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THE RECONSTRUCTION PROBLEM FOR A MULTIVALUED LINEAR OPERATOR'S PROPERTIES

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ABSTRACT. The reconstruction problem for a multivalued linear operator (linear relation) T is viewed as the exploration of some properties of T from those of a restriction of T on an invariant linear subspace.

1. Introduction

Let X be a complex Banach space, M is a subspace of X and let T be a bounded and closed multivalued linear operator (linear relation) on X. We say that M is T-invariant (resp., weakly invariant under T) if $T(M) \subset M$ (resp., for all $x \in M$, $Tx \cap M \neq \varnothing$). The restriction T_M of T on M is defined by $G(T_M) = G(T) \cap (M \times M)$, where $G(T)$ stands for the graph of T. This paper aims at exhibiting the usefulness of the restriction T_M as an instrument to study the properties of T. In this paper, the following statements will be established under some supplementary conditions on M and T.

(i) T_M and $\widetilde{T_M}$ are invertible if and only if T is invertible, where $\widetilde{T_M}$ stands for the linear relation induced by T on X/M.

(ii) T_M and $\widetilde{T_M}$ are Drazin invertible if and only if T is Drazin invertible and either $\operatorname{asc}(\widetilde{T_M}) < \infty$ or $\operatorname{des}(T_M) < \infty$ where $\operatorname{asc}(\cdot)$ and $\operatorname{des}(\cdot)$ stand for the ascent and the descent, respectively.

(iii) For all $\lambda \neq 0$, $R(\lambda - T)$ is closed in X if and only if $R(\lambda - T_M)$ is closed in M and $\lambda - T$ is bounded below if and only if $\lambda - T_M$ is bounded below, where $R(T)$ stands for the range of T.

(iv) For $\lambda \neq 0$, if $\lambda - T$ is relatively regular in X then $\lambda - T_M$ is relatively regular in M and the converse is also true whenever M is dense in X. Some of these statements extend some of the results given in [4, 9, 10] in the context of linear operators.

The paper is organized as follows: Section 2 contains basic notions and some auxiliary results which will be used in the following sections. Essentially, we give some inequalities between the ascent and the descent of T as well as those of the linear relations T_M and $\widetilde{T_M}$. In Section 3, the statements (i) and (ii) are studied.

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Section 4 is devoted to prove the statements (iii). Finally, in Section 5 the statement (iv) is proved.

2. Preliminary results

We begin by recalling some basics from the theory of linear relations in normed linear spaces. We adhere to the notations and terminology of the monograph [7].

Let X, Y and Z be complex normed linear spaces. A linear relation T from X to Y is a mapping from a subspace $D(T) = \{x \in X : Tx \neq \emptyset\}$, called the domain of T, into the collection of nonempty subsets of Y satisfying $T(\alpha x_1+\beta x_2) = \alpha Tx_1+\beta Tx_2$ for all non zero scalars α, β and vectors $x_1, x_2 \in D(T)$. We denote by $LR(X,Y)$ the class of all linear relations everywhere defined from X into Y and we write $LR(X) := LR(X,X)$.

If T maps the points of its domain to singletons, then T is said to be an operator, that is equivalent to $T(0) = \{0\}$. The class of linear bounded operators is denoted by $L(X,Y)$ and we write $L(X) := L(X,X)$.

A linear relation $T \in LR(X,Y)$ is uniquely determined by its graph defined by $G(T) := \{(x,y) \in X \times Y : x \in D(T), y \in Tx\}$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) := \{(y,x) \in Y \times X : (x,y) \in G(T)\}$. The linear subspaces $N(T) := T^{-1}(0) = \{x \in D(T) : 0 \in Tx\}$ and $R(T) := T(D(T)) := \{y : (x,y) \in G(T) \text{ for some } x \in D(T)\}$ are called the nullspace (or the kernel) and the range space of T respectively. We say that T is injective if $N(T) = \{0\}$ and surjective (or onto) if $R(T) = Y$. If T is injective and surjective, then T is said to be bijective.

For $T, S \in LR(X,Y)$ and $\lambda \in \mathbb{C}$, the linear relations $T+S$, λT are respectively defined by $G(T+S) := \{(x, y+z) \in X \times Y : (x,y) \in G(T) \text{ and } (x,z) \in G(S)\}$ and $G(\lambda T) := \{(x, \lambda y) \in X \times Y : (x,y) \in G(T)\}$. For $T \in LR(X)$ and $\lambda \in \mathbb{C}$, the linear relation $T - \lambda I$ is defined by $G(T - \lambda I) := \{(x, y - \lambda x) \in X \times X : (x,y) \in G(T)\}$. For $T \in LR(X,Y)$ and $S \in LR(Y,Z)$, the product linear relation $ST \in LR(X,Z)$ is defined by $G(ST) := \{(x,z) \in X \times Z : (x,y) \in G(T) \text{ and } (y,z) \in G(S) \text{ for some } y \in Y\}$. For a given closed subspace E of X, let Q_E^X or simply Q_E denote the quotient map with domain X onto X/E. We shall denote $Q_{\overline{T(0)}}$ by Q_T for a given linear relation $T \in LR(X,Y)$. It is easy to see that $Q_T T$ is an operator, and thus we can define $\|Tx\| := \|Q_T Tx\|$, $x \in D(T)$, and $\|T\| := \|Q_T T\|$. We say that T is continuous if $\|T\| < \infty$, bounded if it is continuous and everywhere defined, open if its inverse is continuous, and bounded below if it is injective and open. The class of bounded relations will be denoted by $BR(X,Y)$. A linear relation T is said to be closed if its graph is a closed subspace of $X \times Y$. It is well known that T^{-1} is closed if and only if T is closed if and only if $Q_T T$ is a closed operator and $T(0)$ is a closed subspace. If T is closed, then $N(T)$ is closed. It is also known that if X and Y are Banach spaces and T is closed with $D(T) = X$, then T is bounded. This suggests the following notation:

$$BCR(X,Y) := \{T \in LR(X,Y) : T \text{ is closed and everywhere defined}\}.$$

An everywhere defined closed linear relation in a Banach space X is said to be *invertible* if T^{-1} is a bounded operator, or equivalenty, if T is injective and onto.

Let T be a closed linear relation. The *resolvent set* of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective and onto}\}$$

and $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. For $T \in LR(X)$, the kernels and the ranges of the iterates T^n, $n \in \mathbb{N}$, form two increasing and decreasing chains, respectively, i.e., the chain of kernels

$$N(T^0) = \{0\} \subset N(T) \subset N(T^2) \subset \cdots$$

and the chain of ranges

$$R(T^0) = X \supset R(T) \supset R(T^2) \supset \cdots$$

The *ascent* and the *descent* of T are defined, respectively, by

$$\operatorname{asc}(T) := \min\{k \in \mathbb{N} : N(T^k) = N(T^{k+1})\},$$
$$\operatorname{des}(T) := \min\{k \in \mathbb{N} : R(T^k) = R(T^{k+1})\}.$$

The *singular chain manifold* $R_c(T)$ of T is defined as

$$R_c(T) := \Big(\bigcup_{n=1}^{\infty} N(T^n)\Big) \cap \Big(\bigcup_{n=1}^{\infty} T^n(0)\Big).$$

Definition 2.1 ([15, Section 1])**.** For $T \in LR(X)$, a subspace M of X is called *weakly invariant* under T if for all $x \in M$, $Tx \cap M \neq \emptyset$, and is called *nontrivial* if $\{0\} \neq M \neq X$ and $M \neq T(0)$.

