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NEW CHARACTERIZATION OF (b,c)-INVERSES THROUGH POLARITY

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Abstract. Given any ring R with unity 1 and any $a, b, c \in R$, a is called (b,c)-*polar* if there exist two idempotents $p, q \in R$ such that $p \in bRca$, $q \in abRc$, $pb = b$, $cq = c$, $cap = ca$ and $qab = ab$. These p and q are shown to be unique whenever they exist. The existence of $a^{\|(b,c)}$, the (b,c)-inverse of a, is shown to be equivalent to a being (b,c)-polar, and hence $a^{\|(b,c)}$ is itself unique and expressed in terms of p and q. Generalizing results of Koliha–Patrício and Song–Zhu–Mosić, further connections between the (b,c)-polar and (b,c)-invertible properties are found. Applying these results to bounded linear operators on a Banach space, we also generalize some known results in this setting.

1. Introduction

Throughout this paper, R will denote an associative ring with unity 1. An element $a \in R$ is *regular* if $a \in aRa$, i.e., $a = axa$ for some $x \in R$. Any such x is called an *inner inverse* of a. An inner inverse of a will be denoted by a^-. We denote the set of all inner invertible elements in R by R^-, while the group of units in R is denoted by R^{-1} and the set of all left invertible (resp. right invertible) elements in R by R_l^{-1} (resp. R_r^{-1}). For any $a \in R$ we define the *commutant* and *double commutant* of a respectively by

$$\mathrm{comm}(a) = \{x \in R : ax = xa\}$$
$$\mathrm{comm}^2(a) = \{x \in R : xy = yx, \text{ for all } y \in \mathrm{comm}(a)\}.$$

An element a is *quasinilpotent* if $1 + xa \in R^{-1}$ for all $x \in \mathrm{comm}(a)$ [6]. Let R^{nil} and R^{qnil} denote, respectively, the set of all nilpotent and quasinilpotent elements in R.

Following Drazin [3], an element $a \in R$ is said to be *Drazin invertible* if there exists $x \in R$ such that

$$x \in \mathrm{comm}(a), \quad xax = x, \quad \text{and} \quad a^{k+1}x = a^k$$

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for some nonnegative integer k. The element x is unique if it exists and is called the *Drazin inverse* of a and is denoted by a^D. The smallest nonnegative integer k satisfying the above conditions is called the *Drazin index* of a, and denoted by $\operatorname{ind}(a)$. The set of all Drazin invertible elements in R is denoted by R^D. If $\operatorname{ind}(a) \leq 1$, then x is called the *group inverse* of a, denoted by $a^\sharp$. We denote the set of all group invertible elements in R by $R^\sharp$.

Koliha and Patrício [8] extended the notion of Drazin inverse to that of generalized Drazin inverse: an element $a \in R$ is *generalized Drazin invertible* if there exists $b \in R$ such that

$$b \in \operatorname{comm}^2(a), \quad ab^2 = b, \quad \text{and} \quad a^2b - a \in R^{\text{qnil}}. \tag{1.1}$$

Any element $b \in R$ satisfying the conditions in (1.1) is unique and is called the *g-Drazin inverse* of a, denoted by a^{gD}. The set of all g-Drazin invertible elements in R is denoted by R^{gD}. Koliha and Patrício gave a characterization for (generalized) Drazin invertibility via idempotents by introducing the notion of polar and quasipolar elements. An element $a \in R$ is *quasipolar* (resp. *polar*) if there exists an idempotent $p \in R$ such that

$$p \in \operatorname{comm}^2(a), \quad a + p \in R^{-1}, \quad \text{and} \quad ap \in R^{\text{qnil}} \text{ (resp. } ap \in R^{\text{nil}}).$$

The idempotent p is unique and is called the *spectral idempotent* of a, denoted by a^π. It is proved that a is generalized Drazin invertible if and only if it is quasipolar, and that a is Drazin invertible if and only if a is polar. In this case, $a^{gD} = (a+p)^{-1}(1-p)$.

Based on this approach, Wang and Chen [14] introduced the notion of pseudopolarity. An element $a \in R$ is said to be *pseudopolar* if there exists an idempotent $p \in R$ such that

$$p \in \operatorname{comm}^2(a), \quad a + p \in R^{-1}, \quad \text{and} \quad ap \in R^{\text{rad}},$$

where R^{rad} denotes the Jacobson radical of R. Also, the idempotent p is unique if it exists. They also introduced the notion of pseudo Drazin invertibility, which lies between Drazin invertibility and generalized Drazin invertibility: an element a is *pseudo Drazin invertible* if there exists $b \in R$ such that

$$b \in \operatorname{comm}^2(a), \quad bab = b, \quad \text{and} \quad a^k - a^{k+1}b \in R^{\text{rad}}$$

for some nonnegative integer k. Such an element is unique if it exists and is called the *pseudo Drazin inverse* of a. Moreover, a is pseudo Drazin invertible if and only if a is pseudopolar [14].

Mary [10] introduced a generalized inverse using Green's relations. An element $a \in R$ will be said to be invertible along $d \in R$ if there exists $y \in R$ such that

$$yad = d = day, \quad yR \subseteq dR, \quad Ry \subseteq Rd.$$

Such a y is unique if it exists and called *the inverse of a along d*, denoted by $a^{\|d}$. Moreover, if a is invertible along d then d is regular. The set of all elements in R that are invertible along d is denoted by $R^{\|d}$.

