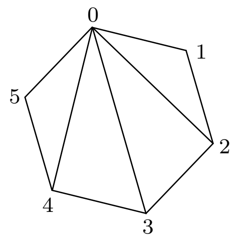
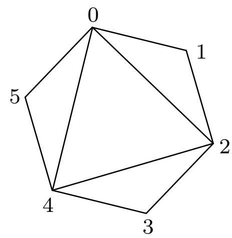
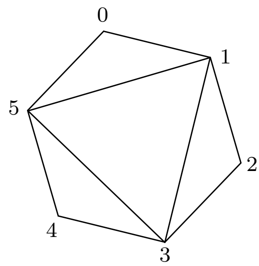
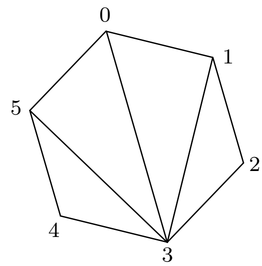
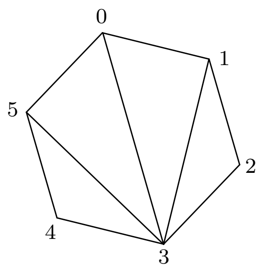
Permutations	$\sigma_1 = 1234$	$\mu_{23}(\sigma_1) = 1324$
Corresponding diagonals	$\{0,2\}$, $\{0,3\}$, $\{0,4\}$	$\{0,2\}$, $\{2,4\}$, $\{0,4\}$
Triangulations	T_1: 	T_1': 

TABLE 1. Illustration of Theorem 3.8.

T_2: 

T_2': 

Let T_2'' be the corresponding triangulation of $\mu_{31}(\sigma_2) = 4213$. According to the cutting procedure in Section 2 we have the following triangulation:

T_2'': 

It is clear that $T_2'' = T_2'$.

3.3. **Cluster algebras and permutations.** A cluster algebra is generated by a set of variables, called *cluster variables*, obtained recursively by a combinatorial process known as mutation starting from a set of initial cluster variables [8, 9].

Let T be any triangulation of polygon P_{n+2}. According to [5, Sections 3.4 and 4.1], the cluster algebra $\mathcal{A}(T)$ associated with the triangulation T of the polygon P_{n+2} is constructed as follows.

Start with the convex polygon P_{n+2} and choose a triangulation T by non-intersecting diagonals. Label these diagonals $x_{i,j}$, $i,j \in \{0,1,\dots,n+1\}$, $i \notin$

$\{j-1, j, j+1\}$; label the sides $y_{i,i+1}$ and the vertices of P_{n+2} as $0, 1, \ldots, n+1$. The labels for the remaining diagonals are obtained by flipping diagonals. The cluster algebra $\mathcal{A}(T)$ is the subalgebra of the field of rational functions in $x_{i,j}$ and $y_{i,i+1}$ generated by all the labels in P_{n+2}. The cluster algebra associated with a triangulation of a polygon P_{n+2} depends only on $n+2$ and is of type $\mathbb{A}_{n-1}$ because this cluster algebra has rank $n-1$.

In this text the variables $y_{i,i+1}$ are taken to be equal to 1.

The cluster algebra associated with a polygon P_{n+2} can be identified with the coordinate ring $\mathbb{C}[\mathrm{Gr}_{2,(n+2)}]$, where $\mathrm{Gr}_{2,(n+2)}$ is the Grassmannian of 2-planes in an $(n+2)$-dimensional vector space [7]. The coordinate ring $\mathbb{C}[\mathrm{Gr}_{2,(n+2)}]$ is generated by the three-term Plücker coordinates $p_{i,j}$ for $0 \leqslant i < j \leqslant n+1$. The relations among the Plücker coordinates are generated by the three-term Plücker relations: for any $0 \leqslant i < j < k < l \leqslant n+1$, one has

$$P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}.$$

By assigning the value of P_{ij} to the variable $x_{i,j}$ associated with the diagonal $\{i, j\}$ of the triangulation T of the polygon P_{n+2}, the three-term Plücker relations correspond to the exchange relation in $\mathcal{A}(T)$. We state the following.

Theorem 3.10. *Let $n \geqslant 1$ be an integer, σ a canonical reading in the permutation group S_n, and T the triangulation associated with σ. The cluster algebras $\mathcal{A}(T)$ of type $\mathbb{A}_{n-1}$ and $\mathcal{A}(\sigma)$ coincide.*

Before giving the proof of Theorem 3.10, let us give the analogous of [7, Corollary 5.3.6].

Proposition 3.11. *Cluster variables in a seed pattern of type $\mathbb{A}_{n-1}$ can be labelled by diagonals of a convex $(n+2)$-gon P_{n+2} so that*

- *clusters correspond to canonical readings;*
- *flips correspond to mutations of permutations.*

Cluster variables labelled by diagonals are distinct, so there are altogether $\frac{(n-1)(n+2)}{2}$ cluster variables and $\frac{1}{n+1}\binom{2n}{n}$ seeds. □

Proof of Theorem 3.10. Each canonical reading produces $n-1$ diagonals which correspond to a seed. According to Theorem 2.11, there are $\frac{1}{n+1}\binom{2n}{n}$ canonical readings, so by Proposition 3.11, these canonical readings correspond to seeds and the mutations of canonical readings correspond to flips, which correspond to cluster mutations. □

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References

[1] P. Caldero, F. Chapoton, and R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Amer. Math. Soc.* **358** no. 3 (2006), 1347–1364. DOI MR Zbl

[2] J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns, *Math. Gaz.* **57** no. 400 (1973), 87–94. DOI MR Zbl

[3] J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns, *Math. Gaz.* **57** no. 401 (1973), 175–183. DOI MR Zbl

[4] S. Eliahou and C. Lecouvey, Signed permutations and the four color theorem, *Expo. Math.* **27** no. 4 (2009), 313–340. DOI MR Zbl

[5] S. Fomin and N. Reading, Root systems and generalized associahedra, in *Geometric combinatorics*, IAS/Park City Math. Ser. 13, American Mathematical Society, Providence, RI, 2007, pp. 63–131. DOI MR Zbl

[6] S. Fomin, M. Shapiro, and D. Thurston, Cluster algebras and triangulated surfaces. I. Cluster complexes, *Acta Math.* **201** no. 1 (2008), 83–146. DOI MR Zbl

[7] S. Fomin, L. Williams, and A. Zelevinsky, *Introduction to cluster algebras. Chapters 4-5*, 2017. arXiv:1707.07190 [math.CO].

[8] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, *J. Amer. Math. Soc.* **15** no. 2 (2002), 497–529. DOI MR Zbl

[9] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification, *Invent. Math.* **154** no. 1 (2003), 63–121. DOI MR Zbl

[10] F. Hurtado and M. Noy, Graph of triangulations of a convex polygon and tree of triangulations, *Comput. Geom.* **13** no. 3 (1999), 179–188. DOI MR Zbl

Kodjo Essonana Magnani
Département de Mathématiques, Université de Lomé, BP 1515 Lomé, Togo
kodjo.essonana.magnani@usherbrooke.ca

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