Let T be a linear relation and let M be a closed subspace of X. We say that T and M satisfy the condition $(\mathcal{H})$ if we have

$$(\mathcal{H}) : \begin{cases} \bullet\, T \in BCR(X) \\ \bullet\, M \text{ is a closed weakly invariant subspace under } T \\ \qquad \text{satisfying the condition (c): } x \in M \text{ if and only if } Tx \cap M \neq \varnothing, \\ \bullet\, M + T(0) \text{ is closed.} \end{cases}$$

In the rest of this section, we suppose that T and M satisfy the condition $(\mathcal{H})$. We denote by T_M the relation defined by

$$G(T_M) = G(T) \cap (M \times M). \tag{2.1}$$

It is clear that $T_M \in BCR(M)$ and if, in addition, $R_c(T) = \{0\}$ then $R_c(T_M) = \{0\}$. Now, we denote by $\widetilde{T_M}$ the linear relation induced by T on X/M and defined by

$$\begin{aligned} \widetilde{T_M} : X/M &\longrightarrow X/M \\ \overline{x} &\longmapsto \widetilde{T_M}\overline{x} = \{\overline{y} : y \in Tx\}. \end{aligned} \tag{2.2}$$

The linear relation $\widetilde{T_M}$ is well defined. We claim that $\widetilde{T_M}$ is a closed relation. Indeed, let $(\overline{x_n}, \overline{y_n}) \in G(\widetilde{T_M})$ be such that $(\overline{x_n}, \overline{y_n}) \xrightarrow[n\to+\infty]{} (\overline{x}, \overline{y})$. Therefore, for all $n \in \mathbb{N}$, there exists α_n, β_n and γ_n in M such that $x_n - \alpha_n \xrightarrow[n\to+\infty]{} x$,

$y_n - \gamma_n \xrightarrow[n\to+\infty]{} y$ and $y_n - \beta_n \in T(x_n - \alpha_n)$. Let $z \in Tx$. We have, $y_n - \beta_n - z \in T(x_n - \alpha_n - x)$. So, $\|T(x_n - \alpha_n - x)\| = d(y_n - \beta_n - z, T(0)) \leqslant \|T\| \, \|x_n - \alpha_n - x\|$. Thus, there exists $\alpha'_n \in T(0)$ such that $\lim\limits_{n\to+\infty} y_n - \beta_n - z - \alpha'_n = 0$. Hence, $\lim\limits_{n\to+\infty} y_n - \gamma_n - y_n + \beta_n + z + \alpha'_n = y$. Therefore, $y - z = \lim\limits_{n\to+\infty} \beta_n - \gamma_n + \alpha'_n$. Since $M + T(0)$ is closed, we have $y - z \in M + T(0)$, and so $y \in M + Tx$. Thus, there exists $m \in M$ such that $y - m \in Tx$. This implies that $(\overline{x}, \overline{y}) \in G(\widetilde{T_M})$ and $\widetilde{T_M}$ is closed as claimed. Now, as $\widetilde{T_M}$ is everywhere defined, it is bounded.

Remark 2.2. We note that if T and M satisfy the condition (c) then for all $n \geqslant 1$, T^n and M still satisfy the same condition.

Indeed, we prove that for $n \geq 1$, $T^n x \cap M \neq \varnothing \Leftrightarrow x \in M$. Let $n \geqslant 1$. Suppose that $T^n x \cap M \neq \varnothing \Leftrightarrow x \in M$, and let us show that $T^{n+1} x \cap M \neq \varnothing \Leftrightarrow x \in M$.

For the first implication, let $y \in T^{n+1}x \cap M$. Then there exists $z \in X$ such that $(x, z) \in G(T^n)$ and $(z, y) \in G(T)$. We have $y \in Tz \cap M$, then by the condition (c) we get $z \in M$. So, $z \in T^n x \cap M$ and $x \in M$. For the reverse implication, let $x \in M$. Then we have $T^n x \cap M \neq \varnothing$. Let $y \in T^n x \cap M$. Thus, $T^n x = y + T^n(0)$. So, $T(T^n x) = Ty + T^{n+1}(0)$. As $y \in M$, we have $Ty \cap M \neq \varnothing$, so $Ty \subset M + T(0)$. Therefore, $T^{n+1}x \subset M + T^{n+1}(0)$. Let $z \in T^{n+1}x$. Then, there exists $\alpha_m \in M$ and $\alpha_0 \in T^{n+1}(0)$ such that $z = \alpha_m + \alpha_0$. So, $\alpha_m = z - \alpha_0 \in T^{n+1}(x) \cap M$. Therefore, $T^{n+1}x \cap M \neq \varnothing$.

In the rest of this section, we will focus on some relations between the ascent and the descent of T and those of the linear relations T_M and $\widetilde{T_M}$.

Lemma 2.3. *Let T and M satisfy the condition $(\mathcal{H})$ and let T_M, $\widetilde{T_M}$ be defined as in* (2.1) *and* (2.2)*. Then, we have*

(1) $R(\widetilde{T_M}) = \dfrac{R(T) + M}{M}$.

(2) $\widetilde{T^n_M} = (\widetilde{T_M})^n$ *and* $R((\widetilde{T_M})^n) = \dfrac{R(T^n) + M}{M}$ *for all* $n \geqslant 1$.

(3) $N((T_M)^n) = N(T^n) \cap M$, $R((T_M)^n) = R(T^n) \cap M$ *and* $(T_M)^n = (T^n)_M$ *for all* $n \geq 1$.

Proof. (1) For the first inclusion, let $\overline{y} \in R(\widetilde{T_M})$. Then there exists $\overline{x} \in X/M$ such that $\overline{y} \in \widetilde{T_M}(\overline{x})$. Hence there exists $z \in X$ such that $\overline{y} = \overline{z}$ and $z \in Tx$. So $y \in R(T) + M$. Thus, $\overline{y} \in \frac{R(T)+M}{M}$. For the reverse inclusion, let $\overline{y} \in \frac{R(T)+M}{M}$. Then there exists $z \in R(T)$ and $m \in M$ such that $y = z + m$. So, there exists $x \in X$ such that $y = z + m \in Tx + m$. Hence $y - m \in Tx$. Thus $\overline{y - m} \in \widetilde{T_M}(\overline{x})$. As $\overline{y - m} = \overline{y}$, we have $\overline{y} \in R(\widetilde{T_M})$.

(2) We proceed by induction. For $n = 1$ there is nothing to prove. Let $n \geqslant 1$ and suppose that $\widetilde{(T^n)_M} = (\widetilde{T_M})^n$. We prove that $\widetilde{(T^{n+1})_M} = (\widetilde{T_M})^{n+1}$. Let $(\overline{x}, \overline{y}) \in G(\widetilde{(T^{n+1})_M})$. Thus, $x = x_1 + m_x$ and $y = y_1 + m_y$ with $m_x, m_y \in M$ and $y_1 \in T^{n+1}x_1$. So there exists $z \in X$ such that $(x_1, z) \in G(T^n)$ and $(z, y_1) \in G(T)$. Hence, $(\overline{x_1}, \overline{z}) \in G(\widetilde{(T^n)_M})$ and $(\overline{z}, \overline{y_1}) \in G(\widetilde{T_M})$. Then, $(\overline{x_1}, \overline{z}) \in G(\widetilde{T_M}^n)$ and

$(\overline{z}, \overline{y_1}) \in G(\widetilde{T_M})$. So, $(\overline{x}, \overline{y}) = (\overline{x_1}, \overline{y_1}) \in G((\widetilde{T_M})^{n+1})$. For the reverse inclusion, let $(\overline{x}, \overline{y}) \in G((\widetilde{T_M})^{n+1})$. Then, there exists $\overline{z} \in X/M$ such that $(\overline{x}, \overline{z}) \in G((\widetilde{T_M})^n)$ and $(\overline{z}, \overline{y}) \in G(\widetilde{T_M})$. Then, $(\overline{x}, \overline{z}) \in G(\widetilde{(T^n)_M})$ and $(\overline{z}, \overline{y}) \in G(\widetilde{T_M})$. So, $z = z_1 + m_z$, $x = x_1 + m_x$ such that $m_z, m_x \in M$ and $z_1 \in T^n x_1$ and $z = z_1' + m_z'$, $y = y_1 + m_y$ such that $m_z', m_y \in M$ and $y_1 \in T z_1'$. On the other hand, $z_1 - z_1' = m_z' - m_z \in M$. So, $z_1' = z_1 + \alpha$ with $\alpha \in M$ and $Tz_1' = Tz_1 + T\alpha$. As $\alpha \in M$, we have $T\alpha \cap M \neq \varnothing$, so $T\alpha = m + T(0)$ where $m \in M$, so $Tz_1' = Tz_1 + m$. Therefore, $y_1 - m \in Tz_1$. Thus, $(x_1, y_1 - m) \in G(T^{n+1})$. Hence, $(\overline{x}, \overline{y}) = (\overline{x_1}, \overline{y_1 - m}) \in G(\widetilde{(T^{n+1})_M})$.

The equality $R((\widetilde{T_M})^n) = \frac{R(T^n)+M}{M}$ follows immediately from part (1).