Recently, to give a new characterization of the invertibility along an element via idempotent elements, Song, Zhu and Mosić [13] provided a definition for the concept of the polarity along an element in R. Let $a, d \in R$; we say that a is *polar along* d if there exists some $p \in R$ such that

$$p = p^2 \in \mathrm{comm}(da), \quad pd = d, \quad \text{and} \quad 1 + da - p \in R^{-1},$$

which is equivalent to

$$p = p^2 \in \mathrm{comm}(da), \quad pd = d, \quad \text{and} \quad p \in daRda.$$

In this case, p is unique and is denoted by $a^{d\pi}$. It is also proved that a is invertible along d if and only if a is polar along d. In this case, the inverse of a along d is given by $a^{\| d} = (1 + da - p)^{-1}d$, and p is also established via $p = a^{\| d}a$. Also, a is invertible along d if and only if a is dually polar along d. Recall that a is *dually polar along* d if there exists some $q \in R$ such that

$$q = q^2 \in \mathrm{comm}(ad), \quad dq = d, \quad \text{and} \quad 1 + ad - q \in R^{-1},$$

which is equivalent to

$$q = q^2 \in \mathrm{comm}(ad), \quad dq = d, \quad \text{and} \quad q \in adRad.$$

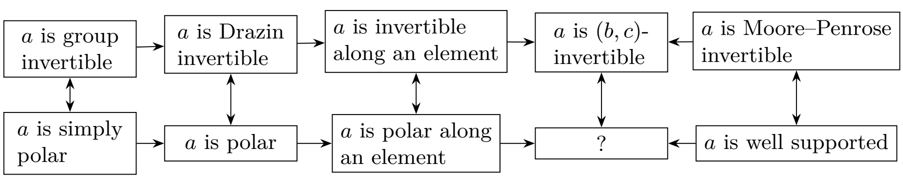
In this case, q is unique and is denoted by $a_{d\pi}$.

In 2012, Drazin [4] introduced a class of outer inverses that extend inverses along elements, and thus extend both Drazin inverses and Moore–Penrose inverses. For any $a, b, c \in R$, a is said to be (b,c)*-invertible* if there exists $y \in R$ such that

$$y \in bR \cap Rc, \quad yab = b, \quad cay = c.$$

If such a y exists, it is unique and is called the (b,c)*-inverse* of a, denoted by $a^{\|(b,c)}$. Also, if a is (b,c)-invertible, then b, c and cab are regular. The set of all (b,c)-invertible elements in R is denoted by $R^{\|(b,c)}$. In the case where $b = e$ and $c = f$ such that e and f are idempotents, we say that a is (e,f)-*Bott–Duffin invertible* if a is (e,f)-invertible [4]. Moreover, the inverse along an element is a special case of the more general class of (b,c)-inverses that occurs when $b = c$; consequently, we have $a^{\| d} = a^{\|(d,d)}$ and $a^D = a^{\|(a^k,a^k)}$, where k is the index of a, and in particular $a^\sharp = a^{\|(a,a)}$.

The approach of introducing new generalized inverses via polarities was used also in [5, 9]. So it is natural to ask whether there exists a kind of polarity that extends polarity and polarity along an element, and also characterizes (b,c)-invertibility (see also [5]). More precisely, the motivation for this paper arises from the following incomplete diagram of the related concepts:



We introduce in this paper the notion of (b,c)-polarity (Definition 2.1). We show that when $b = c$, (b,b)-polarity coincides with polarity along b, which then extends polarity along an element. Moreover, we show that an element a is (b,c)-polar if and only if a is (b,c)-invertible. We give then a new characterization of (b,c)-invertible elements. In Section 3, we introduce the concept of dually (b,c)-polar elements as an extension of dually polar along an element introduced in [13]. Among other things, we show that a is dually (b,c)-polar if and only if a is (c,b)-invertible. The last section is devoted to illustrating (b,c)-polarity in the context of bounded linear operators.

2. The (b,c)-polarity

We start by introducing the concept of (b,c)-polarity.

Definition 2.1. Let a, b, $c \in R$; we say that a is (b,c)-*polar* if there exist p, $q \in R$ such that

(1) $p^2 = p \in bRca$;
(2) $q^2 = q \in abRc$;
(3) $pb = b$, $cq = c$;
(4) $cap = ca$, $qab = ab$.

Any idempotent p (resp. q) satisfying the above conditions is called a *left* (b,c)-*spectral idempotent* of a (resp. a *right* (b,c)-*spectral idempotent* of a).

In the following, we show the uniqueness of the left and right (b,c)-spectral idempotents of a (b,c)-polar element.

Theorem 2.2. *Let a, b, $c \in R$ such that a is (b,c)-polar. Then a has a unique left (b,c)-spectral idempotent and a unique right (b,c)-spectral idempotent.*

Proof. Suppose that p and p' are two left (b,c)-spectral idempotents of a, and q and q' are two right (b,c)-spectral idempotents of a. As $p \in bRca$, we have $p = btca$ for some $t \in R$. It follows that

$$p - p'p = btca - p'btca = (b - p'b)tca = 0 \quad (\text{since } b = pb = p'b).$$

So we obtain

$$p = p'p.$$

Similarly, $p' - pp' = 0$, and we get

$$p' = pp'.$$

On the other hand, we have $cap = ca = cap'$, so $p - pp' = btca - btcap' = btca - btca = 0$. Hence $p = pp'$ and thus $p = p'$.

Similarly, we show that $q = q'$. □

If a is (b,c)-polar then we denote the left (b,c)-spectral idempotent p by $a_l^{(b,c)\pi}$ and the right (b,c)-spectral idempotent q by $a_r^{(b,c)\pi}$.

Example 2.3. Let $R = \mathcal{M}_2(\mathbb{Z})$, and $a, b, c \in R$ be given by

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then a is (b,c)-polar with

$$p = a_l^{(b,c)\pi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = a_r^{(b,c)\pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Indeed, with a quick check we obtain

(i) $p^2 = p$, $q^2 = q$.

(ii) $p = b\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} ca$ and $q = ab\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} c$.

(iii) $pb = b$, $cq = c$.

(iv) $cap = ca$, $qab = ab$.

Here we show that the (b,c)-polarity is an extension of the polarity along an element.

Proposition 2.4. *Let a and $b \in R$. Then a is (b,b)-polar if and only if a is polar along b.*

Proof. If a is (b,b)-polar then we have

$$p = p^2 \in bRba, \quad q = q^2 \in abRb, \quad pb = b = bq, \quad \text{and} \quad bap = ba = pba,$$

which implies $p = p^2 \in \operatorname{comm}(ba)$ and $pb = b$. Also, since $p = bxba$ for some $x \in R$, we obtain $p = bqxba \in babRbxba \subseteq baRba$. Hence, a is polar along b by [13, Theorem 2.4].