(3) A straightforward calculation shows that $N((T_M)^n) = N(T^n) \cap M$ and $R((T_M)^n) = R(T^n) \cap M$ for all $n \geq 1$. To prove the last equality, using [3, 2.02], it is sufficient to show that $(T_M)^n \subseteq (T^n)_M$ for all $n \geq 1$. To do this, it suffices to reason by induction. ☐

In a similar way we show the following:

Lemma 2.4. *Let T and M satisfy the condition ($\mathcal{H}$) and let T_M, $\widetilde{T_M}$ be defined as in* (2.1) *and* (2.2). *Then, we have*

(1) $N(\widetilde{T_M}) = \dfrac{N(T)+M}{M}$.

(2) $N((\widetilde{T_M})^n) = \dfrac{N(T^n)+M}{M}$ *for all* $n \geqslant 1$.

Proposition 2.5. *Let T and M satisfy the condition ($\mathcal{H}$) and let T_M, $\widetilde{T_M}$ be defined as in* (2.1) *and* (2.2). *Then*

$$\operatorname{asc}(T_M) \leqslant \operatorname{asc}(T) \quad \textit{and} \quad \operatorname{des}(\widetilde{T_M}) \leqslant \operatorname{des}(T).$$

Proof. Let $n \geqslant \operatorname{asc}(T)$. We prove that $N((T_M)^{n+1}) = N((T_M)^n)$. Let $x \in N((T_M)^{n+1})$. Then part (3) of Lemma 2.3 implies that $x \in N(T^{n+1}) \cap M$. Since $n \geqslant \operatorname{asc}(T)$ and $x \in N(T^{n+1})$, we have $x \in N(T^n)$. Therefore $x \in N(T^n) \cap M$. By part (3) of Lemma 2.3, we get $x \in N((T_M)^n)$. This proves that $\operatorname{asc}(T_M) \leqslant \operatorname{asc}(T)$.

Let $n \geqslant \operatorname{des}(T)$. We claim that $R((\widetilde{T_M})^{n+1}) = R((\widetilde{T_M})^n)$. Let $\overline{y} \in R((\widetilde{T_M})^n)$. Then part (2) of Lemma 2.3 implies that $\overline{y} \in \frac{R(T^n)+M}{M}$. Since $n \geqslant \operatorname{des}(T)$, we have $R(T^{n+1}) = R(T^n)$. Thus, $\overline{y} \in \frac{R(T^{n+1})+M}{M}$. Using the part (2) of Lemma 2.3, we get $\overline{y} \in R((\widetilde{T_M})^{n+1})$.

Therefore, $\operatorname{des}(\widetilde{T_M}) \leqslant \operatorname{des}(T)$. ☐

Proposition 2.6. *Let T and M satisfy the condition* ($\mathcal{H}$) *and let T_M, $\widetilde{T_M}$ be defined as in* (2.1) *and* (2.2). *Then,*

(1) $\operatorname{asc}(T) \leqslant \operatorname{asc}(T_M) + \operatorname{asc}(\widetilde{T_M})$,

(2) $\operatorname{des}(T) \leqslant \operatorname{des}(T_M) + \operatorname{des}(\widetilde{T_M})$.

Proof. (1) Let $n \geqslant \operatorname{asc}(T_M) + \operatorname{asc}(\widetilde{T_M})$. We have that $n \geqslant \operatorname{asc}(T_M)$ and $n \geqslant \operatorname{asc}(\widetilde{T_M})$. So, $N((T_M)^n) = N((T_M)^{n+1})$ and $N((\widetilde{T_M})^n) = N((\widetilde{T_M})^{n+1})$. Hence, by

Lemmas 2.3 and 2.4, $N(T^n)\cap M = N(T^{n+1})\cap M$ and $\frac{N(T^n)+M}{M} = \frac{N(T^{n+1})+M}{M}$. We claim that $N(T^{n+1}) = N(T^n)$. Suppose, on the contrary, that $N(T^n) \subsetneq N(T^{n+1})$. Then, there exists $x \in N(T^{n+1})$ such that $x \notin N(T^n)$. On the other hand, we have $\overline{x} \in \frac{N(T^{n+1})+M}{M} = \frac{N(T^n)+M}{M}$. So, there exists $x' \in N(T^n) + M$ such that $\overline{x} = \overline{x'}$. Thus $x - x' \in M$ and hence $x \in N(T^n) + M$. Therefore, there exists $\alpha \in N(T^n) \subset N(T^{n+1})$ such that $x - \alpha \in M$. Hence $x - \alpha \in N(T^{n+1})\cap M$. Therefore, $x \in N(T^n)$ which is absurd. Thus, $N(T^{n+1}) = N(T^n)$ and so $\operatorname{asc}(T) \leqslant \operatorname{asc}(T_M) + \operatorname{asc}(\widetilde{T_M})$.

(2) The result can be proved by similar arguments. □

3. Reconstruction of the invertibility and the Drazin invertibility properties

The main intent of this section is to provide proofs of the statements (i) and (ii) announced in Section 1. To do this, we begin by recalling some definitions and properties of Drazin invertible linear relations.

Definition 3.1 ([12])**.** Let $T \in BCR(X)$. T is said to be *Drazin invertible of degree* $k \in \mathbb{N}$ if there exists a bounded operator $T^D \in L(X)$ such that

(i) $TT^D = T^DT + T(0)$;
(ii) $T^DTT^D = T^D$; and
(iii) $T^{k+1}T^D = T^k + T^{k+1}(0)$.

T^D is called the *Drazin inverse* of T.

Notice that if $R_c(T) = \{0\}$, then T is Drazin invertible of degree 0 if and only if T is invertible.

Lemma 3.2 ([12, Lemma 3.2])**.** *Let $T \in BCR(X)$ be Drazin invertible and T^D be a Drazin inverse of T. Then*

(i) *T^DT is an operator.*
(ii) *For all $k \in \mathbb{N}$, $(T^D)^kT^k = T^DT = (T^DT)^k$.*

Lemma 3.3 ([12, Theorem 3.3])**.** *Let $T \in BCR(X)$ be such that $\rho(T) \neq \varnothing$. The following statements are equivalent:*

(i) *T is Drazin invertible;*
(ii) *$\operatorname{asc}(T)$ and $\operatorname{des}(T)$ are finite;*
(iii) *There exist two closed subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is an invertible linear relation and T_N is a bounded nilpotent operator.*

Definition 3.4 ([14, Definition 2, Lemma 3])**.** Let $T \in BCR(X)$. We say that T is a *generalized Drazin invertible* linear relation if there exists $B \in L(X)$ such that

$$TB = BT + T(0), \quad BTB = B, \quad T - TBT = N_1 + T(0),$$

with $T^2(0) \subseteq N(N_1)$ and N_1 being a bounded quasinilpotent operator. We say that B is a *generalized Drazin inverse* of T.

Proposition 3.5 ([14, Theorem 19])**.** *Let $T \in BCR(X)$ be such that $R_c(T) = \{0\}$. Then T is generalized Drazin invertible if and only if 0 is an isolated point of the spectrum of T.*

The following theorem provides the equivalence between the invertibilities of T and those of T_M and $\widetilde{T_M}$.

Theorem 3.6. *Let T and M satisfy the condition $(\mathcal{H})$ and let T_M and $\widetilde{T_M}$ be defined as in (2.1) and (2.2). Then, we have*

$$T_M \text{ and } \widetilde{T_M} \text{ are invertible} \Longleftrightarrow T \text{ is invertible.}$$

Proof. For the direct implication, assume that T_M and $\widetilde{T_M}$ are invertible. First, we prove that T is surjective. Let $y \in X$. Since $\widetilde{T_M}$ is surjective, there exists $x \in X$ such that $\overline{y} \in \widetilde{T_M}\overline{x}$. Hence, there exists $z \in X$ such that $\overline{y} = \overline{z}$ and $z \in Tx$. So, $y - z \in M$. Then, $y = z + m$ for some $m \in M$. Hence, $y \in Tx + m$. Since $m \in M$ and T_M is surjective, there exists $x' \in M \subset X$ such that $m \in Tx'$. Therefore, $y \in Tx + Tx' = T(x + x')$.

Next assume that $0 \in Tx$. Then, $\overline{0} \in \widetilde{T_M}\overline{x}$. Since $\widetilde{T_M}$ is injective, we have $\overline{x} = \overline{0}$. Then $x \in M$. Since T_M is invertible and $0 \in T_M x$, we have $x = 0$. This proves that T is invertible.

The reverse implication follows immediately from Lemmas 2.3 and 2.4. □

As a consequence of Theorem 3.6, we get the following lemma.

Lemma 3.7. *Let T and M satisfy the condition $(\mathcal{H})$. Then we have:*

(i) $\sigma(T) \subset \sigma(T_M) \cup \sigma(\widetilde{T_M})$;
(ii) $\sigma(T_M) \subset \sigma(T) \cup \sigma(\widetilde{T_M})$;
(iii) $\sigma(\widetilde{T_M}) \subset \sigma(T) \cup \sigma(T_M)$.

Remark 3.8. By Lemma 3.7, we deduce that $\sigma(T) = (\sigma(T_M) \cup \sigma(\widetilde{T_M})) \setminus V$ where $V \subset \sigma(T_M) \cap \sigma(\widetilde{T_M})$.

Corollary 3.9. *Let T and M satisfy the condition $(\mathcal{H})$. Then the following properties hold:*

(i) *If $\lambda \in (\sigma(T_M) \cup \sigma(\widetilde{T_M})) \setminus \sigma(T)$, then $\lambda \in \sigma(T_M) \cap \sigma(\widetilde{T_M})$;*
(ii) *If $\lambda \in (\sigma(T) \cup \sigma(\widetilde{T_M})) \setminus \sigma(T_M)$, then $\lambda \in \sigma(T) \cap \sigma(\widetilde{T_M})$;*
(iii) *If $\lambda \in (\sigma(T) \cup \sigma(T_M)) \setminus \sigma(\widetilde{T_M})$, then $\lambda \in \sigma(T) \cap \sigma(T_M)$.*

We now give the first theorem connecting the Drazin invertibility of T with that of T_M (resp., $\widetilde{T_M}$) in the case where $\widetilde{T_M}$ (resp., T_M) is invertible.

Theorem 3.10. *Let T and M satisfy the condition $(\mathcal{H})$ and let T_M and $\widetilde{T_M}$ be defined as in (2.1) and (2.2). If $\widetilde{T_M}$ (resp., T_M) is invertible, then*

(i) $\operatorname{asc}(T_M) = \operatorname{asc}(T)$ *and* $\operatorname{des}(T_M) = \operatorname{des}(T)$ *(resp.,* $\operatorname{asc}(\widetilde{T_M}) = \operatorname{asc}(T)$ *and* $\operatorname{des}(\widetilde{T_M}) = \operatorname{des}(T)$*).*
(ii) *If moreover $\rho(T) \neq \varnothing$ then T is Drazin invertible if and only if T_M (resp., $\widetilde{T_M}$) is Drazin invertible.*

Proof. (i) Consider the case where $\widetilde{T_M}$ is an invertible relation. Then, $\operatorname{asc}(\widetilde{T_M}) = \operatorname{des}(\widetilde{T_M}) = 0$ and by Propositions 2.5 and 2.6 we have $\operatorname{asc}(T) = \operatorname{asc}(T_M)$ and $\operatorname{des}(T) \leqslant \operatorname{des}(T_M)$.

Now, let $n \geqslant \operatorname{des}(T)$. We prove that $R((T_M)^n) = R((T_M)^{n+1})$. Let $y \in R((T_M)^n)$, then $y \in R(T^n) \cap M$. Since $n \geqslant \operatorname{des}(T)$, we get $R(T^n) = R(T^{n+1})$. Therefore, $y \in R(T^{n+1}) \cap M$. Hence, by part (3) of Lemma 2.3 we have $y \in R((T_M)^{n+1})$.

Next, we consider the case where T_M is an invertible relation. Then, $\operatorname{asc}(T_M) = \operatorname{des}(T_M) = 0$ and by Propositions 2.5 and 2.6 we have $\operatorname{des}(T) = \operatorname{des}(\widetilde{T_M})$ and $\operatorname{asc}(T) \leqslant \operatorname{asc}(\widetilde{T_M})$.

Now, let $n \geqslant \operatorname{asc}(T)$. Pick $\overline{x} \in N((\widetilde{T_M})^{n+1})$. Then $\overline{x} \in \frac{N(T^{n+1})+M}{M}$. Since $n \geqslant \operatorname{asc}(T)$, we get $N(T^n) = N(T^{n+1})$. Therefore, $\overline{x} \in \frac{N(T^n)+M}{M} = N((\widetilde{T_M})^n)$. This shows that $\operatorname{asc}(\widetilde{T_M}) \leqslant \operatorname{asc}(T)$.

(ii) This is immediate from part (i) and Lemma 3.3. □

Definition 3.11 ([5, Definition 2.1])**.** A closed subspace M of X is said to be *strongly invariant* under a relation $T \in BCR(X)$ with nonempty resolvent set $\rho(T)$ if for all $\lambda \in \rho(T)$ we have $(\lambda - T)^{-1}(M) \subset M$.

Remark 3.12 ([5])**.** Note that if M is a closed subspace of X that is strongly invariant under $T \in BCR(X)$, then it is also weakly invariant under T. The converse is not always true even for bounded operators.

Remark 3.13. If M is a closed subspace of X strongly invariant under $T \in BCR(X)$, then for all $\lambda \in \rho(T)$ we have that $(\lambda - T)$ and M satisfy the condition (c). That is, for $\lambda \in \rho(T)$, we have

$$x \in M \iff (\lambda - T)x \cap M \neq \varnothing.$$

Notation. Let T and M satisfy the condition $(\mathcal{H})$. If M is strongly invariant under T, then we say that T and M satisfy the condition $(\mathcal{H}')$.

Example 3.14. As an example, we can see that if $T \in BCR(X)$ is semi regular (that is, $R(T)$ is closed and for all $n, m \in \mathbb{N}$, we have $N(T^n) \subset R(T^m)$, see [1]) and $M = R^\infty(T) := \bigcap_{n\in\mathbb{N}} R(T^n)$, then T and M satisfy the condition $(\mathcal{H}')$.

Lemma 3.15. *Let T and M satisfy the condition $(\mathcal{H}')$. Then, for all $\lambda \in \mathbb{C}$ we have*

$$(\lambda - T)_M = \lambda - T_M \quad \textit{and} \quad \widetilde{(\lambda - T)}_M = \lambda - \widetilde{T_M}.$$

Proof. We have

$$\begin{aligned} G((\lambda - T)_M) &= G(\lambda - T) \cap (M \times M) \\ &= \{(x, \lambda x - y) : (x, y) \in G(T) \cap (M \times M)\} \\ &= G(\lambda - T_M). \end{aligned}$$

We claim that $\widetilde{(\lambda - T)}_M = \lambda - \widetilde{T_M}$. Indeed, let $(\overline{x}, \overline{y}) \in G(\widetilde{(\lambda - T)}_M)$. Thus, $x = x_1 + m_x$ and $y = y_1 + m_y$ with $m_x, m_y \in M$ and $y_1 \in (\lambda - T)x_1$. Therefore, $(x_1, -y_1 + \lambda x_1) \in G(T)$. Hence, $(\overline{x}, \overline{-y + \lambda x}) = (\overline{x_1}, \overline{-y_1 + \lambda x_1}) \in G(\widetilde{T_M})$. Therefore, $(\overline{x}, \overline{y}) \in G(\lambda - \widetilde{T_M})$.