Conversely, if a is polar along b then there exists a unique $p = p^2 \in R$ such that $p \in \operatorname{comm}(ba)$, $pb = b$ and $p \in baRba$. Then we have

$$\begin{cases} p \in baRba \subseteq bRba \\ pb = b \\ bap = pba = ba. \end{cases} \tag{2.1}$$

On the other hand, we have that a is polar along b if and only if a is dually polar along b. Then there exists a unique $q = q^2 \in R$ such that $q \in \operatorname{comm}(ab)$, $bq = b$, and $q \in abRab$. So

$$\begin{cases} q \in abRab \subseteq abRb \\ bq = b \\ qab = abq = ab. \end{cases} \tag{2.2}$$

Now, from (2.1) and (2.2), we can see that a is (b,b)-polar. □

Lemma 2.5. *Let $a, b, c \in R$. If a is (b,c)-polar, then a, c, and cab are regular.*

Proof. Suppose that a is (b,c)-polar. We have $b = a_l^{(b,c)\pi} b \in bRcab \subseteq bRb$, and also $c = ca_r^{(b,c)\pi} \in cabRc \subseteq cRc$ and $cab = ca_r^{(b,c)\pi} aa_l^{(b,c)\pi} b \in cabRcabRcab \subseteq cabRcab$. This means that b, c and cab are regular and admit inner inverses denoted respectively by b^-, c^- and $(cab)^-$. □

The following theorem shows the equivalence between the (b,c)-polarity and the (b,c)-invertibility.

Theorem 2.6. *Let $a,b,c \in R$. Then a is (b,c)-polar if and only if a is (b,c)-invertible. In this case, we have*

(i) $p = a_l^{(b,c)\pi} = a^{\|(b,c)}a$.
(ii) $q = a_r^{(b,c)\pi} = aa^{\|(b,c)}$.
(iii) $a^{\|(b,c)} = (1+p-bb^-)_l^{-1}b(cab)^-c = b(cab)^-c(1+q-c^-c)_r^{-1}$. *Here* $(1+p-bb^-)_l^{-1}$ *is a left inverse of* $1+p-bb^-$, *and* $(1+q-c^-c)_r^{-1}$ *is a right inverse of* $1+q-c^-c$.

Proof. Suppose that a is (b,c)-polar. To prove that a is (b,c)-invertible, it suffices to prove that $b \in Rcab$ and $c \in cabR$ by [1, Lemma 1]. We have

$$a_l^{(b,c)\pi} \in bRca, \quad a_r^{(b,c)\pi} \in abRc, \quad ca_r^{(b,c)\pi} = c, \quad \text{and} \quad a_l^{(b,c)\pi}b = b.$$

It follows that

$$a_l^{(b,c)\pi}b \in bRcab \;\Rightarrow\; b \in bRcab \subseteq Rcab,$$

and

$$ca_r^{(b,c)\pi} \in cabRc \;\Rightarrow\; c \in cabRc \subseteq cabR.$$

So a is (b,c)-invertible.

By Lemma 2.5, we know that b,c and cab are regular with b^-, c^- and $(cab)^-$ as inner inverses of b,c and cab, respectively. Moreover, we have that $1+a_l^{(b,c)\pi}-bb^-$ is left invertible and $1+a_r^{(b,c)\pi}-c^-c$ is right invertible. Indeed, since $a_l^{(b,c)\pi} \in bRca \subseteq bR$, $a_l^{(b,c)\pi} = bt$ for some $t \in R$ and we write $a_l^{(b,c)\pi} = bt = bb^-bt = bb^-a_l^{(b,c)\pi}$. So

$$(bb^- + 1 - a_l^{(b,c)\pi})(1 + a_l^{(b,c)\pi} - bb^-) = 1,$$

which means that $1 + a_l^{(b,c)\pi} - bb^-$ is left invertible, and we denote a left inverse of $1 + a_l^{(b,c)\pi} - bb^-$ by $(1 + a_l^{(b,c)\pi} - bb^-)_l^{-1}$. Similarly for $1 + a_r^{(b,c)\pi} - c^-c$, as $a_r^{(b,c)\pi} \in abRc \subseteq Rc$ we write $a_r^{(b,c)\pi} = xc$ for some $x \in R$, so $a_r^{(b,c)\pi} = xc = xcc^-c = a_r^{(b,c)\pi}c^-c$, which implies

$$(1 + a_r^{(b,c)\pi} - c^-c)(c^-c + 1 - a_r^{(b,c)\pi}) = 1.$$

Hence $1 + a_r^{(b,c)\pi} - c^-c$ is right invertible, and we denote a right inverse of $1 + a_r^{(b,c)\pi} - c^-c$ by $(1 + a_r^{(b,c)\pi} - c^-c)_r^{-1}$.

Now set $y = (1+a_l^{(b,c)\pi} - bb^-)_l^{-1}b(cab)^-c = b(cab)^-c(1+a_r^{(b,c)\pi} - c^-c)_r^{-1}$. Then $yab = b$, $cay = c$ and $y \in bR \cap Rc$. Therefore, y is the (b,c)-inverse of a.

Conversely, suppose that a is (b,c)-invertible with y as the (b,c)-inverse of a. Set $p = ya$ and $q = ay$. Then p is the left (b,c)-spectral idempotent of a, and q is the right (b,c)-spectral idempotent of a. Indeed, we have

$$p^2 = yaya = ya = p \quad \text{and} \quad q^2 = ayay = ay = q.$$

On the other hand, as y is the (b,c)-inverse of a, we have $y \in bR \cap Rc$. Thus

$$p = ya \in bRa \subseteq bR \quad \text{and} \quad p = ya \in Rca, \quad \text{so } p = p^2 \in bRca,$$

and

$$q = ay \in abR \quad \text{and} \quad q = ay \in aRc \subseteq Rc, \quad \text{so } q = q^2 \in abRc.$$

We have $pb = yab = b$, $cq = cay = c$, $cap = caya = ca$, and $qab = ayab = ab$. Finally, a is (b,c)-polar. ☐

Combining Proposition 2.4 and Theorem 2.6, we retrieve the main result of [13].

Corollary 2.7. *Let $a, b \in R$. Then a is polar along b if and only if a is invertible along b if and only if a is (b,b)-invertible if and only if a is (b,b)-polar.*

Definition 2.8. Given $a, e, f \in R$, we say a is (e,f)-*Bott–Duffin polar* if a is (e,f)-polar and e and f are idempotents.