For the reverse inclusion, let $(\overline{x}, \overline{y}) \in G(\lambda - \widetilde{T_M})$. Then, $\overline{y} \in (\lambda - \widetilde{T_M})\overline{x}$. So, $-\overline{y} + \lambda\overline{x} \in \widetilde{T_M}\overline{x}$. Hence, there exists $z \in X$ such that $-\overline{y} + \lambda\overline{x} = \overline{z}$ and $z \in Tx$. Thus, $z = -y + \lambda x + m$ with $m \in M$. Therefore, $y + m \in (\lambda - T)x$. Hence, $(\overline{x}, \overline{y}) = (\overline{x}, \overline{y + m}) \in G(\widetilde{(\lambda - T)}_M)$. □

Lemma 3.16. *Let T and M satisfy the condition $(\mathcal{H}')$. Then,*

$$\rho(T) \subset \rho(T_M) \cap \rho(\widetilde{T_M}).$$

Proof. It readily follows from Theorem 3.6 and Lemma 3.15. □

Lemma 3.17. *Let T and M satisfy the condition $(\mathcal{H}')$. Then,*

(i) *If $0 \notin \operatorname{acc}\sigma(T_M) \cup \operatorname{acc}\sigma(T)$ then $0 \notin \operatorname{acc}\sigma(\widetilde{T_M})$;*
(ii) *If $0 \notin \operatorname{acc}\sigma(\widetilde{T_M}) \cup \operatorname{acc}\sigma(T)$ then $0 \notin \operatorname{acc}\sigma(T_M)$.*

Proof. (i) If we suppose that $0 \in \operatorname{acc}\sigma(\widetilde{T_M})$, then by Lemma 3.7 (i) we get that $0 \in \sigma(T) \cup \sigma(T_M)$. Therefore, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \sigma(T) \cup \sigma(T_M)$ such that $\lambda_n \neq \lambda_{n'}$ for all $n \neq n'$ and $\lambda_n \xrightarrow[n\to+\infty]{} 0$. Thus, $A = \{n \in \mathbb{N} : \lambda_n \in \sigma(T)\}$ or $B = \{n \in \mathbb{N} : \lambda_n \in \sigma(T_M)\}$ is infinite. If we suppose that A is infinite then, we can extract a subsequence $\rho_1(n)$ such that $(\lambda_{\rho_1(n)})_{n\in\mathbb{N}} \in \sigma(T) : \lambda_{\rho_1(n)} \neq \lambda_{\rho_1(n')}$ for all $n \neq n'$ and $\lambda_{\rho_1(n)} \xrightarrow[n\to+\infty]{} 0$. Therefore, $0 \in \operatorname{acc}\sigma(T)$. However, this contradicts the fact that $0 \notin \operatorname{acc}\sigma(T)$. If we suppose that B is infinite then, in a similar way we show that $0 \in \operatorname{acc}\sigma(T_M)$. However this contradicts the fact that $0 \notin \operatorname{acc}\sigma(T_M)$. So $0 \notin \operatorname{acc}\sigma(\widetilde{T_M})$

(ii) The result can be proved by similar arguments. □

Remark 3.18. Let T and M satisfy the condition $(\mathcal{H}')$. Then,

$$\sigma(T) = \sigma(T_M) \cup \sigma(\widetilde{T_M}).$$

Indeed, by Lemma 3.16, $\rho(T) \subset \rho(T_M) \cap \rho(\widetilde{T_M})$. Hence $\sigma(T_M) \cup \sigma(\widetilde{T_M}) \subset \sigma(T)$. On the other hand, by Lemma 3.7, $\sigma(T) \subset \sigma(T_M) \cup \sigma(\widetilde{T_M})$.

In the following theorem we characterize the Drazin invertibility of T using those of T_M and $\widetilde{T_M}$.

Theorem 3.19. *Let T and M satisfy the condition $(\mathcal{H}')$ and $\rho(T) \neq \varnothing$. Then, the following assertions are equivalent:*

(1) *T_M and $\widetilde{T_M}$ are Drazin invertible;*
(2) *T is Drazin invertible and $\operatorname{des}(T_M) < \infty$;*
(3) *T is Drazin invertible and $\operatorname{asc}(\widetilde{T_M}) < \infty$.*

Proof. Assume that (1) holds. Then, by Lemmas 3.16 and 3.3, we have that $\operatorname{asc}(T_M)$, $\operatorname{asc}(\widetilde{T_M})$, $\operatorname{des}(T_M)$ and $\operatorname{des}(\widetilde{T_M})$ are all finite. By Propositions 2.5 and 2.6, it follows that $\operatorname{asc}(T)$ and $\operatorname{des}(T)$ are finite. Furthermore, from the hypothesis $\rho(T) \neq \varnothing$ and Lemma 3.3, it follows that T is Drazin invertible. This verifies that (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3).

(2) $\Longrightarrow$ (3): We need to show that $\operatorname{asc}(\widetilde{T_M}) < \infty$. By Proposition 2.5, we have $\operatorname{asc}(T_M) < \infty$. Then, $\operatorname{asc}(T_M)$ and $\operatorname{des}(T_M)$ are both finite. On the other hand, since $\rho(T) \neq \varnothing$, by Lemma 3.16 we get $\rho(T_M) \neq \varnothing$. It follows from Lemma 3.3 that T_M is Drazin invertible. On the other hand, by [6, Lemma 2.6] we have $R_c(T_M) = \{0\}$. Then, using Proposition 3.5 we get that $0 \notin \operatorname{acc}\sigma(T_M) \cup \operatorname{acc}\sigma(T)$. So, by Lemma 3.17, $0 \notin \operatorname{acc}\sigma(\widetilde{T_M})$. Then it follows by Proposition 3.5, that $\widetilde{T_M}$ is generalized Drazin invertible. We claim that $\widetilde{T_M}$ is Drazin invertible. Indeed, as $\widetilde{T_M}$ is generalized Drazin invertible, by [14, Theorem 28], there exist two closed subspaces M_1 and M_2 of X/M such that $X/M = M_1 \oplus M_2$ and $\widetilde{T_M} = \widetilde{T_M}_1 \oplus \widetilde{T_M}_2$ where $\widetilde{T_M}_1$ is an invertible linear relation and $\widetilde{T_M}_2$ is a bounded quasinilpotent operator. On the other hand, we have $\operatorname{des}(\widetilde{T_M}) \leqslant \operatorname{des}(T) < \infty$. Therefore, $\operatorname{des}(\widetilde{T_M}) = \operatorname{des}(\widetilde{T_M}_2) < \infty$. Using [16], we get that $\widetilde{T_M}_2$ is a bounded nilpotent operator. Then, it follows from Lemma 3.3 that $\widetilde{T_M}$ is Drazin invertible and therefore, $\operatorname{asc}(\widetilde{T_M}) < \infty$.

(3) $\Longrightarrow$ (1): Since $\rho(T) \neq \varnothing$ and T is Drazin invertible, by Lemma 3.3, $\operatorname{asc}(T)$ and $\operatorname{des}(T)$ are both finite. Suppose that $\operatorname{asc}(\widetilde{T_M}) < \infty$. This, taken together with the fact that $\operatorname{des}(\widetilde{T_M}) \leqslant \operatorname{des}(T) < \infty$, and the use of Lemma 3.3 and Lemma 3.16, implies that $\widetilde{T_M}$ is Drazin invertible. On the other hand, we infer from [6, Lemma 2.6] and Proposition 3.5 that $0 \notin \operatorname{acc}\sigma(\widetilde{T_M})$. So, $0 \notin \operatorname{acc}\sigma(T_M)$. Then, it follows from Proposition 3.5 that T_M is generalized Drazin invertible. Now, in a similar way as in the proof of the precedent implication we show that T_M is Drazin invertible. □

4. Reconstruction of the closed range property

We need a larger class of invariant subspaces than that of closed subspaces. Specially, we consider throughout this section the class of those subspaces which are paraclosed. Fundamental results concerning paraclosed subspaces can be found in [2, 8, 11].

Definition 4.1 ([2, Definition 3.1])**.** Let X be a Banach space. A subspace M of X is said to be *paraclosed* if M has a structure of its own, which makes the inclusion of $M \hookrightarrow X$ a continuous mapping.

Clearly, every closed subspace is paraclosed, but the class of paraclosed subspaces is strictly larger.

Remark 4.2. A subspace M of X is paraclosed if and only if there is a Banach space norm $\|\cdot\|_M$ and a constant $k > 0$ such that $\|m\|_X \leqslant k\|m\|_M$ for all $m \in M$.

We gather in the following lemma some useful results that are a direct consequence of [8, Proposition 3.9].

Lemma 4.3. *Let X be a Banach space and let M be a paraclosed subspace of X. Let also N be a closed subspace of X. Then,*

(i) *$M \cap N$ is a closed subspace of M;*
(ii) *if moreover $N \subset M$, then M/N is a paraclosed subspace of X/N.*

Let $T \in BCR(X)$ and let M be a paraclosed subspace of X. We say that M is T-invariant if $T(M) \subset M$. Denote by T_M the restriction of T to M. This section explores the questions:

- If the range of T is closed in X, then under what conditions it the same for T_M in M?
- If the range of T_M is closed in M, then under what conditions it the same for T in X?

Proposition 4.4. *Let $S \in BCR(X)$ and let M be an S-invariant paraclosed subspace of X. The following statements are equivalent:*

(a) *$S^{-1}(M) = M + N(S)$;*
(b) *$R(S_M) = R(S) \cap M$.*

When (b) holds and $R(S)$ is closed, we also get that $R(S_M)$ is closed in M.