Proposition 2.9. *Let $a, b, c \in R$. If a is (b,c)-polar then a is (e,f)-Bott–Duffin polar with $e = a_l^{(b,c)\pi}$ and $f = a_r^{(b,c)\pi}$.*

Proof. Set $a_l^{(e,f)} = a_l^{(b,c)}$ and $a_r^{(e,f)} = a_r^{(b,c)}$. Then we obtain the result. ☐

Corollary 2.10. *Let $a, e, f \in R$. Then a is (e,f)-Bott–Duffin polar if and only if a is (e,f)-Bott–Duffin invertible.*

3. The dual (b,c)-polarity

Definition 3.1. Let a, b and $c \in R$. We say that *a is dually (b,c)-polar* if there exist $r, s \in R$ such that

(1) $r^2 = r \in acRb$;
(2) $s^2 = s \in cRba$;
(3) $br = b$ and $sc = c$;
(4) $rac = ac$ and $bas = ba$.

Any idempotent r (resp. s) satisfying the above conditions is called a *dual right (b,c)-spectral idempotent* of a (resp. a *dual left (b,c)-spectral idempotent* of a).

Theorem 3.2. *Let $a, b, c \in R$ such that a is dually (b,c)-polar. Then a has a unique dual right (b,c)-spectral idempotent and a unique dual left (b,c)-spectral idempotent.*

Proof. Suppose that r and r' are two dual right (b,c)-spectral idempotents of a. Then we have

$$r - r'r = actb - r'actb = actb - actb = 0$$

for some $t \in R$. Then $r = r'r$. Also,

$$r' - rr' = acxb - racxb = acxb - acxb = 0$$

for some $x \in R$. Hence $r' = rr'$. Consequently, we have

$$r'r - r' = acxbr - acxb = acx(br - b) = 0,$$

so $r'r = r'$. Therefore we get $r' = r$.

By the same way we prove the uniqueness of the dual left (b,c)-spectral idempotent of a. ☐

We denote the dual right (b,c)-spectral idempotent of a by $r = a^r_{(b,c)\pi}$ and the dual left (b,c)-spectral idempotent of a by $s = a^l_{(b,c)\pi}$.

Lemma 3.3. *Let $a, b, c \in R$. If a is dually (b,c)-polar then b, c and bac are regular.*

Proof. It is similar to the proof of Lemma 2.5. □

Theorem 3.4. *Let $a, b, c \in R$. Then a is dually (b,c)-polar if and only if a is (c,b)-invertible. In this case, we have*

(i) $r = a^r_{(b,c)\pi} = aa^{\|(c,b)}$.
(ii) $s = a^l_{(b,c)\pi} = a^{\|(c,b)}a$.
(iii) $a^{\|(c,b)} = (1 + a^l_{(b,c)\pi} - cc^-)^{-1}_l c(bac)^- b = c(bac)^- b(1 + a^r_{(b,c)\pi} - b^- b)^{-1}_r$.

Proof. Suppose that a is dually (b,c)-polar. Then there exist $r, s \in R$ such that

$$br = b \quad \text{and} \quad r \in acRb,$$
$$sc = c \quad \text{and} \quad s \in cRba.$$

Then $br \in bacRb$, which implies that $b \in bacRb \subseteq bacR$. Also, $sc \in cRbac$, which means that $c \in cRbac \subseteq Rbac$. Thus a is (c,b)-invertible.

To obtain the formulas for the (c,b)-inverse of a, we follow the same procedure as in the proof of Theorem 2.6.

Conversely, suppose that a is (c,b)-invertible. Then we set $r = aa^{\|(c,b)}$ and $s = a^{\|(c,b)}a$. Following the same procedure as in the proof of Theorem 2.6, we obtain that a is dually (b,c)-polar with r (resp. s) its dual right (b,c)-spectral idempotent (resp. dual left (b,c)-spectral idempotent). □

It may be that a is dually (b,c)-polar but not (b,c)-polar, as shown by the following example.

Example 3.5. Let $R = \mathcal{M}_2(\mathbb{Z})$, and $a, b, c \in R$ be given by

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then a is dually (b,c)-polar with $r = a^r_{(b,c)\pi} = s = a^l_{(b,c)\pi} = a$. Indeed, with a quick check we obtain

- $r^2 = r$;
- $r = ac \begin{pmatrix} x_1 & x_2 \\ x_3 & 1 \end{pmatrix} b$, where x_1, x_2 and x_3 are arbitrary elements of $\mathbb{Z}$;
- $s^2 = s$; $s = c \begin{pmatrix} t_1 & t_2 \\ t_3 & 1 \end{pmatrix} ba$, where t_1, t_2 and t_3 are arbitrary elements of $\mathbb{Z}$;
- $br = b$ and $sc = c$;
- $rac = ac$ and $bas = ba$.

Notice that in this example, a is not (b,c)-polar because a is not (b,c)-invertible since $ab = 0$.

Corollary 3.6. *Let $a,b,c \in R$. Then a is (b,c)-polar if and only if a is dually (c,b)-polar.*

Proof. This follows from Theorems 2.6 and 3.4. □

Theorem 3.7. *Let $a,b,c \in R$. If a is both (b,c) and (c,b)-invertible such that $aba \in \operatorname{comm}(c)$ and $aca \in \operatorname{comm}(b)$, then we have*

(1) $a_l^{(b,c)\pi} = a^r_{(b,c)\pi}$.
(2) $a_r^{(b,c)\pi} = a^l_{(b,c)\pi}$.
(3) $a^{\|(b,c)} = ba(caba + 1 - (aba)^{c\pi})^{-1}c = (baca + 1 - (aca)^{c\pi})^{-1}bac$.
(4) $a^{\|(c,b)} = (caba + 1 - (aba)^{c\pi})^{-1}cab = ca(baca + 1 - (aca)^{b\pi})^{-1}b$.

Proof. To prove this, we first write the formulas for $a^{\|(b,c)}$ and $a^{\|(c,b)}$. We have $a \in R^{\|(b,c)} \cap R^{\|(c,b)}$. By [12, Theorem 1] we get

$$a^{\|(b,c)} = ba(aba)^{\|c} = (aca)^{\|b}ac \tag{3.1}$$

and

$$a^{\|(c,b)} = (aba)^{\|c}ab = ca(aca)^{\|b}. \tag{3.2}$$

So we obtain

$$a_l^{(b,c)\pi} = a^{\|(b,c)}a = (aca)^{\|b}aca;$$
$$a_r^{(b,c)\pi} = aa^{\|(b,c)} = aba(aba)^{\|c};$$
$$a^r_{(b,c)\pi} = aa^{\|(c,b)} = aca(aca)^{\|b};$$
$$a^l_{(b,c)\pi} = a^{\|(c,b)}a = (aba)^{\|c}aba.$$

As $aba \in \operatorname{comm}(c)$ and $aca \in \operatorname{comm}(b)$, we have $(aba)^{\|c}aba = aba(aba)^{\|c}$ and $aca(aca)^{\|b} = (aca)^{\|b}aca$ by [10, Theorem 10], and it follows that

$$a_l^{(b,c)\pi} = a^r_{(b,c)\pi} \quad \text{and} \quad a_r^{(b,c)\pi} = a^l_{(b,c)\pi}.$$

Using [13, Theorem 2.8], we obtain

$$(aba)^{\|c} = (caba + 1 - (aba)^{c\pi})^{-1}c$$

and

$$(aca)^{\|b} = (baca + 1 - (aca)^{b\pi})^{-1}b.$$

Substituting in (3.1) and (3.2), we obtain the result of items (3) and (4). □

Remark 3.8. (1) We can also write $a^{\|(b,c)}$ and $a^{\|(c,b)}$ by using the result of [15, Theorem 2.6] or [12, Proposition 5] as follows:

$$a^{\|(b,c)} = bac(abac)^{\sharp} = ba(caba)^{\sharp}c = b(acab)^{\sharp}ac = (baca)^{\sharp}bac,$$

$$a^{\|(c,b)} = cab(acab)^{\sharp} = ca(baca)^{\sharp}b = c(abac)^{\sharp}ab = (caba)^{\sharp}cab.$$

(2) If a is only (c,b)-polar, this does not allow us to have the equalities in the previous theorem, because we may not have the right and left (b,c)-spectral idempotents of a, since a may not be (b,c)-polar, as shown in Example 3.5.

An *involution* $*$ is a bijection $x \mapsto x^*$ on R that satisfies the following conditions for all $a, b \in R$:

(i) $(a^*)^* = a$;
(ii) $(ab)^* = b^* a^*$;
(iii) $(a+b)^* = a^* + b^*$.

We say that R is a $*$-ring if there is an involution on R.

Proposition 3.9. *Let R be a $*$-ring, and let $a, b, c \in R$. Then a is (b,c)-polar if and only if a^* is dually (b^*, c^*)-polar. In this case, we have*

$$(a_l^{(b,c)\pi})^* = (a^*)^r_{(b^*,c^*)\pi},$$
$$(a_r^{(b,c)\pi})^* = (a^*)^l_{(b^*,c^*)\pi}.$$

Proof. We have a that is (b,c)-polar if and only if a is (b,c)-invertible, by Theorem 2.6. Suppose that $y = a^{\|(b,c)}$, so $y \in bR \cap Rc$, $yab = b$, and $cay = c$. By involution we get $y^* \in c^* R \cap Rb^*$, $b^* a^* y^* = b^*$, and $y^* a^* c^* = c^*$, which means that a^* is (c^*, b^*)-invertible with inverse y^*, i.e., $y^* = (a^{\|(b,c)})^* = a^{*\|(c^*,b^*)}$. Moreover, a^* is (c^*, b^*)-invertible if and only if a^* is dually (b^*, c^*)-polar. And we have

$$(a_l^{(b,c)\pi})^* = (a^{\|(b,c)}a)^* = a^*(a^{\|(b,c)})^* = a^* a^{*\|(c^*,b^*)} = (a^*)^r_{(b^*,c^*)\pi},$$
$$(a_r^{(b,c)\pi})^* = (aa^{\|(b,c)})^* = (a^{\|(b,c)})^* a^* = a^{*\|(c^*,b^*)} a^* = (a^*)^l_{(b^*,c^*)\pi}. \qquad \square$$

Proposition 3.10. *Let $a, b, c \in R$ such that a is (b,c)-polar and let $k \geq 1$. If $a \in \mathrm{comm}(b) \cap \mathrm{comm}(c)$ and $ba = ca$, then we have*

(1) *a is polar along b and $a^{b\pi} = a_l^{(b,c)\pi}$.*
(2) *a is dually polar along c and $a_{c\pi} = a_r^{(b,c)\pi}$.*
(3) *a^k is (b^k, c^k)-polar, with $(a^k)_l^{(b^k,c^k)\pi} = a_l^{(b,c)\pi}$ and $(a^k)_r^{(b^k,c^k)\pi} = a_r^{(b,c)\pi}$.*

Proof. Since a is (b,c)-polar, there exist $p = a_l^{(b,c)\pi}$ and $q = a_r^{(b,c)\pi}$ such that

$$p = p^2 \in bRca, \quad q = q^2 \in abRc, \quad pb = b, \quad cq = c, \quad cap = ca, \quad \text{and} \quad qab = ab.$$

(1) and (2): As $ba = ca = ac = ab$, we have $bap = ba = pba$, which means that $p \in \mathrm{comm}(ba)$ and $acq = ac = qac$, which means that $q \in \mathrm{comm}(ac)$. Also, $p \in bRba \subseteq Rba$ and $q \in abRc = acRc \subseteq acR$. Then $p = tba$ for some $t \in R$ and $q = acx$ for some $x \in R$. Then

$$\begin{aligned}(ptp + 1 - p)(ba + 1 - p) &= ptpba + ptp(1-p) + (1-p)ba + 1 - p \\ &= ptba + 0 + 0 + 1 - p = p^2 + 1 - p \\ &= 1.\end{aligned}$$

Hence $(ba+1-p)\in R_l^{-1}$. And

$$\begin{aligned}(ac+1-q)(qxq+1-q)&=acqxq+ac(1-q)+(1-q)qxq+1-q\\&=acxq+0+0+1-q\\&=q^2+1-q\\&=1.\end{aligned}$$

Hence $ac+1-q\in R_r^{-1}$.