Proof. Suppose that (a) holds. The inclusion $R(S_M) \subseteq R(S) \cap M$ always holds (as long as M is S-invariant). For the reverse inclusion, suppose that $y \in R(S) \cap M$. So, there exists an element $x \in X$ such that $y \in Sx$. Then $x \in S^{-1}(M)$. The statement (a) implies that, $x = z_1 + z_2$ where $z_1 \in M$ and $z_2 \in N(S)$. Then $y \in Sz_1$, so $y \in R(S_M)$. Now suppose that (b) holds. Let $y \in S^{-1}(M)$. Then, there exists $m_1 \in M$ such that $m_1 \in Sy$. So $m_1 \in R(S) \cap M$. Then, $m_1 \in Sm_2$ where $m_2 \in M$. Then $Sy = m_1 + S(0) = Sm_2$. Therefore, $y = m_2 + (y - m_2)$ with $(y - m_2) \in N(S)$. On the other hand, it is clear that $M + N(S) \subset S^{-1}(M)$. Thus, (a) holds.

Assume now that (b) holds and that $R(S)$ is closed. Then, by Lemma 4.3, $R(S) \cap M = R(S_M)$ is closed in M. □

Let $n \in \mathbb{N}^*$. We denote in the remainder of this paper:

$$\mathcal{P}_n = \{(M,T) : T \in BCR(X),\, T^2(0) = T(0),\, M \text{ is a paraclosed subspace of } X,\\ M \text{ is } T\text{-invariant, and } T^n x \subset M \text{ for all } x \in X\}$$

and

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}^*} \mathcal{P}_n.$$

Proposition 4.5. *Let $(M,T) \in \mathcal{P}$. Fix $n \geq 1$, such that $(M,T) \in \mathcal{P}_n$. Then for all $\lambda \neq 0$, $(\lambda - T)^{-1}(M) = M$. Moreover, we have:*

$$\textit{for all } \lambda \neq 0, \textit{ if } R(\lambda - T) \textit{ is closed in } X \textit{ then } R(\lambda - T_M) \textit{ is closed in } M. \quad (4.1)$$

Proof. Fix $\lambda \neq 0$. Let $m \in M$. Since M is T-invariant, there exists $y \in M$ such that $y \in Tm$. Then, $\lambda m - y \in (\lambda - T)m$. So $m \in (\lambda - T)^{-1}(\lambda m - y)$. Hence, $m \in (\lambda - T)^{-1}(M)$. For the reverse inclusion, set $S = \lambda^{-1}T$. Let $y \in (I-S)^{-1}(M)$. Thus, $(I - S)y \subset M$. Now as by [7] and [3, Theorem 2.3] we have $(I - S^n) = (I - S)(I + S + \cdots + S^{n-1})$, we get $(I - S^n)y = (I + S + \cdots + S^{n-1})(I - S)y \subset (I + S + \cdots + S^{n-1})(M) \subset M$. Since $S^n y \subset M$, we get $y \in M$. This proves that $(\lambda - T)^{-1}(M) = (I - S)^{-1}(M) = M$.

Let $x \in N(\lambda - T)$. Then, $x \in \frac{1}{\lambda}Tx$ and for n we have $x \in \frac{1}{\lambda^n}(T^n x)$. This implies that $x \in M$. So $N(\lambda - T) \subset M$. Therefore, the assertion (4.1) follows from Proposition 4.4. □

Proposition 4.6. *Let $(M, S) \in \mathcal{P}_1$. If $\lambda \neq 0$ and $R(\lambda - S_M)$ is closed in M, then $R(\lambda - S)$ is closed in X.*

Proof. We may assume that $\lambda = 1$, so our hypothesis is that $Z \equiv R(I - S_M)$ is closed in M. For convenience, set $N = N(I - S_M)$ and also we note that $N = N(I - S)$ (this follows from the fact that, for all $x \in X$, $Sx \subset M$).

Now we complete the proof that $R(I - S)$ is closed by showing that the linear map

$$\begin{aligned} \psi : R(I - S) &\longrightarrow X/N \\ y &\longmapsto \{\overline{z} : z \in (I - S)^{-1}y\} \end{aligned}$$

is continuous and closed.

We have $\psi(0) = \{\overline{z} : z \in (I - S)^{-1}0\} = \{\bar{0}\}$. Therefore, ψ is a linear operator.

Let $y_n \in R(I-S)$ such that $y_n \xrightarrow[n\to+\infty]{X} 0$. We claim that $\psi y_n \xrightarrow[n\to+\infty]{X/N} \bar{0}$. Indeed, let $z_n \in (I - S)^{-1}y_n$. We have $\psi y_n = \overline{z_n}$. Let $t_n \in Sz_n \cap M$. Then, there exists $\alpha_n \in S(0)$ such that $z_n - y_n = t_n + \alpha_n$.

Therefore,

$$\psi y_n = \overline{z_n} = \overline{z_n - t_n - \alpha_n} + \overline{t_n + \alpha_n} = \overline{y_n} + \overline{t_n + \alpha_n}.$$

We have $\|\overline{y_n}\|_{X/N} = d(y_n, N) \leqslant \|y_n\|_X$. Hence, $\lim\limits_{n\to+\infty} \|\overline{y_n}\|_{X/N} = 0$.

We claim that $\overline{t_n + \alpha_n} \xrightarrow[n\to+\infty]{X/N} 0$. Indeed, we have $t_n + \alpha_n \in Sz_n + S(0) = Sz_n$. Now, let us define the relation

$$\widehat{I - S_M} : M/N \to Z$$

by

$$(\widehat{I - S_M})(m + N) = (I - S_M)m \quad \text{for all } m \in M.$$

It is clear that $\widehat{I - S_M}$ is bijective. On the other hand, as $I - S$ is a bounded relation, for all $m \in M$ and $\alpha \in N$ we have

$$\|\widehat{I - S_M}(\overline{m})\|_Z = \|(I - S)(m + \alpha)\| \leqslant \|(I - S)\| \, \|m + \alpha\| \leqslant \|(I - S)\| \, \|\overline{m}\|_{M/N}.$$

So, $\widehat{I - S_M}$ is a bounded and closed invertible linear relation. We have

$$\widehat{I - S_M}(\overline{t_n + \alpha_n}) = (I - S_M)(t_n + \alpha_n) \subset (I - S_M)Sz_n \subset S(I - S)z_n \subset Sy_n.$$

Let $\beta_n \in \widehat{I-S_M}(\overline{t_n+\alpha_n}) \subset Sy_n$. So,

$$\begin{aligned}\|\widehat{I-S_M}(\overline{t_n+\alpha_n})\|_Z &= d_Z(\beta_n, S(0)) = \inf_{\alpha\in S(0)} \|\beta_n-\alpha\|_Z \\ &= d_X(\beta_n, S(0)) = \|Sy_n\| \leqslant \|S\|\,\|y_n\| \xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

On the other hand, $\overline{t_n+\alpha_n} = (\widehat{I-S_M})^{-1}(\widehat{I-S_M})(\overline{t_n+\alpha_n})$. Then,

$$\left\|\overline{t_n+\alpha_n}\right\|_{M/N} \leqslant \left\|(\widehat{I-S_M})^{-1}\right\|_{B(Z,M/N)} \left\|(\widehat{I-S_M})(\overline{t_n+\alpha_n})\right\|_Z \xrightarrow[n\to+\infty]{} 0.$$

Thus, $\|\overline{t_n+\alpha_n}\|_{M/N} \to 0$ and therefore, by Lemma 4.3 (ii), also in X/N. So, ψ is continuous.