By Jacobson's lemma and the expression of $p=a^{\|(b,c)}a$ and $q=aa^{\|(b,c)}$, we have

$$1+ba-p\in R_l^{-1}\iff ac+1-q\in R_l^{-1}$$

and

$$1+ac-q\in R_r^{-1}\iff ba+1-p\in R_r^{-1}.$$

Thus we obtain $ba+1-p\in R^{-1}$ and $ac+1-q\in R^{-1}$. Consequently, a is polar along b with $a^{b\pi}=p=a_l^{(b,c)\pi}$, and a is dually polar along c with $a_{c\pi}=q=a_r^{(b,c)\pi}$.

(3): Since p and q are idempotents and $a\in\operatorname{comm}(b)\cap\operatorname{comm}(c)$, we have

$$c^ka^kp=(ca)^kp^k=(ca)^k=c^ka^k$$

and

$$qa^kb^k=q^k(ab)^k=(ab)^k=a^kb^k.$$

On the other hand, we have $pb^k=pbb^{k-1}=bb^{k-1}=b^k$ and $c^kq=c^{k-1}cq=c^{k-1}c=c^k$.

Using (1) and (2) we have

$$\begin{aligned}ba+1-p\in R^{-1}&\Longrightarrow(ba+1-p)^k\in R^{-1}\\&\iff(ba)^k+1-p\in R^{-1}\\&\iff p\in(ba)^kR(ba)^k\quad\text{by [13, Theorem 2.4]}\\&\iff p\in b^ka^kR(ca)^k\subseteq b^kRc^ka^k.\end{aligned}$$

Similarly,

$$\begin{aligned}ac+1-q\in R^{-1}&\Longrightarrow(1+ac-q)^k\in R^{-1}\\&\iff(ac)^k+1-q\in R^{-1}\iff q\in(ac)^kR(ac)^k\\&\iff q\in(ab)^kRa^kc^k\subseteq a^kb^kRc^k.\end{aligned}$$

Finally, a^k is (b^k,c^k)-polar with $(a^k)_l^{(b^k,c^k)\pi}=p=a_l^{(b,c)\pi}$ and $(a^k)_r^{(b^k,c^k)\pi}=q=a_r^{(b,c)\pi}$. □

Theorem 3.11. *Let $a,b,c,d\in R$ such that a is (b,c)-polar. Then the following conditions are equivalent:*

(1) *d is (b,c)-polar such that $a_l^{(b,c)\pi}=d_l^{(b,c)\pi}$ and $a_r^{(b,c)\pi}=d_r^{(b,c)\pi}$.*

(2) $$\begin{cases}cda_l^{(b,c)\pi}=cd,\ a_l^{(b,c)\pi}\in bRcd,\ \text{and } a_l^{(b,c)\pi}b=b,\\ a_r^{(b,c)\pi}db=db,\ a_r^{(b,c)\pi}\in dbRc,\ \text{and } ca_r^{(b,c)\pi}=c.\end{cases}$$

(3) $\begin{cases} cda_l^{(b,c)\pi} = cd, \ a_l^{(b,c)\pi} \in bRcd \cap bRca, \ \textit{and}\ a_l^{(b,c)\pi} b = b, \\ a_r^{(b,c)\pi} db = db, \ a_r^{(b,c)\pi} \in abRc \cap dbRc, \ \textit{and}\ ca_r^{(b,c)\pi} = c. \end{cases}$

(4) *d is (b,c)-polar, $cda_l^{(b,c)\pi} = cd$, and $a_r^{(b,c)\pi} db = db$.*

Proof. (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (1). By Definition 2.1 and Theorem 2.2.

(2) $\Rightarrow$ (3). We have $a_l^{(b,c)\pi} \in bRca$ and $a_r^{(b,c)\pi} \in abRc$, and we also have, by hypothesis, $a_l^{(b,c)\pi} \in bRcd$ and $a_r^{(b,c)\pi} \in dbRc$. Hence $a_l^{(b,c)\pi} \in bRcd \cap bRca$ and $a_r^{(b,c)\pi} \in abRc \cap dbRc$.

(3) $\Rightarrow$ (2). It is obvious.

(1) $\Rightarrow$ (4). It is clear.

(4) $\Rightarrow$ (1). We prove that $a_l^{(b,c)\pi} = d_l^{(b,c)\pi}$ and $a_r^{(b,c)\pi} = d_r^{(b,c)\pi}$. First we have $d_l^{(b,c)\pi} \in bRcd$, so $d_l^{(b,c)\pi} = bxcd$ for some $x \in R$, and $a_l^{(b,c)\pi} \in bRca$, so $a_l^{(b,c)\pi} = btca$ for some $t \in R$. Moreover, $cdd_l^{(b,c)\pi} = cd = cda_l^{(b,c)\pi}$ and $a_l^{(b,c)\pi} b = b = d_l^{(b,c)\pi} b$, thus

$$d_l^{(b,c)\pi} - d_l^{(b,c)\pi} a_l^{(b,c)\pi} = bxcd - bxcda_l^{(b,c)\pi} = bxcd - bxcd = 0.$$

Hence

$$d_l^{(b,c)\pi} = d_l^{(b,c)\pi} a_l^{(b,c)\pi}.$$

Also,

$$a_l^{(b,c)\pi} - d_l^{(b,c)\pi} a_l^{(b,c)\pi} = btca - d_l^{(b,c)\pi} btca = btca - btca = 0.$$

So

$$a_l^{(b,c)\pi} = d_l^{(b,c)\pi} a_l^{(b,c)\pi}.$$

Consequently, $a_l^{(b,c)\pi} = d_l^{(b,c)\pi}$.

Similarly, we show that $d_r^{(b,c)\pi} = a_r^{(b,c)\pi}$. □

4. The (B,C)-polarity for bounded linear operators

Let X be a complex Banach space, and let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators. Let $A \in \mathcal{B}(X)$. We denote by $\mathcal{N}(A) = \{x \in X : Ax = 0\}$ the null space of A and by $\mathcal{R}(A) = \{Ax : x \in X\}$ the range of A, and we write $I \in \mathcal{B}(X)$ for the identity operator.

Let A, B, $C \in \mathcal{B}(X)$. Then A is (B,C)-polar if there exist two projections $P, Q \in \mathcal{B}(X)$ such that

(1) $P \in B\mathcal{B}(X)CA$;
(2) $Q \in AB\mathcal{B}(X)C$;
(3) $PB = B$, $CQ = C$;
(4) $CAP = CA$, $QAB = AB$.

A closed subspace M of X is a complemented subspace of X if there exists a closed subspace N of X such that $X = M \oplus N$.

Recall that an operator $A \in \mathcal{B}(X)$ is regular if and only if $\mathcal{R}(A)$ is closed and a complemented subspace of X, and $\mathcal{N}(A)$ is a complemented subspace of X (see [11, Proposition 13.1]).

Theorem 4.1. *Let A, B, $C \in \mathcal{B}(X)$ such that B, C and CAB are regular. Then the following assertions are equivalent:*

(1) *A is (B,C)-invertible;*
(2) *A is (B,C)-polar;*
(3) *There exist projections $P, Q \in \mathcal{B}(X)$ such that*
  (i) *$\mathcal{R}(P) = \mathcal{R}(B)$;*
  (ii) *$\mathcal{N}(Q) = \mathcal{N}(C)$;*
  (iii) *$\mathcal{R}(Q) = \mathcal{R}(AB)$;*
  (iv) *$\mathcal{N}(P) = \mathcal{N}(CA)$.*

Proof. (1) $\Leftrightarrow$ (2). By Theorem 2.6.
(2) $\Rightarrow$ (3).

(i) From $P \in B\mathcal{B}(X)CA$ we have $\mathcal{R}(P) \subseteq \mathcal{R}(B)$. Also, from $PB = B$, we see that $\mathcal{R}(B) \subseteq \mathcal{R}(P)$. Hence
$$\mathcal{R}(P) = \mathcal{R}(B).$$
(ii) $CQ = C \Rightarrow \mathcal{N}(Q) \subseteq \mathcal{N}(C)$.
Now let $x \in X$ such that $Cx = 0$. Since $Q \in AB\mathcal{B}(X)C$, we have $Qx = 0$ and so $\mathcal{N}(C) \subseteq \mathcal{N}(Q)$, thus
$$\mathcal{N}(Q) = \mathcal{N}(C).$$
(iii) $Q \in AB\mathcal{B}(X)C \Rightarrow \mathcal{R}(Q) \subseteq \mathcal{R}(AB)$. Also, $QAB = AB \Rightarrow \mathcal{R}(AB) \subseteq \mathcal{R}(Q)$. Hence
$$\mathcal{R}(Q) = \mathcal{R}(AB).$$
(iv) $CAP = CA \Rightarrow \mathcal{N}(P) \subseteq \mathcal{N}(CA)$, and $P \in B\mathcal{B}(X)CA \Rightarrow \mathcal{N}(CA) \subseteq \mathcal{N}(P)$. Hence
$$\mathcal{N}(P) = \mathcal{N}(CA).$$

(3) $\Rightarrow$ (1). To show that A is (B,C)-invertible, it suffices to prove that $\mathcal{N}(B) = \mathcal{N}(CAB)$ and $\mathcal{R}(C) = \mathcal{R}(CAB)$ by virtue of [2, Theorem 4.1].

First we can see that $PB = B$ and $CQ = C$. Indeed, let $x \in X$. As $Bx \in \mathcal{R}(B) = \mathcal{R}(P)$, we have $P(Bx) = Bx$ and so $PB = B$.

We have $Cx = C(x - Q(x)) + CQx = 0 + CQx = CQx$ as $x - Qx \in \mathcal{N}(Q) = \mathcal{N}(C)$. Hence $C = CQ$.

Obviously we have $\mathcal{R}(CAB) \subseteq \mathcal{R}(C)$. Let $y \in \mathcal{R}(C)$. Then there exists some $x \in X$ such that $y = Cx$. As $CQ = C$, we get $y = CQx$ and we can write $y = Cz$ for some $z = Qx \in \mathcal{R}(Q) = \mathcal{R}(AB)$, so $z = ABt$ for some $t \in X$. Thus we obtain $y = Cz = CABt$, which implies that $y \in \mathcal{R}(CAB)$. Hence $\mathcal{R}(C) \subseteq \mathcal{R}(CAB)$ and consequently
$$\mathcal{R}(C) = \mathcal{R}(CAB).$$

On the other hand, we obviously have $\mathcal{N}(B) \subseteq \mathcal{N}(CAB)$. Suppose that $s \in \mathcal{N}(CAB)$. Then $CABs = 0$. Set $z = Bs$. We obtain $CAz = 0$, which means that $z \in \mathcal{N}(CA) = \mathcal{N}(P)$, and hence $Pz = 0 = PBs$. Since $PB = B$, we obtain $Bs = 0$, which gives $s \in \mathcal{N}(B)$. Hence $\mathcal{N}(CAB) \subseteq \mathcal{N}(B)$, and we conclude that
$$\mathcal{N}(B) = \mathcal{N}(CAB).$$
Therefore A is (B,C)-invertible. □

Remark 4.2. Assume that A is (B,C)-invertible. Then, with respect to the decompositions $X=\mathcal{R}(P)\oplus\mathcal{N}(P)$ and $X=\mathcal{R}(Q)\oplus\mathcal{N}(Q)$, we have the following matrix representation of A:

$$A=\begin{bmatrix} QA & QA \\ (I-Q)A & (I-Q)A \end{bmatrix} : \mathcal{R}(P)\oplus\mathcal{N}(P)\longrightarrow \mathcal{R}(Q)\oplus\mathcal{N}(Q)$$
$$x_1+x_2\longmapsto A(x_1+x_2).$$

Indeed, with respect to the Peirce decomposition, we have

$$A=\begin{bmatrix} QAP & QA(I-P) \\ (I-Q)AP & (I-Q)A(I-P) \end{bmatrix}.$$

Then for $x_1\in\mathcal{R}(P)$ and $x_2\in\mathcal{N}(P)$ we have

$$QAP:\mathcal{R}(P)\longrightarrow\mathcal{R}(Q),\quad x_1\longmapsto QAPx_1=QAx_1;$$
$$(I-Q)AP:\mathcal{R}(P)\longrightarrow\mathcal{N}(Q),\quad x_1\longmapsto (I-Q)APx_1=(I-Q)Ax_1;$$
$$QA(I-P):\mathcal{N}(P)\longrightarrow\mathcal{R}(Q),\quad x_2\longmapsto QAx_2;$$
$$(I-Q)A(I-P):\mathcal{N}(P)\longrightarrow\mathcal{N}(Q),\quad x_2\longmapsto (I-Q)Ax_2.$$