On the other hand, we note that for all $x \in X, \psi(I-S)x = \overline{x}$. Let us consider the bounded quotient map operator $Q : X \to X/N$ defined by $Qx = \overline{x}$. We have $\psi(I-S) = Q$. Hence, $\psi = Q(I-S)^{-1}$. As $S \in BCR(X)$, we have that $I-S$ is closed, and then $(I-S)^{-1}$ is also closed. Thus, $Q(I-S)^{-1} = Q_{(I-S)^{-1}(0)}(I-S)^{-1}$ is closed. Then, ψ is closed. As ψ is continuous and closed, $R(I-S)$ is closed. □

Lemma 4.7. *Let $T \in BCR(X)$ be such that $T^2(0) = T(0)$ and let M be a paraclosed T-invariant subspace of X. Set $V = T^{-1}(M)$, so $M \subseteq V \subseteq X$. Define a norm on V by*

$$\|v\|_V = \|v\|_X + \|\widetilde{T}v\|_M \quad \textit{for } v \in V,$$

where $\widetilde{T} : (V, \|\cdot\|_X) \to (M, \|\cdot\|_M)$ is the linear relation induced by T and

$$\begin{aligned}\|\widetilde{T}v\|_M &= d_M(y, T(0));\ \textit{for some } y \in \widetilde{T}v \\ &= \inf_{\alpha\in T(0)} \|y-\alpha\|_M = \|Q_{\widetilde{T}}\widetilde{T}v\|_{M/T(0)}.\end{aligned}$$

Then, we have:

(a) *V is a T-invariant subspace and $T(V) \subseteq M$;*
(b) *$\|\cdot\|_V$ is a complete norm on V;*
(c) *M is paraclosed in V, and V is paraclosed in X.*

Proof. (a) Let $x \in V$. Then $x \in T^{-1}m$ for some $m \in M$. So $Tx \subset M$. Thus, $T(V) \subseteq M \subseteq V$. Therefore, V is a T-invariant subspace and $T(V) \subseteq M$.

(b) First, by Lemma 4.3, $M/T(0)$ is complete. Now, let $\{v_n\} \subseteq V$ be a Cauchy sequence in the norm $\|\cdot\|_V$. Then, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$ we have

$$\|v_n - v_m\|_V = \|v_n - v_m\|_X + \|Q_{\widetilde{T}}\widetilde{T}v_n - Q_{\widetilde{T}}\widetilde{T}v_m\|_{M/T(0)} \leqslant \varepsilon.$$

So, $\{v_n\}$ is Cauchy in $\|\cdot\|_X$ and $\{Q_{\widetilde{T}}\widetilde{T}v_n\}$ is Cauchy in $\|\cdot\|_{M/T(0)}$. Then, there exists $v_0 \in X$ and $w_0 \in M$ such that $\|v_n - v_0\|_X \xrightarrow[n\to+\infty]{} 0$ and $\|Q_{\widetilde{T}}\widetilde{T}v_n - Q_{\widetilde{T}}w_0\|_{M/T(0)} \xrightarrow[n\to+\infty]{} 0$. Thus, $Q_{\widetilde{T}}\widetilde{T}v_0 = Q_{\widetilde{T}}\widetilde{T}w_0$. In addition,

$$\begin{aligned}\|v_n - v_o\|_V &= \|v_n - v_o\|_X + \|Q_{\widetilde{T}}\widetilde{T}v_n - Q_{\widetilde{T}}\widetilde{T}v_0\|_{M/T(0)} \\ &= \|v_n - v_o\|_X + \|Q_{\widetilde{T}}\widetilde{T}v_n - Q_{\widetilde{T}}w_0\|_{M/T(0)} \xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

As $v_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} v_0$, we have $Q_T T v_n \xrightarrow[n\to+\infty]{\|\cdot\|_{X/T(0)}} Q_T T v_0$. On the other hand, we have

$$Q_{\widetilde{T}} \widetilde{T} v_n \xrightarrow[n\to+\infty]{\|\cdot\|_{M/T(0)}} Q_{\widetilde{T}} w_0.$$

Hence $Q_T T v_n \xrightarrow[n\to+\infty]{\|\cdot\|_{X/T(0)}} Q_T w_0$. Therefore, $\|v_n - v_o\|_V \xrightarrow[n\to+\infty]{} 0$.

(c) It is clear that there exist $k_1 > 0$ and $k_2 > 0$ such that $\|m\|_V \leqslant k_1 \|m\|_M$ for all $m \in M$ and $\|v\|_X \leqslant k_2 \|v\|_V$ for all $v \in V$. Then, by Definition 4.1, we have the desired result. □

Theorem 4.8. *Let $n \in \mathbb{N}^*$ and $(M,T) \in \mathcal{P}_n$. Then, for all $\lambda \neq 0$,*
1. *$R(\lambda - T)$ is closed in X if and only if $R(\lambda - T_M)$ is closed in M;*
2. *$N(\lambda - T) = N(\lambda - T_M)$;*
3. *$\lambda - T$ is bounded below if and only if $\lambda - T_M$ is bounded below.*

Proof. (1) For convenience, set $V_0 = M$, $V_n = X$ and $\|m\|_0 = \|m\|_M$ for $m \in M$. Let $V_1 = T^{-1}(M)$ be equipped with the complete norm $\|v\|_1 = \|v\|_X + \|\widetilde{T}v\|_0$. By Lemma 4.7, we have $M \subseteq V_1 \subseteq X$, the embeddings are continuous, and $T(V_1) \subseteq V_0$.

Continuing inductively in this fashion, using Lemma 4.7, construct V_k, $1 \leqslant k \leqslant n-1$, with norm $\|v\|_k = \|v\|_X + \|\widetilde{T}v\|_{k-1}$, having the properties:

(a) $T(V_k) \subseteq V_{k-1}$ for $1 \leqslant k \leqslant n$;
(b) $(V_k, \|v\|_k)$ is a Banach space, $1 \leqslant k \leqslant n-1$;
(c) $M = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{n-1} \subseteq V_n = X$, and each of the embeddings are continuous.

Let T_k be the restriction of T to V_k, $0 \leqslant k \leqslant n$. It is clear that $(V_{k+1}, T_k) \in \mathcal{P}_1$ for all $k \in \{0, \ldots, n-1\}$. It follows from Propositions 4.5 and 4.6 that for all $\lambda \neq 0$ and $1 \leqslant k \leqslant n$, $R(\lambda - T_{k-1})$ is closed if and only if $R(\lambda - T_k)$ is closed. Therefore, $R(\lambda - T)$ is closed if and only if $R(\lambda - T_M)$ is closed. This proves (1).

(2) First note that $N(\lambda - T_M) \subseteq N(\lambda - T)$. Now suppose that $\lambda \neq 0$ and let $x \in N(\lambda - T)$. Then, $x \in \frac{1}{\lambda} T x$, and for n we have $x \in \frac{1}{\lambda^n}(T^n x)$. This implies that $x \in M$. Therefore, $N(\lambda - T) = N(\lambda - T_M)$.

(3) Follows from (1) and (2). □

5. Reconstruction of the relative regularity property

This section will shed light on some definitions, auxiliary results and properties of relatively regular linear relations that will be needed in the what follows.

Definition 5.1. A bounded linear operator S will be called a *pseudo inverse* of $T \in BCR(X)$ or g_1*-inverse* of T if

$$TST = T \quad \text{and} \quad T(0) \subset N(S).$$

We then say that T is a *relatively regular* linear relation.

Remark 5.2 ([13, Remark 1.1])**.** Let $T \in BCR(X)$. If $T(0)$ is topologically complemented, then it is sufficient to verify that there exists $S \in L(X)$ such that $TST = T$ to prove that T is a relatively regular linear relation.

Proposition 5.3 ([13, Theorem 2.2])**.** *Let $T \in BCR(X)$ be such that $T(0)$ is topologically complemented. Then T has a g_1-inverse if and only if both the null space $N(T)$ and the range space $R(T)$ are topologically complemented in X.*

Definition 5.4 ([7, Definition I.5.1])**.** Let $T \in BCR(X)$ and $A \in L(X)$. A is called a *selection* (or *single valued part*) of T if

$$T = A + T - T.$$

If A is a selection of T, then we have, for all $x \in X$, $Tx = Ax + T(0)$. So $R(T) = R(A) + T(0)$.