Corollary 4.3. *Let $A,B\in\mathcal{B}(X)$, with B regular. Then the following assertions are equivalent:*

(1) *A is invertible along B.*
(2) *A is polar along B.*
(3) *There exists a projection $P\in\mathcal{B}(X)$ such that*
  (i) *$\mathcal{N}(P)=\mathcal{N}(BA)=\mathcal{N}(B)$;*
  (ii) *$\mathcal{R}(P)=\mathcal{R}(AB)=\mathcal{R}(B)$.*
(4) *$\mathcal{R}(B)$ is closed and a complemented subspace of X, $\mathcal{R}(AB)$ is closed with $X=\mathcal{R}(AB)\oplus\mathcal{N}(B)$, and $A|_{\mathcal{R}(B)}:\mathcal{R}(B)\to\mathcal{R}(AB)$ is invertible.*

Proof. The equivalence between (1), (2) and (3) follows from Theorem 4.1. The equivalence between (1) and (4) is [7, Theorem 2]; however, we can give another proof by showing that (2) or (3) is equivalent to (4).

Indeed, assume that A is polar along B; then by (3) $\mathcal{R}(B)=\mathcal{R}(P)$, which is closed and complemented in X, since P is a bounded projection. Also, $\mathcal{R}(AB)=\mathcal{R}(P)$ is closed and $X=\mathcal{R}(P)\oplus\mathcal{N}(P)=\mathcal{R}(AB)\oplus\mathcal{N}(B)$.

The operator $A|_{\mathcal{R}(B)}$ is surjective by construction. So let $x\in\mathcal{R}(B)$ such that $ABx=0$; then $BABx=0$ and hence $Bx\in\mathcal{N}(BA)=\mathcal{N}(P)$ by (3). Thus $0=PBx=Bx$. Therefore, $A|_{\mathcal{R}(B)}$ is injective.

Conversely, assume that (4) holds. Let P be the bounded projection onto $\mathcal{R}(AB)$. Let $x\in X$ with $x=x_1+x_2$ such that $x_1\in\mathcal{R}(AB)=\mathcal{R}(P)$ and $x_2\in\mathcal{N}(B)=\mathcal{N}(P)$. Then

$$BPx=Bx_1=B(x_1+x_2)=Bx.$$

So $BP=B$. Also,

$$ABPx=ABx=ABx_1=PABx_1=PABx.$$

Hence $P\in\text{comm}(AB)$.

Now, to deduce that A is dually polar along B, it remains to show that $AB+I-P$ is invertible. We have $(AB+I-P)x = ABx_1 + x_2$. Then if $(AB+I-P)x = 0$, we get $ABx_1 = 0$ and $x_2 = 0$. Since the operator $A|_{\mathcal{R}(B)}$ is invertible, we deduce that $x_1 = 0$. Thus $AB+I-P$ is injective.

Let $y = y_1+y_2$ such that $y_1 \in \mathcal{R}(AB)$ and $y_2 \in \mathcal{N}(B)$. Then $y_1 = ABx = ABx_1$ for some $x \in X$. Set $z = x_1 + y_2$. Then $(AB+I-P)z = ABx_1 + y_2 = y$. Hence $AB+I-P$ is surjective. Therefore A is dually polar along B, hence A is polar along B. □

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References

[1] X. Chen and J. Chen, The (b,c)-inverse in semigroups and rings with involution, *Front. Math. China* **15** no. 6 (2020), 1089–1104. DOI MR Zbl

[2] C. Deng and R. Liu, The existence and expressions of the inverse along operators B and C, *Oper. Matrices* **12** no. 4 (2018), 1027–1042. DOI MR Zbl

[3] M. P. Drazin, Pseudo-inverses in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506–514. DOI MR Zbl

[4] M. P. Drazin, A class of outer generalized inverses, *Linear Algebra Appl.* **436** no. 7 (2012), 1909–1923. DOI MR Zbl

[5] M. P. Drazin, Generalized inverses: Uniqueness proofs and three new classes, *Linear Algebra Appl.* **449** (2014), 402–416. DOI MR Zbl

[6] R. Harte, On quasinilpotents in rings, *Panamer. Math. J.* **1** (1991), 10–16. MR Zbl

[7] G. Kantún-Montiel and S. V. Djordjević, Invertibility along an operator, *Oper. Matrices* **11** no. 2 (2017), 347–354. DOI MR Zbl

[8] J. J. Koliha and P. Patricio, Elements of rings with equal spectral idempotents, *J. Aust. Math. Soc.* **72** no. 1 (2002), 137–152. DOI MR Zbl

[9] T. Li, F. Peng, and H. Zhu, Generalized inverses of elements and their polarities in rings, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **115** no. 3 (2021), Paper No. 109. DOI MR Zbl

[10] X. Mary, On generalized inverses and Green's relations, *Linear Algebra Appl.* **434** no. 8 (2011), 1836–1844. DOI MR Zbl

[11] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, second ed., Operator Theory: Advances and Applications 139, Birkhäuser, Basel, 2007. MR Zbl

[12] G. Shi and J. Chen, Symmetric properties of (b,c)-inverses, *Mathematics* **10** no. 16 (2022), Paper No. 2948. DOI

[13] Y. Song, H. Zhu, and D. Mosić, The polarity along an element in rings, *Rocky Mountain J. Math.* **53** no. 3 (2023), 937–949. DOI MR Zbl

[14] Z. Wang and J. Chen, Pseudo Drazin inverses in associative rings and Banach algebras, *Linear Algebra Appl.* **437** no. 6 (2012), 1332–1345. DOI MR Zbl

[15] C. Wu and J. Chen, On (b, c)-inverses and (c, b)-inverses, *Comm. Algebra* **49** no. 10 (2021), 4313–4323. DOI MR Zbl

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