Proposition 5.5 ([7, Proposition I.5.2])**.** *Let $T \in BCR(X)$ be such that $T(0)$ is topologically complemented. If P is a bounded linear projection with kernel $T(0)$ then PT is a selection of T. Conversely, if A is a selection of T and $A(X) \cap T(0) = \{0\}$, then any bounded linear projection P defined on X with kernel $T(0)$ and such that $A(X) \subset P(X)$, satisfies $A = PT$.*

Lemma 5.6. *Let $T \in BCR(X)$ be such that $T^2(0) = T(0)$ and $T(0)$ is topologically complemented and P be a bounded projection with kernel $T(0)$. Assume that G_0 is a g_1-inverse of $I - T$. Let A be the selection of T given by $A = PT$. Let $G_1 = P + A + AG_0T$. Then G_1 is a g_1-inverse of $I - T$.*

Proof. To begin the proof, we note that $TA = AT + T(0)$. Introduce the following equation

$$\begin{aligned}
&(I - T)G_1(I - T) \\
&\qquad = (I - T)(P + A + AG_0T)(I - T) \\
&\qquad = (I - T)(P + A)(I - T) + (I - T)AG_0(I - T)T \\
&\qquad = (I - T)(P - A) + (I - T)A(I - T) + A(I - T)G_0(I - T)T + T(0) \\
&\qquad = P - A - TP + TA + A - TA - AT + TAT + AT - AT^2 + T(0) \\
&\qquad = P - TP + AT^2 - AT^2 + T(0).
\end{aligned}$$

Now, as $T^2(0) = T(0)$ and $A = PT$, we have $AT^2 - AT^2 = 0$. So, $(I - T)G_1(I - T) = (I - T)P = I - T$. On the other hand, it is easy to see that $T(0) \subset N(G_1)$ and using [7, Corollary II.3.13] we get that AG_0T is bounded. Therefore, G_1 is a g_1-inverse of $I - T$. ☐

Proposition 5.7. *Let $T \in BCR(X)$ be such that $T^2(0) = T(0)$ and $T(0)$ is topologically complemented and P be a bounded projection with kernel $T(0)$. Assume that G_0 is a g_1-inverse of $I - T$. Let A be the selection of T given by $A = PT$. Then for each integer $m \geqslant 1$, there exists a polynomial p_m with $p_m(0) = 0$ such that*

$$G_m = P + p_m(A) + A^m G_0 T^m \text{ is a } g_1\text{-inverse of } I - T.$$

Also, for $m \geqslant 0$, $G_{m+1} = P + A + AG_mT$.

Proof. Let $p_1(x) = x$. Then $G_1 = P + p_1(A) + AG_0T$ is a g_1-inverse for $I - T$ by Lemma 5.6. Now assume that for some $m \geqslant 1$, p_m is a polynomial with the properties in the statement of the proposition, and that $G_m = P + p_m(A) + A^m G_0 T^m$ is a g_1-inverse of $I - T$. Computing, we get

$$\begin{aligned} P + A + AG_mT &= P + A + A(P + p_m(A) + A^m G_0 T^m)T \\ &= P + A + A^2 + A^2 p_m(A) + A^{m+1} G_0 T^{m+1} \\ &= P + p_{m+1}(A) + A^{m+1} G_0 T^{m+1}, \end{aligned}$$

where $p_{m+1}(x) = x + x^2 + x^2 p_m(x)$. Let $G_{m+1} = P + p_{m+1}(A) + A^{m+1} G_0 T^{m+1}$. Since G_m is a g_1-inverse of $(I - T)$, we have that $G_{m+1} = P + A + AG_mT$ is a g_1-inverse of $I - T$ by Lemma 5.6. □

Remark 5.8. Let T and $R \in BCR(X)$ be such that $T_M = R_M$ where M is a dense subspace of X invariant under T and R. Then, $T = R$ on X. Indeed, let $x \in X$. Then, there exists $(x_n)_{n\in\mathbb{N}} \subset M$ such that $(x_n) \xrightarrow[n\to+\infty]{} x$. We have $T_M x_n = R_M x_n$, then $Tx_n = Rx_n$. Hence, $Q_T T x_n = Q_T R x_n$. On the other hand, as $T_M = R_M$, we get $T(0) = R(0)$. Therefore, $Q_T T x_n = Q_R R x_n$. Since $Q_T T$ and $Q_R R$ are bounded operators, and $x_n \xrightarrow[n\to+\infty]{} x$, we have $Q_T T x = Q_R R x = Q_T R x$. Therefore, $Q_T(Tx - Rx) = \overline{0}$. Then, $Tx - Rx \subset T(0)$. Thus, $Tx = Rx$.

We state now the main theorem of this section.

Theorem 5.9. *Let $(M, T) \in \mathcal{P}$ be such that $T(0)$ is topologically complemented and P be a bounded projection with kernel $T(0)$. Set $A = PT$. For $\lambda \neq 0$, if $\lambda - T$ has a g_1-inverse in $L(X)$, then $\lambda - T_M$ has a g_1-inverse in $L(M)$. Conversely, if M is dense in X and $\lambda - T_M$ has a g_1-inverse in $L(M)$, then $\lambda - T$ has a g_1-inverse in $L(X)$.*

Proof. First, we verify that it suffices to prove the result when $\lambda = 1$. Note that for $\lambda \neq 0$, $\lambda^{-1}T \in \mathcal{P}$, and that $(I - \lambda^{-1}T)$ has a g_1-inverse G, so

$$(I - \lambda^{-1}T)G(I - \lambda^{-1}T) = (I - \lambda^{-1}T) \text{ if and only if } (\lambda - T)(\lambda^{-1}G)(\lambda - T) = (\lambda - T).$$

Now, fix $m \geqslant 1$ such that $T^m x \subset M$ for all $x \in X$. Assume that $I - T$ has a g_1-inverse G in $L(X)$. From Proposition 5.7 it follows that there exists a polynomial p_m such that $F = P + p_m(A) + A^m G T^m$ is a g_1-inverse of $I - T$. We have $T(M) \subset M$ and $T^m x \subset M$ for all $x \in X$. We claim that $A(M) \subset M$. Indeed, $A(M) = PT(M) \subset P(M)$. On the other hand, $X = T(0) \oplus R(P)$, where $T(0) = N(P)$. Let $x \in M$. Then, $x = x_0 + x_1$ with $x_0 \in T(0)$ and $x_1 \in R(P)$. Then, $P(x) = x_1 = x - x_0 \in M$. Hence $A(M) \subset M$. Furtheremore, $A^m x \in T^m x \subset M$ for all $x \in X$. Hence $F(M) \subset M$.

Conversely, assume that M is dense in X and $\lambda - T_M$ has a g_1-inverse G in $L(M)$.

First of all, we need to verify that T_M is in $BCR(M)$. To do this, as T_M is everywhere defined in M, by the closed graph theorem [7, Theorem III.4.2] it is enough to prove that $G(T_M)$ is closed in M. Let $(x_n, y_n) \in G(T_M)$ be such that $(x_n, y_n) \xrightarrow[n\to+\infty]{M\times M} (x, y)$. Thus, $(x, y) \in M$ and $\|(x_n, y_n) - (x, y)\|_{X\times X} \leqslant M\|(x_n, y_n) - (x, y)\|_{M^2} \xrightarrow[n\to+\infty]{} 0$. Thus, $(x_n, y_n) \xrightarrow[n\to+\infty]{X\times X} (x, y)$. Now, as $G(T)$

is closed in $X \times X$, we have $(x, y) \in G(T)$. Hence, $G(T_M)$ is closed in $M \times M$. On the other hand, we have $M = (T(0) \oplus R(P)) \cap M = T(0) \oplus R(P) \cap M$. We note that $T(0)$ and $R(P) \cap M$ are closed in M. Let P_1 be the bounded projection in M with kernel $T(0)$ and range $R(P_1) = R(P) \cap M$. We note that $P_1 = P_M$. Let $A_1 = P_1 T_M$. Then by Proposition 5.7, there exists a polynomial p_m such that $F = P_1 + p_m(A_1) + A_1^m G T_M^m \in L(M)$ is a g_1-inverse of $I - T_M$.

Let $\overline{F} = P + p_m(A) + A^m G T^m$. We claim that $\overline{F} \in L(X)$. We have $P \in L(X)$, $p_m(A) \in L(X)$ and we claim that $T^m \in BR(X, M)$. Indeed, as T^m is everywhere defined and $T^m x \subset M$ for all $x \in X$, to conclude that $T^m \in BCR(X, M)$ it suffices to notice that $G(T^m)$ is closed in $X \times M$. Thus, $A^m G T^m \in B(X, M)$. Then, as M is continuously embedded in X, $A^m G T^m \in L(X)$. So $\overline{F} \in L(X)$. Also, we have $F = \overline{F}_M$. Hence $\overline{F}$ is an extension of F.

Note that since M is dense in X and $(I - T_M)F(I - T_M) = (I - T_M)$ on M, it follows by Remark 5.8 that $(I - T)\bar{F}(I - T) = (I - T)$ on X. □

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